Fluid Mechanics
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Lecture - 31
Incompressible Viscous Flows Part - III

Good morning. I welcome you all to this session of fluid mechanics. Now last class we were discussing about the velocity profiles in a quiet flow in exact solution of Navier-Stokes equations. Let us continue the discussion now.

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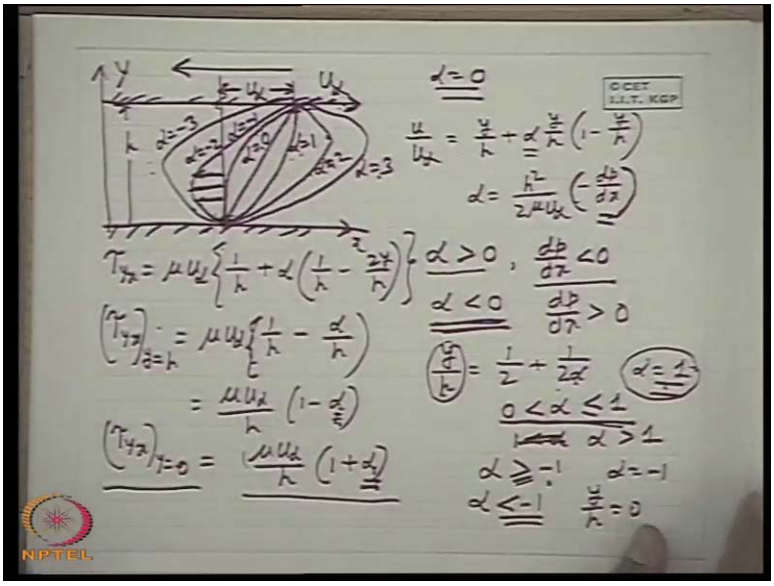

We will see that in quiet flow in the last class. We discussed the velocity profile as a function of y in this fashion. This is y direction; this is x 1 of the plate is moving. Now y by h plus alpha y by h, 1 minus y by h, where alpha is the non-dimensional pressure gradient which is defined as h square by 2 mu u infinity minus d p d x; h is the distance between the plates.

Now, you see from here that when alpha is equal to 0 is a simple case then u by u infinity is y by h. This is the curve which is plane quiet flow or simple quiet flow as I discussed earlier. So, from 0 at this plate which is addressed to u infinity linearly, here the velocity varies. Now, before going to find out the velocity profiles for different values of alpha as parameters, we discussed the different regimes. One regime is alpha greater than 0 when d p d x is less than 0; that means, this is a favorable pressure gradient the pressure decreases in the direction of flow. And another regime is that alpha less than 0 that is, negative value which is associated with the positive value of the pressure gradient; that means pressure increases in the direction of flow which is the adverse pressure gradient situation.

Now one thing was noticed that, this curve has both a maximum and minimum and it is found that when alpha is greater than 0 for negative pressure gradient, the curve shows a maximum; while alpha less than 0 for a positive pressure gradient, the curve shows a minimum. And for maximum and minimum condition was satisfied by making del u del y 0 and we arrived at a condition with respect to y, y by h half plus half alpha. This we already recognized in the last class.

Now, here we have seen that when alpha greater than 0, but less than 1; that means, in this regime of alpha, we have seen less than equal to 1, y by h, if you put here is more than 1; that means, more than the flow regime beyond the flow regime. So, they are put in this case, the maximum velocity is u infinity though a mathematical maximum may not exist. But exactly at alpha is equal to 1, we get a mathematical maximum at y a by h is equal to 1 that means, at this point there is a mathematical maximum; that means, that alpha is equal to 1 the maximum value of the velocity is u infinity because here the velocity is u infinity. But it shows a maximum value; that means, that del u del y or d u d y is 0 here; that is the curve alpha is equal to 1.

And we see that alpha greater than 1; that means, in these regime alpha greater than 1 or we can write this way alpha greater than 1, we see that y by h is lower than 1. Alpha greater than 1 from this equation y by h is less than 1. So, there both the maximum velocity occurs somewhere here. So, that has to be more the u infinity because, here the velocity will be u infinity. So, it will be more than this than it goes on decreasing. This is the value curve for different alpha 3.

Similarly, it is found that in this range when alpha less than 0 it has been found that, so long alpha is greater than minus 1, greater than is equal minus 1. Then the minimum will be found at a distance, lower than 0, less than 0 which is not visible. So, therefore, in this regime that means, alpha greater is equal to minus 1, the minimum velocity is the 0 velocity exactly at alpha is equal to minus 1. We get y by h 0; that means, that alpha is equal to minus 1. We get a 0 gradient here; that means, a minimum condition. And the minimum velocity is 0; that means, this will be alpha is equal to minus 1.

So, when alpha is still less than minus 1; that means, minus 2 minus 3 we, if we put here, we get a value of y by h greater than 0 which means slightly away from this fixed; that means, it comes in the flow regime a minimum velocity comes. And that minimum velocity will be lower than 0 velocity because here, the velocity will be 0 which means, a negative velocity appears somewhere here, which is the, which implies a reversal of flow in this direction. This is for alpha is minus 2 alpha is minus 3. So, this is precisely the velocity distribution characteristics that we discussed yesterday.

Now, you want see the nature of the shear stress distribution. Now you see if we write the shear stress from here, tau y x is simply mu d u d y. So, mu u infinity. So, if you made d u d y, it will be 1 by h plus alpha into it will be 1 by h minus y square 2, y by h. So, simply this is the expression for the shear stress.

Now, if we are interested for shear stress at 2 walls then first of all I write tau x y at y is equal to h. Then what we get? mu infinity 1 by h, sorry 1 by h. So, if we put y is equal to h, 2 by h, 1 by h. So, simply minus alpha by h; that means, we can write mu u infinity by h into 1 minus alpha. So, from here we see, clearly one thing that, when alpha is equal to 1 then this is 0; so you see, when alpha greater than, less than 1, when alpha less than 1, alpha between 0 to 1 in this regime, this shear stress is positive. Alpha is equal to once shear stress is 0; this curve. So, from the curve itself it is clear and when alpha greater than 1 the shear stress goes on changing its direction. What does it mean; that means, when alpha is equal to 0 there is no shear stress; that means the plate, moving plate does not experience any drag force. Obviously, in that case the favorable pressure gradient which is imposed in the flow at the value of alpha is equal to 1, can cause a velocity u infinity at this position y is equal to h. So, that if your plate is kept here, plate will automatically we dragged with u infinity velocity. You have understood?

Now, below a value of alpha is 1 that pressure gradient or even without pressure gradient one has to exertive force to drag the plate with a velocity u infinity in this direction, which will indicate that shear stress is positive here. So, when the value of the favorable pressure gradient, that minus d p d x or the value of alpha whichever you say, is more than 1, then what happens, the velocity here, this shear stress changes its direction, negative; that means, the drag force acting on this plate will be in the opposite direction which physically implies. That means, to maintain a velocity u infinity of the plate, this moving plate; we have to given forcing the opposite direction. Why? Because, in that case the favorable pressure gradient corresponding to the value of alpha is equal to 2, 3 will cause a velocity at y is equal to h more than u infinity; that means, you have plate is kept free it will move with a velocity more than u infinity. So, that your retarded force or a force in the opposite direction has to be kept to make the plate moving with u infinity for which the shear stress direction changes. And alpha is equal to 0 there is no shear stress there is no drag force required to move the plate.

In a similar fashion if we see the shear stress distribution; that means, tau y x at y is equal to 0. Then we see that y is equal to 0. It will be the same expression similar type expression 1 plus alpha. Here also we see that so long in this regime that alpha less than 0, so long alpha is greater than minus 1. What happened? Then, alpha is greater than minus 1 this is positive, but when alpha is equal to minus 1 this is 0. That means, the shear stress is 0 and alpha is less than minus 1, these shear stress direction changes. Why? Because the flow direction changes. So, here the change in shear stress direction on this plate means that, there is a change in the flow direction. That is, flow reversal takes place. So, up to alpha is equal to 0, the flow is in the positive direction. Exactly, sorry, alpha is equal to minus 1. This is alpha is equal to minus 1.

So, at exactly alpha is equal to minus 1 flow is 0; that means, the fluid particles moves in this direction alpha is equal to minus 1 is the point onset of flow reversal; that means, this is the onset point of the flow reversal after which, if you increase the negative pressure gradient. The value of the alpha is lesser and lesser than minus 1. Then the flow reversal will take place, so, shear stress will change its direction. So, this way we can find the shear stress distribution at the two plates in case of a quiet flow.

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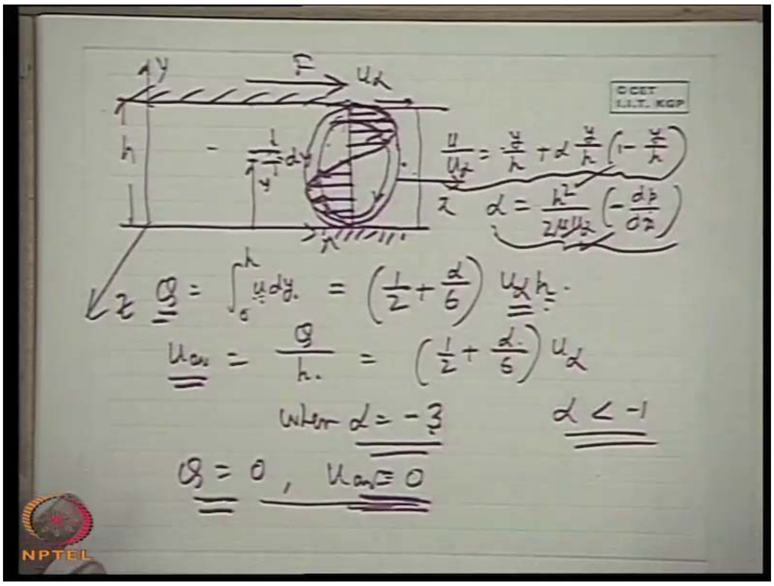

Now, another interesting feature comes for quiet flow is this, that if we find out the average velocity, the average flow velocity or the flow rate. Now, in a quiet flow, if you see again I am drawing this. Now in a quiet flow how to find out the, now this x, this is y, this is y. Now, u by u infinity is y by h plus alpha y by h, 1 minus y by h. So, it has to be kept in mind always, alpha is h square by 2 mu u infinity into minus d p d x.

Now, to find out the flow rate, it will be u d y; that means, we take an element small element d y at a distance y. So, if this is d y; and if we take a unit depth in the z direction in a direction perpendicular to the plane of the paper; so u d y, 0 2 h that means, if I substitute this, here this function and integrate you will get this equation, half plus alpha by 6 into u infinity. If you just make the integration. It is very simple and now doing it here so that means, u is equal to u infinity times, this if you substitute the value of u the function and integrate from 0 to h. You will get the value u infinity h half alpha plus 6. Alpha is the parameter which is non dimensional pressure gradient.

So, therefore, u average a average velocity in fluid mechanics is the flow rate divided by the cross sectional area which is simply h, because we are taking unit depth in the z direction; that means, in this direction z; perpendicular to the plane of the paper. So, it simply comes half, now, this gives a very interesting result. So, therefore, we see the average flow velocity or the flow rate is very much a functional alpha. So, the value depends upon the value of alpha that is, the pressure gradient along with the value of u infinity and h.

Now, you see one interesting thing that, when alpha is equal to minus 3, that is a advanced pressure gradient case; that means, when d p d x is positive and its value is such that alpha becomes minus 3. Then Q is equal to 0 or u average is equal to 0; that means, there is no flow in the positive x direction. This is a situation that there no flow takes place. What happens in this case, at any cross section we have seen that when alpha is less than minus 1 the flow reversal takes place.

So, alpha is minus as equal to minus 3, this situation is like this; that means, there is a flow reversal zone which is extended to a distance from the lower plate or the fixed plate and there is a forward velocity zone. So, these two zones cancel each other; that means, a local flow reversal zones takes place like this, a re circulatory flow takes place. So, the flow goes in the forward direction in the upper half, almost in the upper part of this thing, near the moving plate and the flow goes in this direction near the fixed plate or the lower plate. So, therefore, a local re circulatory flow takes place so that there is no net flow in this x direction where, the flow rate becomes 0 or the average flow velocity becomes 0.

That is one interesting fact. This is attained when the alpha is minus 3; that means, this is attained for a given value of h for a given value of u infinity for a given value of mu a liquid viscosity. In advanced pressure gradient if you impose; that means, which will be balancing the dragging effect, impose whose value equal to alpha is equal to minus 3, corresponds to an advance pressure gradient, in the situation where this advanced pressure gradient, which counter balancing or counter balance the dragging effect then the dragging effect that is a plate is dragged some force is acted on the plate and the advance pressure gradient both counter balance each other. So, that their net flow or the average flow velocity becomes 0.

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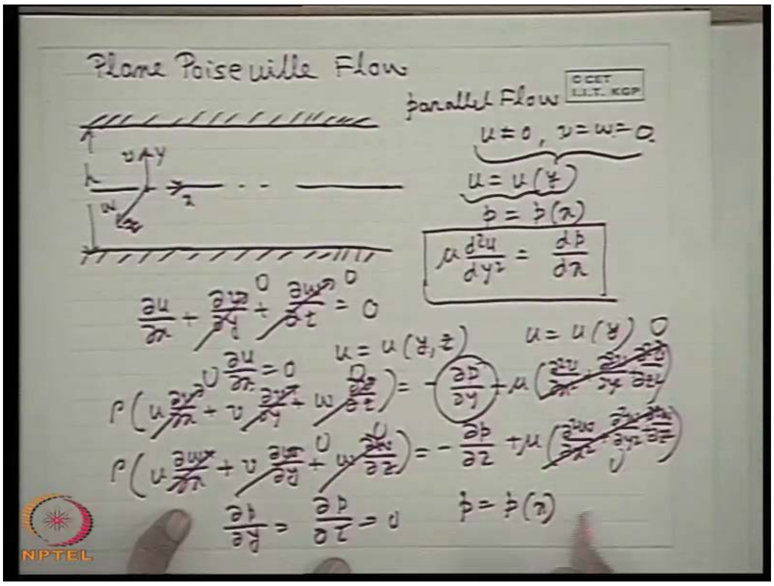

Now I will start another, I will solve another class of parallel flow problem which is plane Poiseuille flow, which is known as plane Poiseuille flow. Now how do you define it? Plane Poiseuille flow is defined like this that this is the flow between two plates, but both the plates are fixed, two fixed plates. In earlier case the one plate was moving with respect to another. Here the two plates are fixed. Let us consider the distance between the plates the h.

Now, when the two plates are fixed and we consider a parallel flow, we consider a parallel flow, parallel viscous fully developed flow. Do you what fully developed will be understood afterwards? The viscous force and inertia force are prominent; that means, throughout the flow the viscous force is dominant. So, this is a parallel incompressible study viscous flow. So, through two plates we, the two fixed plates, two fixed plates. So, for the from, from the symmetry of the problem we consider the middle axis, axis that is a middle point, middle axis between the plates as the x direction and we take this is as y. So, this origin from any, in at any point; so this is x and y.

Now, in this case how do you derive the velocity profile? If you remember that for parallel flow, this is a parallel flow, where u is not equal to 0, but v is equal to w is equal to 0; that means, y component and z component velocities are 0, but u naught is equal to 0. Now, this type of situation we have always earlier derive that, if we consider a parallel flow where only one component x, this other two components are 0 and if we consider the z direction dimension is much larger compared to this y direction dimension; that means, this h distance between the plates.

Then the final expression we can write like this we get certain conclusion or certain result that, u is a function of y only and pressure p is a function of x only. The direction, flow direction and we get this relationship that mu d square u d y square is equal to d p d x. So, this was derived earlier, but I can again recapitulate it, that this we get how. First of all we can write the compressible flow continuity equation in Cartesian coordinate del u del x plus del v del y plus del w del z is equal to 0.

Now, since v is equal to w is equal to 0, we can tell, this is equal to 0 and then del u del x is equal to 0. So, therefore, u(y,z) because u cannot be a function of x. Then if the z direction is so large, compared to the dimensions in the y direction so that, the changes in u with respect to z can be neglected in comparison to change in u with y. So, that we can write u(y). So, you arrive to this corollary from continuity.

Then, if we look to the pressure the Navier-Stokes equation in x and y direction, we will get the simplified equation. Let us first write the y direction Navier-Stokes equation it will be better. So, u del v del x plus v del v del y plus w del v del z is equal to minus del p del y plus mu del square v del x square plus del square v del y square plus del square v del z square. If I write the equations for z direction, I will write in x direction at last. So, it will be u del w del x plus v del w del y, this is the inertial term or the acceleration per unit volume, w del w del z is equal to minus del p del z Laplacian, gradient plus mu this same thing Laplacian of w del square w del x square del square w del y square plus del square w del z square.

Now, you see in two directions that v is 0. So, this all v w 0. So, in both the cases this is 0. So, the entire viscous term is 0; v w 0 which gives that del p del y is equal to 0; that means, del p del y they from this two equation del p del z is equal to 0; that means, p is a function of x only.

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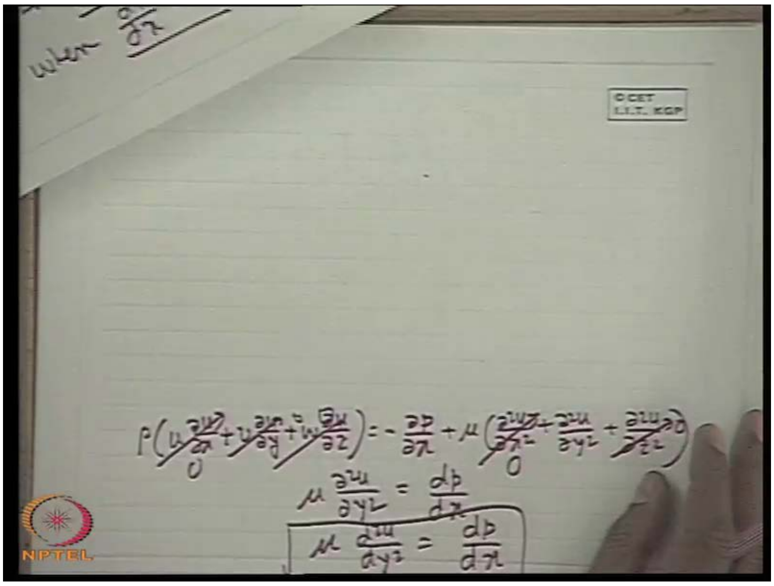

So, with this now, if we write the x direction equation of motions, then we get rho u del u del x plus v del u del y plus w del u del z is equal to minus del p del x plus mu del square u del x square plus del square u del y square plus del square u del z square. Here we see that del u del x is 0 from continuity; u is a function of y only v 0 w 0. And here u is not a function of x 0. Now among this two, we have considered that u is a function of y because z direction dimension is so large. So, we can make it simply. So, from here we get mu del square u del y square. Now p is a function of x only, so, we can write it d p d x and u is a function of y only. So, we can write mu, we can write mu d square u d y square is equal to d p d x. This is the equation for any parallel flow where only one component of velocity exist; that means, mu d square u d y square is equal to d p d x.

Now, the argument is that u is a function of y only p is a function of x only. So, therefore, the equality can only be valid provided d square u d y square and d p d x both are constant; that means, p is a linear function of x, u is a linear function of y.

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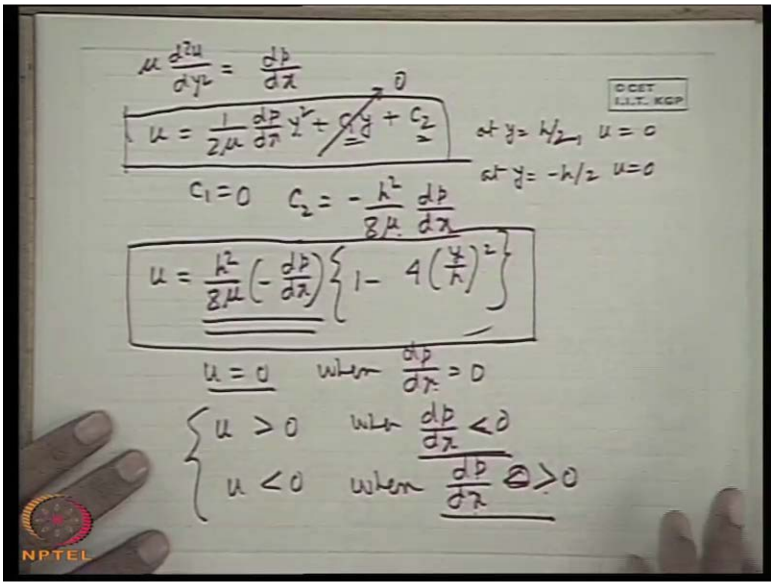

Now the next task which exists, is now the solution of this equation solution of this mu with respect to this. Let me do. mu d square u d y square is equal to d p d x, so, solved it. Now, simply if you make it the integration two times, we get 1 by 2 mu d p d x y square plus c 1 y plus, 2 integration constants. So, we get straight way the profile for u.

Now, the routine calculation is that you have to calculate c 1 c 2 from the boundary condition. What are the boundary conditions? Boundary conditions we have taken the coordinate axis this way; that means, this is h by 2 and this is h by 2. So that, at y is equal to h by 2; that means, this plate u is equal to 0. Similarly, at y is equal to minus h by 2 is equal to 0. So, this is symmetric about this x axis. So, if you put this boundary condition then you get c 1 0.

So, from the first very in, first inspection, first time inspection tells that c 1 cannot be there. Because, if this is symmetric about this axis x axis; that means, there should not be any term containing a (()) power of y. So, therefore, c 1 has to be 0, but if you put it this boundary condition you will also get c 1 0 and c 2 if you put any one of this boundary conditions. So, you put c 2 is minus h square by 8 mu into d p d x. So, this is the value of c 2.

So, now if I put this value of c 1 and c 2, I get the expression u is equal to h square by 8 mu. So, c 1 is 0. If I put the value of c 2 minus d p d x minus d p d x is written in this fashion, 1 minus 4 into y by h whole square. So, precisely this is the expression for the velocity distribution for the plane Poiseuille flow; that means, the flow is between two fixed plates the distance between the plate is h. So, this is the expression where the coordinate axis is taken at the middle axis, middle axis as the x axis from which the y is measured. So, this the velocity profile.

So, if we put this velocity, you can see probably I hope so. So, if we, if we put this velocity plot the velocity profile. So, a number of velocity profiles or a family of curves can be drawn with this as the parameter. So, velocity profile is like this, but one interesting thing here is that u is 0 when d p d x is 0. Similarly, u is greater than 0 when d p d x is less than 0 and u is less than 0 when d p d x is, u is less than 0 when d p d x is, sorry greater than 0.

What does it mean that when d p d x is 0 u is equal to 0. There is no flow when pressure gradient is 0; that means, this flow is totally, totally caused by the pressure gradient because, there is no other effect where the flow is caused; that means, if there is no pressure gradient there will be no flow. So, flow is possible only because of the pressure gradient; that means, the pressure depending upon the direction of the pressure gradient the flow will take place, but in the earlier case in the quiet flow the one plate was moving. So, the external energy was supplied to drag one of the plates to cause the flow. So, therefore, in that case, even at 0 pressure gradient on an adverse pressure gradient a forward direction flow was possible. But here when the pressure gradient you know there will no flow and this two condition implies that the flow takes place express only because of pressure gradient and takes the deduction of pressure gradient. When d p d x is less than 0; that means, pressure here is high pressure, here is low downstream pressure is low; that means, a negative pressure gradient the flow will take place in the positive x direction.

Similarly, when d p d x is greater than 0, there is a negative pressure gradient, sorry positive pressure gradient. The flow in higher in the downstream direction than, in the half stream direction. So, therefore, flow will take place in the direction of favorable pressure force; that means, in the opposite direction that is, the velocity flow of flow is negative. So, therefore, flow velocity takes place depending upon the direction of the pressure gradient. So, pressure gradient is the sole cause of the flow in this case.

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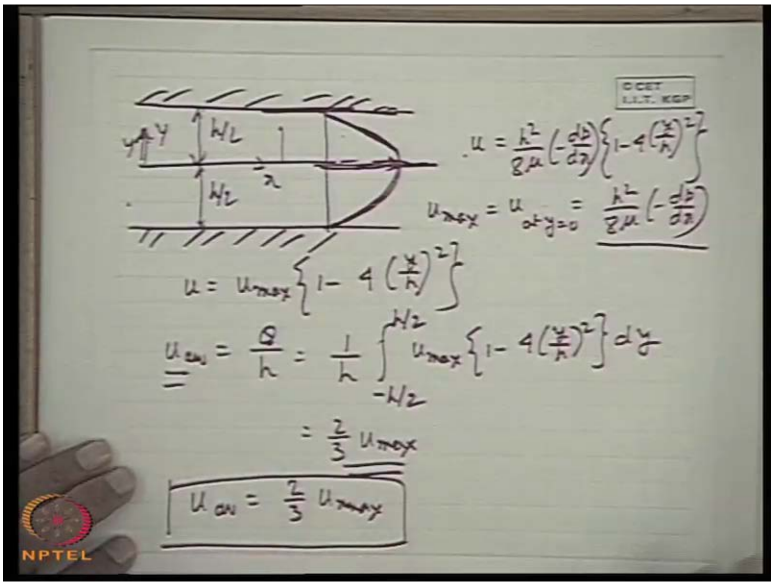

Now, we will go for the routine calculations of shear stresses and flow rate. Now, here if we find out the, want to find out the average flow or average flow velocity, let me this is the axis. So, this is x this is y. So, this is the distance h by 2, this is the distance 8 by 2, let this is the distribution of velocity. I am sorry, this is the distribution, the drawing is not that good. However, so this is the middle point.

So, therefore, we see that the velocity distribution u is equal to h square by 8 mu minus d p d x into 1 minus 4 y by h whole square. So obviously, maximum velocity takes place at y is equal to 0; when y is equal to 0 u is u max; that means, u at y is equal to 0 and its value is h square by 8 mu minus d p d x and this is the diagram. So, maximum velocity is at the central line. So, therefore, u can be written in terms of u max as u max into 1 minus 4 y by h whole square.

Now, if we want to find out the average velocity from the flow rate in the similar fashion. So, u average is equal to q by h; that means, 1 by h q means the flow rate which can be found out integration of u, u d y; that means, u max I just put the expression of u d y, u max is constant is not a function of y. So, it is h square by 8 mu minus d p d x. So, it is a typical pressure gradient combined with h square by 8 mu. So, u max it is the, it has the unit of velocity d y and it is integrated for minus h by 2 to h by 2 plus 8 by 2. Because axis is here so, y is measured in this direction y. So, minus h by 2 to plus h by 2 cover the entire flow regime.

So, if you do so, you will see that it will be coming two-third u max; that means, you take u max outside of the integration and you perform this integration from minus h by 2 to h by 2. So, then it will becoming two-third h, h, h will cancel. So, ultimately you will find u average is two-third of u max. So, this is one, two-third of u max. So, u max is given by this expression. Now, if we write the expression for shear stress, if we write the expression for shear stress what will be the expression for shear stress?

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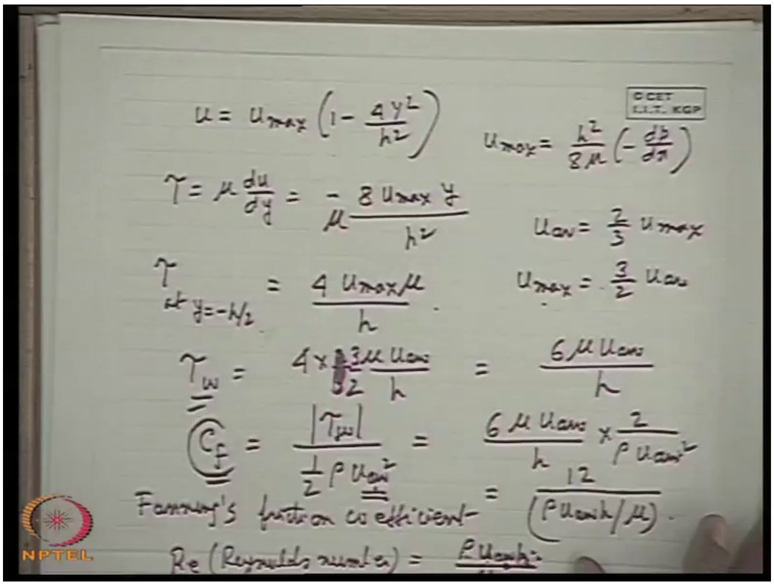

Again I write u is equal to u max, in terms of u max if I write. Then, it will be 1 minus 4 y square by h square. This is the expression; u max you can write h square by 8 mu into minus d p d x.

Now, you find out shear stress; it is mu du d y. So, what is this minus 8 u max y by h square, 2 y that is 8 y by h square. Now, if I want to find out tau at, for example at the lower plate, at y is equal to minus h by 2. So, what we get that h by 2; that means, we get 4 u max. So, 4 u max by h. So, this is the tau, sorry tau is mu d u d y. So, I am sorry d u d y is there will be one mu, mu into this u max mu, 4 u max mu by h.

Now, we are already developed an expression with u average and u max that u average is equal to two-third u max. We have already developed this from integration of this u average is q by h two-third u max. So, u average is two-third u max. So, u max is equal to 3 by 2 u average. So, u max is equal to 3 by 2 u average. So, therefore, tau at the wall 1 minus y is equal to minus h by 2 one of the walls, another wall y is equal to plus h by 2 the sign will be changed the value of the tau will be alright. So, if I replace it, it will be 4 into two-third, mu into u average by h. So, sorry 3 by 2, it is not two-third, 4 into 3 by 2, u max is 3 by 2 u average; that means, 6 mu u average by h. So, this is the value of tau w.

Now, if a friction coefficient c f is defined, it is known as fanning friction coefficient which is defined as, tau w by half rho u (()). Here in this case it will be defined as u average square. Now, in this context, I like to tell you afterwards, we will be discussing the pipe flow, flow passed bodies. So, this is a very good thing to remember that when the flow takes passed, flow takes place over a body or within a duct. So, we define a Skin friction coefficient, this is known as Fanning's friction coefficient which is due to the shear stress at the solid surface.

So, this friction coefficient is defined always at this fashion, the absolute value of the shear stress at the wall. Because, this is a scalar quantity without any sign dived by the dynamic head, half rho into velocity square. And this characteristic velocity will be the average velocity, in case of flow through a duct or channel. In this case we have taken the average velocity. But when there is a flow passed a body, flow passed a surface then this velocity will be the freestream velocity. So, therefore, this is the general expression for the Fanning's friction coefficient that, this is the friction coefficient defined because of this shear stress due to the friction, between the fluid and the solid surface which is equal to the shear stress at the wall; that means, that is solid surface divided by the dynamic head, based on the characteristic velocity which is the average velocity in case of a confined flow; and freestream velocity in case of a flow past bodies.

So this is the definition. So, if with this definition, if we just now do mu average by h and then half rho, half rho u average square; that means, two will goes and rho u average square. Then, we will come to a value 12 divided by rho u average h by mu. So, this is defined as Reynolds number. So, Reynolds number which will be discussed afterwards, Reynolds number. So, Reynolds number which is defined in case of a confined flow, flow through the duct, as rho mu these are the rheological properties density and viscosity and numerator this is the characteristic velocity which is the average velocity of flow times the characteristic dimension. So, Reynolds numbers is defined as rho times, the characteristic velocity times, the characteristic dimension divided by the viscosity. This characteristic velocity is the average flow velocity in case of an confined flow and h this characteristic dimension. Here is the h it is actually the hydraulic diameter which is defined as the cross sectional area divided by the weighted perimeter. In this case it is h. So, that the Reynolds number is based on the distance between the two plates; that means, this is the distances between the, distances between the two plates. So, this is the Reynolds number.

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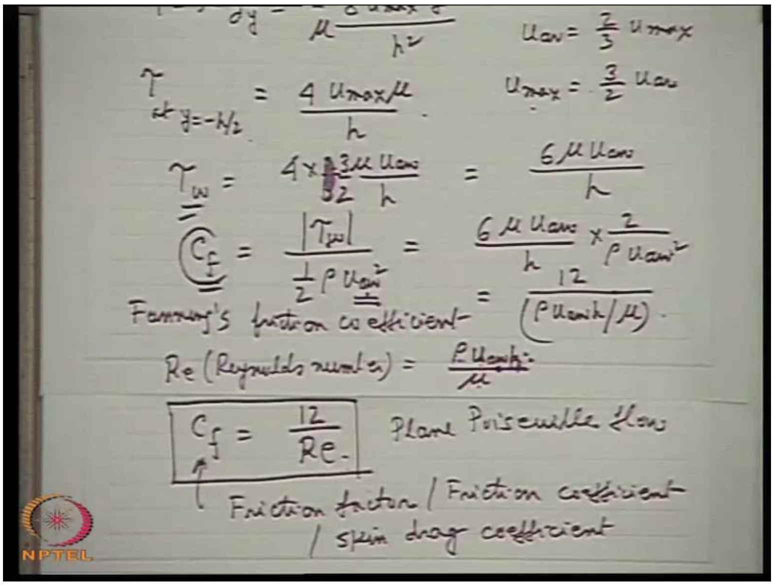

So, therefore, if we define the Reynolds number in this basis we get the free, this is one very important deduction for a confined flow that friction factor or this fact Fanning's friction coefficient. This is known as friction factor or friction coefficient. Fanning's friction factor or friction coefficient or sometimes this is known as drag coefficient, Skin drag coefficient which is used in case flow passed bodies. Skin drag coefficient there are different drag forces for flow passed bodies that will be discussed afterwards. Skin drag, that is why it is known as Skin drag. There is another kind of drag known as form drag Skin drag coefficient.

So, this is a function of Reynolds number only and in case of a plane Poiseuille flow, this is plane Poiseuille flow, in case of a plane Poiseuille flow, this is 12 by R e. So, this is one very important deduction 12 by R e. So, this is the. So, we have derived the Skin friction coefficient formula for plane Poiseuille flow along with the velocity distribution for the plane Poiseuille flow.

So, next class I will start the Hagen-Poiseuille flow another class of parallel flow which is the flow between a cylinder through a pipe; that means, the area is fixed, but the cross sectional area is a circular one.

Thank you.