Computer Numerical Control of Machine Tools and Processes
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Lecture 17
Curved Surface Geometry

Welcome viewers to the 17th lecture in the open online course "Computer numerical control of machine tools and processes". So in today's lecture will be covering certain aspects of Curved surface geometry.

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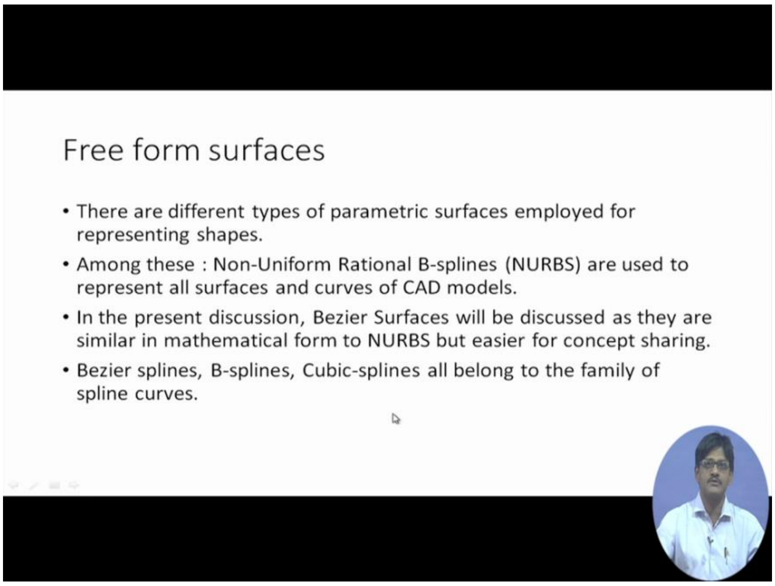

So to start with, there are different types of parametric surfaces which are employed for representing shapes in connection with modeling and machining. And among these the non-uniform rational B-splines or NURBS surfaces are used to represent all surfaces and curves of CAD models because they can represent all sorts of geometries. However, in today's discussion we will be restricting our discussion mainly to Bezier surfaces because 1st of all Bezier surfaces are easier for sharing concepts, not as difficult and complex as non-uniform rational B-splines and still the mathematical basic mathematical form of Bezier surfaces, B-splines, non-uniform rational B-splines, basic mathematical form is the same. So Bezier splines, B-splines, cubic splines, NURBS, et cetera, they all belong to the family of splines curves.

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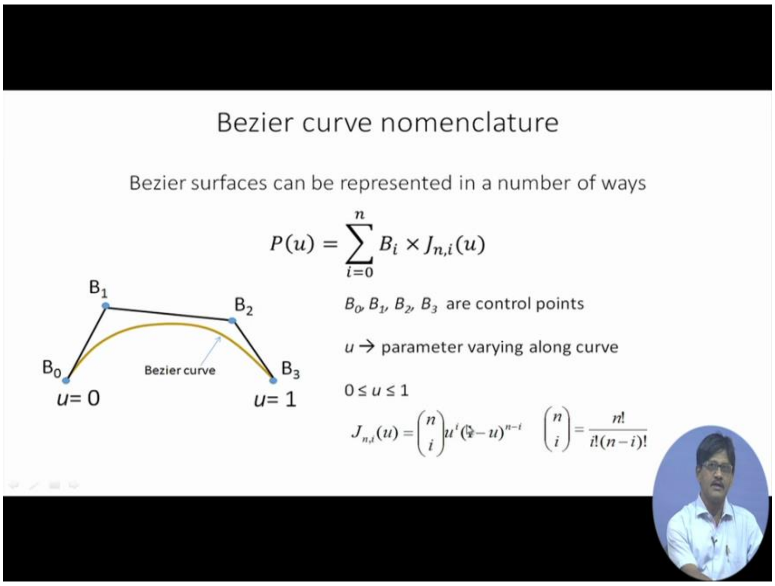

So let us have a quick look this we have already seen once, but let us have a quick look that is the Bezier curve 1st of all is being represented here and there are 4 Control points B 0, B 1, B 2, B 3, which are basically controlling the shape of this particular curve, in what way? They are having multipliers as shown that as a submission, the curve is being expressed as a submission of the control point coordinates that means B 0, B 1, B 2, et cetera multiplied by weight function. This weight function is called Bernstein polynomial and the expression, the way in which Bernstein polynomial is calculated, it is given at the bottom. That means basically it is n c i u to the power 1 - u to the power n - I, okay.

And what is u? u is a parameter which takes up values between 0 and 1 proportionate to the position that it has with respect to the beginning point and end point. If you measure the length along the curve up till the point under consideration, then the fraction of the length of the curve divided by the total length of the curve gives us the value of u. So in this way i is the particular term under consideration starting from 0 ending up in n, where n is one less than the total number of control points and Bernstein polynomial is calculated this way and multiplied with the control point coordinates, so if the control point coordinates be X, Y and Z that means if it is a three-dimensional curve.

In that case with each of these coordinate values this Bernstein polynomial will be multiplied and it will be contributing to the X, Y or Z value of the particular coordinate point being calculated with way on the curve. So this is a point on the curve coming out after calculation in X, Y and Z, B i control point coordinates being given as X, Y and Z and the Bernstein pronominal being calculated with as a function of the parameter u and dependent upon the total number of control points.

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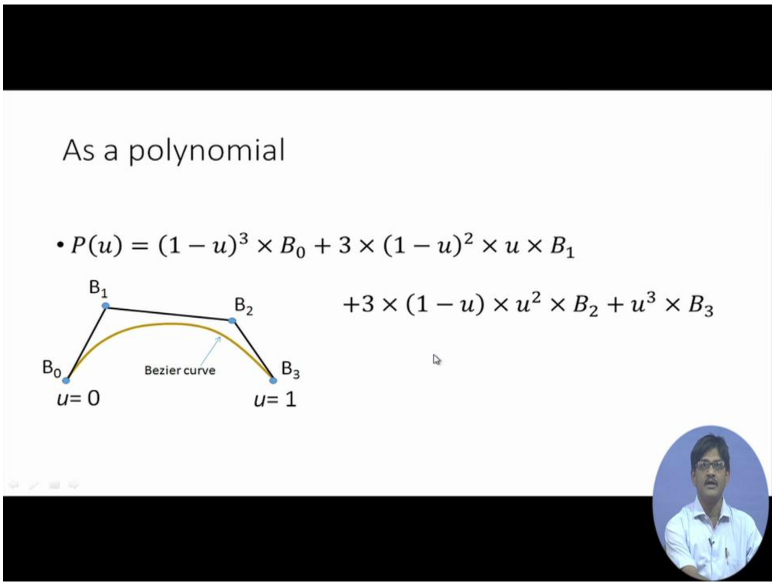

Bezier curves and surfaces can be represented in different ways, this is the submission form, this is the polynomial form that means after working out all those Bernstein polynomial expressions, it can also be represented this way. If you remember u to the power i into 1 - u, n - i, et cetera those things are appearing here. 1 - u with a power 3 and 1 - u to the power 2 into u like, so this is the polynomial form.

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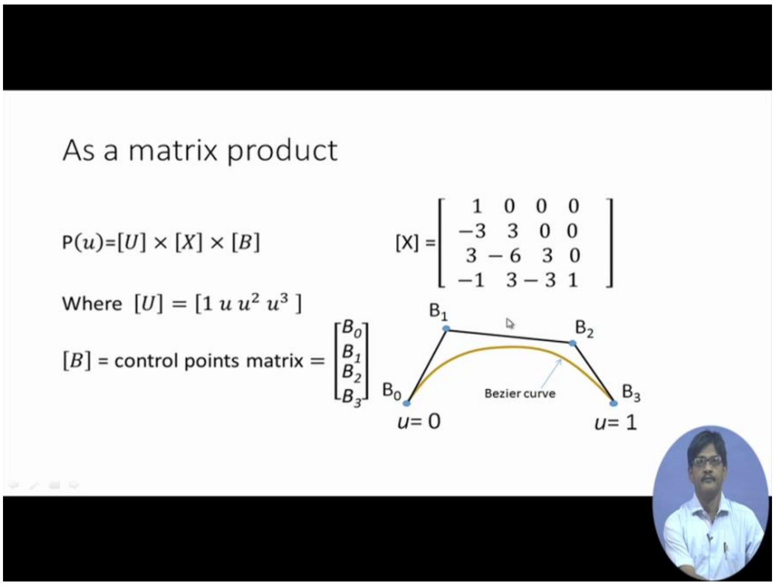

It can also be expressed in matrix form where this particular U matrix capital U matrix will be containing terms only as functions of u, the intermediate X matrix called the coefficient matrix that will be containing some constants depending upon the number of number of control points that we are considering and the B matrix will be containing the control points that we are considering. In this case if there are 4 Control points, they are put as a column for this particular Bezier curve expression.

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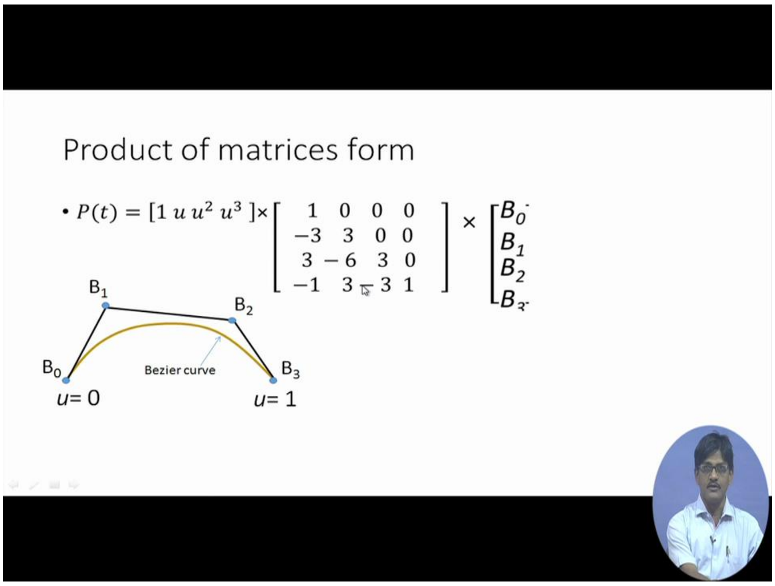

So, as a product of matrices it looks finally like this, U matrix, question matrix and the control point matrix, I am sorry the other side of the script I mean the bracket has somehow got obliterated, it is very much there.

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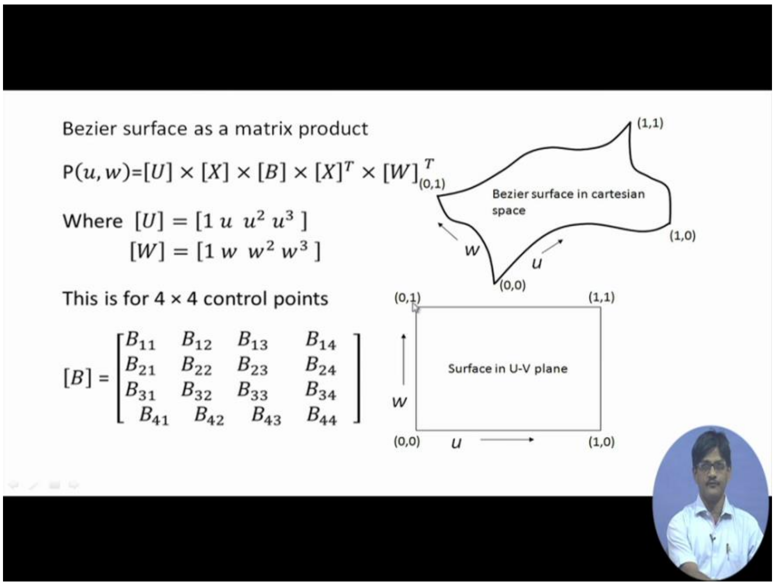

So if we similarly we look at the equation of the sorry formula of the Bezier surface, it looks this way. The Bezier has this U parameter varying from 0 along the edge up till this point where it takes up the value of 1. Here also allow all these lines moving from this side to that side, the U value takes up the value 0 at the start and moves along increases its value and ultimately reaches a value of 1 here. So at intermediate positions also there are lines which are called isoparametric line along which the other parameter which is mentioned here as W that will remain constant and U can be increased so as to produce a curve right from this side to that side, U varying but W remaining constant and that line being known as an isoparametric line.

So if we express the expression of the Bezier surface as a product of matrices, it will appear as just as before the U matrix, the coefficient matrix for the U side, the control point matrix, the coefficient of coefficient matrix for W side transpose and the W matrix that means the other parameter W okay transpose, so U matrix coming out as so it is clear from this expression that if we are looking for terms containing U or its higher powers, it will be only be present here in this matrix. So we have a successfully segregated the all the terms in different matrices, W only here, control points only here like that. So when we come to these 2 okay, U and W they are containing only u terms or only w terms and this is containing only the control point coordinate terms.

So if we express the surface, I mean if we try to draw the surface this is the Cartesian space representation of the surface. If we draw the surface on U-V plane that means just like a graph paper, where X is replaced by U and W replaces Y coordinates, in that case the surface will look exactly like a rectangle or rather a square in this case because both sides of the rectangle I mean adjacent sides of rectangle, they are equal. So if that be so, this is the surface and it stretches from 0 to 1 on this side and stretches from 0 to 1 on that side, so this is exactly what the surface looks like. And interestingly, isoparametric lines will come, straight parallel lines to the X on the Y-axis as the case made.

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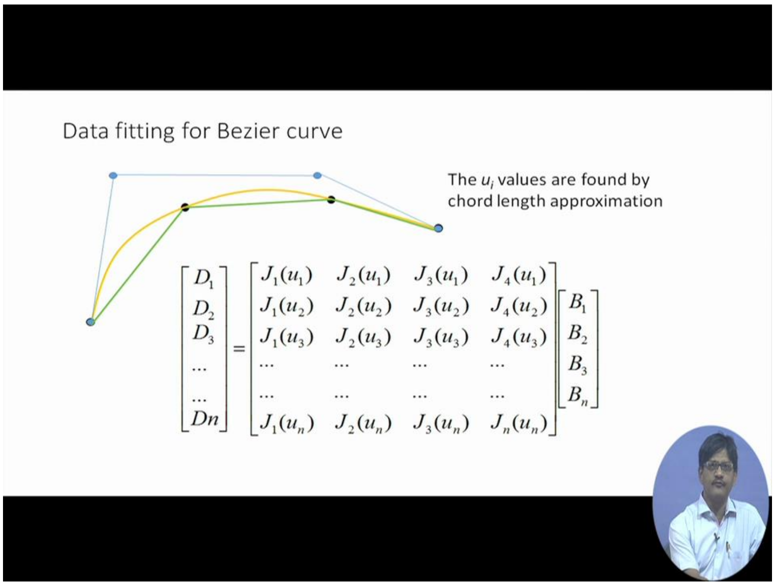

So we will have grid lines X parallel to Y parallel to X appearing here, which will be the isoparametric lines. Coming to data fitting, what we mean by data fitting? Data fitting means that control points may be controlling the surface but designers might be interested to have a curve, which initially passes through a number of points. Let us have a look at at this particular case.

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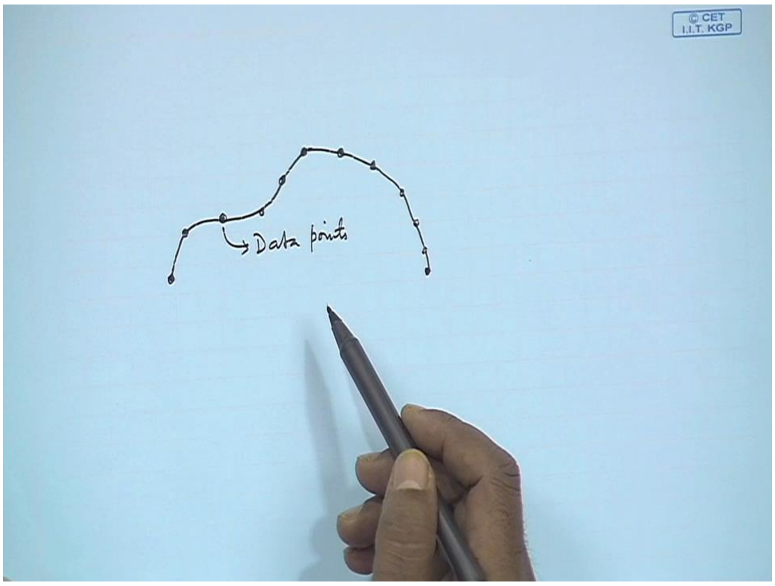

Suppose there is a designer who has designed a very nice looking car body and he asks the asks the person who is handling the geometrical aspects of the design, pass a curve through these points, so these are now not the control points but points through which the which the curve has to essentially pass and therefore, after successfully drawing of the curve, the Bezier curve could look like this it should have positional continuity that means it should pass through all these positions and it should have directional continuity that means commonness of the slopes and curvature continuity through all these points.

So if we have a Bezier curve of this type which passes through all the points, these cannot essentially be the control points. Control points are points which are controlling the surface, but they are not the point through which necessarily the surface has to or the curve has to pass. So these we are calling as data points okay, so these we are calling as data points. If these are the data points then how do we handle I mean how do we obtain the curve and its control points, if we do not get the control point we do not have the curve, so let us have a quick look how this is done.

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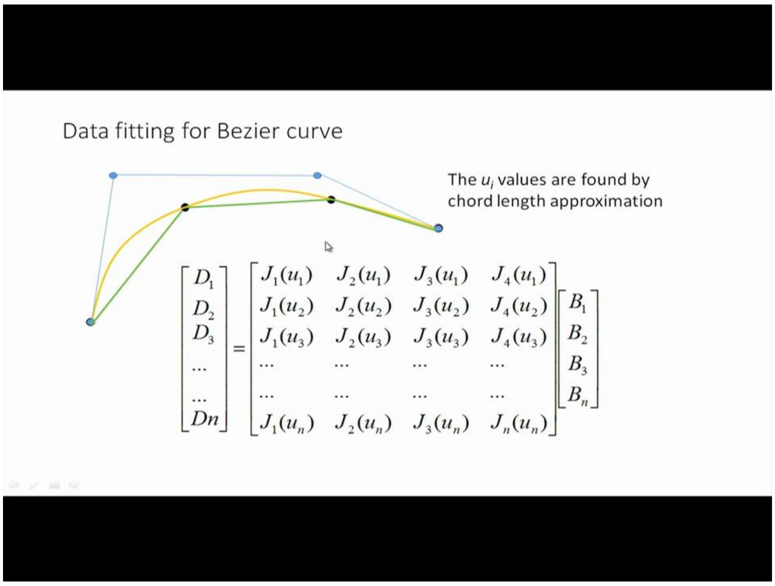

Coming back here, so these points which we have drawn on the curve, these are the data points, it is provided to us and I have to find out the control points which are not known to me. Of course the 1st and the last points data points are also the control points because in case of Bezier curve, the last and the 1st point they are shared between data and control points, so we are faced with a problem in which I have to find out the control points and I am simply given a number of data points through which the curve is supposed to pass. So in that case we can frame a matrix representation of the conditions this way that these are the data points in X, Y and Z, so each what you call it letter here D 1, D 2, etc, they represent X 1, Y 1, Z 1 or X 2, Y 2, Z 2, like that.

And each of these terms, they are the Bernstein polynomials corresponding to a particular value of the parameter that means the parameter value corresponding to this point is given in the 2nd line okay, U 2 corresponds to the parameter value here. Now the problem is, if I do not have the curve, if I cannot measure the length along the curve then how can I possibly assign a particular a parameter value to this point? The solution to this is that we can make an approximation in which the chord length can replace the arc length, instead of taking arc length that means length along the curve, we will take the chord length this way and approximate the value of U 1, U 2, U 3 like that is called Chord length approximation.

So so far so good therefore, I can have numbers against all these terms, this part of the problem is solved, what about this particular matrix? None of these points are known, these are actually the unknowns, which have to be solved. So I do not know this particular matrix, I do know these points, yes I know these values as numbers because I have been able to assign certain values against U 1, U 2, U 3, U 4 by approximating the arc by the chord length and that way I have been able to find out some parameter values corresponding to these points, do I know these points? Yes these are given, these are the point which the design engineer has given to me and he wants the control points or rather he wants me to draw the curve after getting the control points.

So in this case, if there are 4 data points I will have 4 such rows that means 4 equations can be formed and from that definitely I can I can solve for 4 unknowns therefore, if the number of points and the number of control I mean number of data points and the number of control points if they match, I do not have a problem I can exactly solve this problem and I get these points. The problem occurs when there are more number of data points than the control points, why so?

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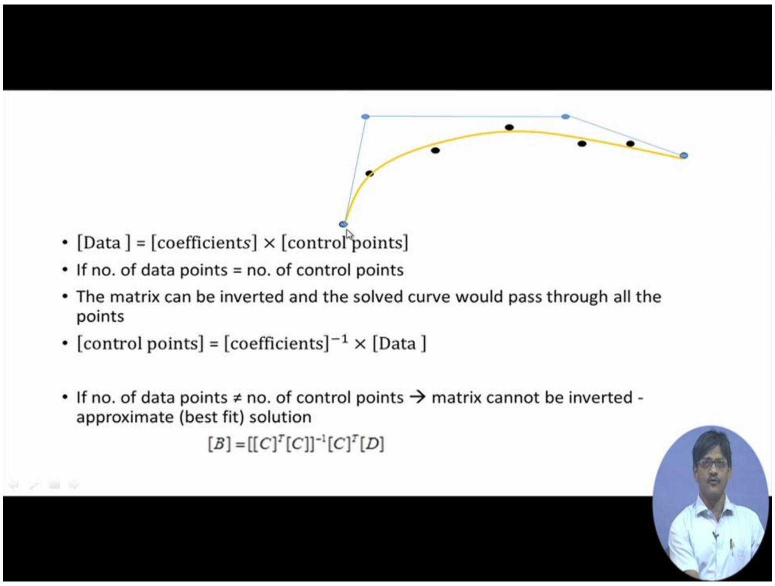

This is because say for a particular curve which is belonging to part or whole of the car body, there will be so many number of points which design engineer would be providing. So as a simple approximation, simple representation of the problem let us say how many points are here? 1 2 3 4 5 6 7, 7 points are here and 1 2 3 4 control points are here. If I take 7 control points my problem is solved, I can have an exact solution and the curve will exactly pass through all these points, but if I have 4 points then the curve can only be solved in a mean sense or a least square fit you can say and averaging or um what you call it, the sum of the square of the errors will be the least but it will not pass necessarily through all the points.

Now the question is, why should I go in this direction which is obviously creating a problem for me, why not make the number of points control points equal to the number of data points? The problem is, if there are huge number of control points I have the highest power of the polynomial in the Bezier curve to be equal to number of points - 1, if you have 4 points you get a cubic Bezier curve that is the highest power is the is U to the power 3. If you have 100 points 100 data points and you solve for 100 control points you will get U to the power 99, just imagine U would the power 99 and the computer has to crunch all these numbers again and again through different calculations, tangents, normals, this that, et cetera.

And you are going to just lose time for just vim of yours that you have to have equal number of control points and data points, so we have to go for this approximation that particular requirement for accuracy just cannot be taken care of with Bezier curves and surfaces. So what we do is, we solve for this in this manner that is we multiply the transpose of the coefficient matrix on both sides after that it can be inverted and after that we get the solution of the control points as a least square fit okay, so this is about curve fitting with Bezier curves.

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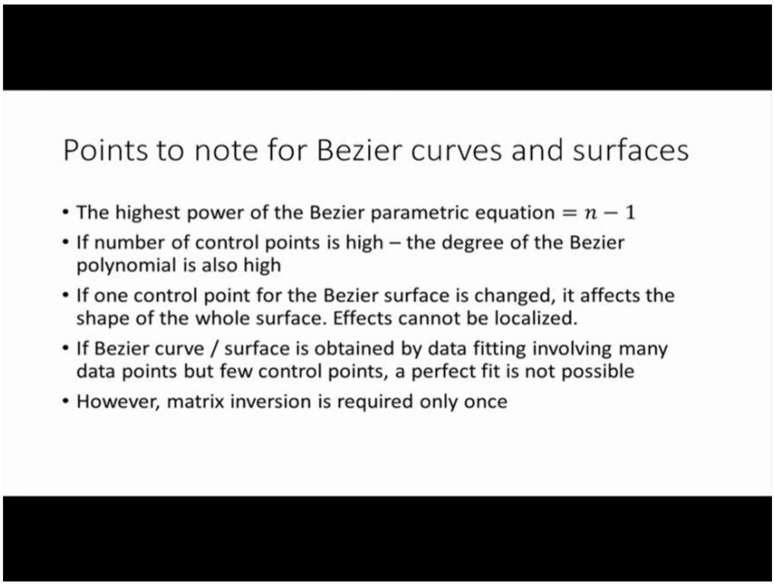

Now, therefore what do we arrive at? We understand that the highest power of the Bezier parameter equation is n - 1, where n is the total number of points and if number of control points is high, we have a problem the degree of the Bezier polynomial is also high. If one control point on the Bezier surface is changed, this is very important it affects the whole shape of the whole curve therefore, F x cannot be localised. Is this a problem in case of B-splines or non-uniform rational B-splines? No. In case of B-splines, rational B-splines, et cetera, we have another yet another curve modifying parameter which is called the knot vector, the knot vector comes in and it essentially divides the curve into different segments and these segments can be individually controlled.

So this problem of the Bezier surface that is changed of the position of location of 1 point affects the surface globally that is not a problem in case of higher more advance curves of this family. And if we have data fitting which we discussed just now, perfect fit is not possible if you want to go for less number of control points in connection with more number of data points. However, one thing is good that is once you have made the matrix inversion obtain the control points, then if you want to make some aesthetic changes, et cetera say to the car body, then you can simply change the locations of the control points, matrix inversion at that stage is not required.

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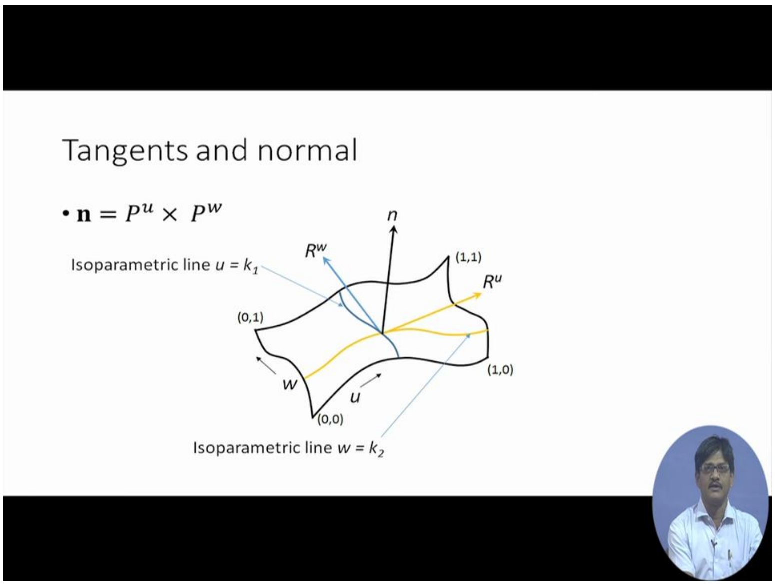

So let us now have a look at tangents, definition of tangents, normal, et cetera, with respect to the Bezier surface, what is so what do we have here? We have the Bezier surface where one side is having the parameter U associated with it, the other side is having the parameter W associated with it instead of X and Y that means the basic formula that we observe for this curve surfaces, et cetera, they are having a variable not in X, Y and Z, but in the form of parameters U, W, etc because of which they are referred to as parametric equations. Now coming to the inside of the surface, what does this orange line represents orange curve?

The orange curve represents a line along which the parameter W is a constant, so we call it the Isoparametric curve okay along this W is going to remain constant and U is going to constantly increase from 0 up to 1 here. It is easy to find out a the points on an isoparametric line because we can simply need a Do loop or a repeat loop inside a program go on incrementing the value of U and go on finding out the X, Y, Z points on the surface and that will give us the isoparametric line. In the same manner you can also have you can also have an isoparametric line I mean curve in the other direction where U is made a constant, so if U is constant you will get isoparametric curve in this direction starting from 0 here and ending up in 1 at this point.

So if we derive the expression of the Bezier curve it is a function of 2 parameters U and W, if we if we derive it partially with respect to U we will get a derivative which will come out as a vector as shown here. This one is the partial of the surface with respect to U and we are assigning a symbol R superscript u to represent that derivative of the surface with respect to U, it is a partial derivative. In the same manner if we derive the expression of the surface with respect to W, we are getting this particular derivative and in space it will look like this one.

So these are essentially 2 tangents to the surface, the surface has infinite number of tangents in a plane called the tangent plane touching this particular point on the surface and it it has a 1 normal. Infinite of tangents, one particular normal and if we take the cross product of R u with R v then we will get this particular normal okay. Now let us have a quick look about the way in which we are supposed to find out these tangents and normals.

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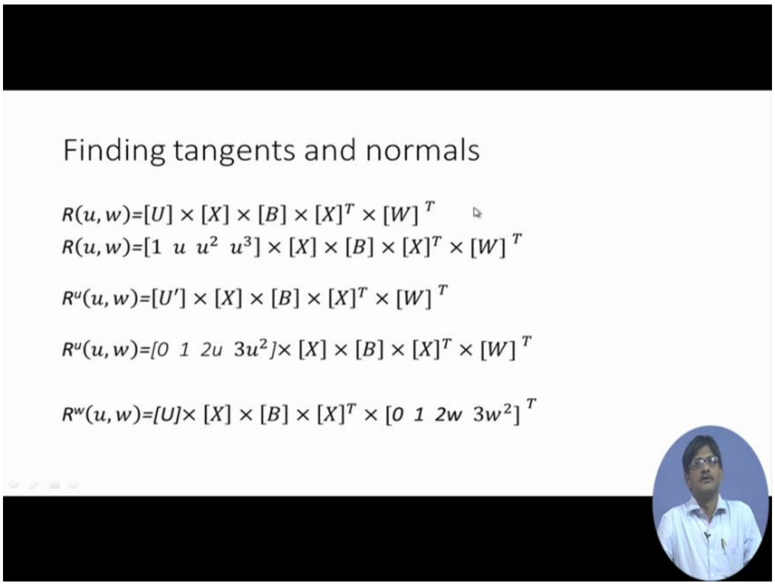

For example, the surface expression is written at the top and it is expanded ones to show how the U matrix looks like, so the U matrix contains 1, U, U square, U cube, et cetera. If we now derive it with respect to U, the duty of it will be only one matrix will be affected, only the U matrix. So we do not really have to concern ourselves with complex mathematical or calculus operations when we are driving the expression of the surface with one of the parameters. Simply derive the matrix, I mean simply derive the terms individually inside the matrix which contains the relevant terms with respect to that parameter. So if we are deriving with respect to U, the 1st matrix capital U gets derived and therefore, we will have instead of 1 U, U square, U cube, we are going to have 0, 1, 2u and 3u square that is it.

We have found out the expression of the partial derivative of the surface with respect to the U parameter and it looks like this one. And the last expression gives us the partial derivative of the surface with respect to that W parameter and it looks like the last row expression which has been provided. So finding out tangents that way is not a very difficult task, I mean the 2 specific tangents that we are talking about. One multiple-choice question, at the bottom we can see that a vertical plane is intersecting a parametric surface, we will call this a curved surface okay curved surface, a curved surface is being intersected by a vertical plane. And at this point we have drawn the tangent to that particular intersection curve.

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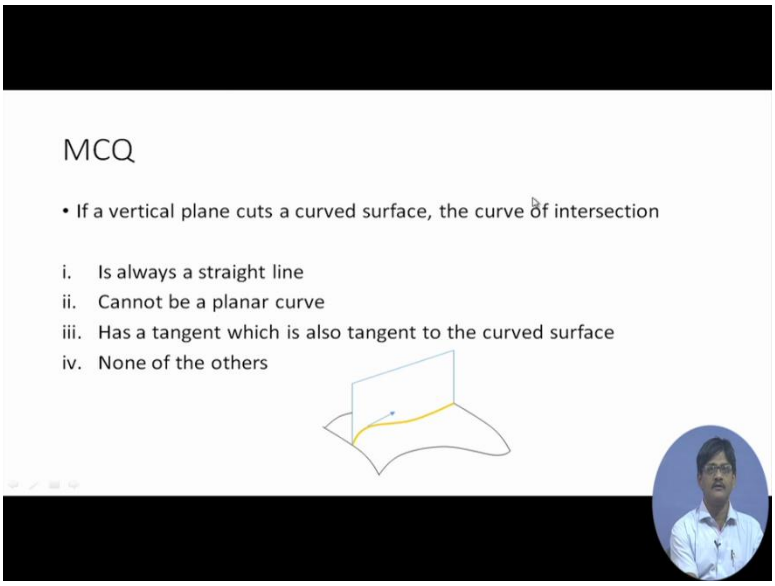

Now, the MCQ reads if a vertical plane cuts a curved surface if a vertical plane cuts a curved surface as shown, the curve of intersection… is always a straight line, cannot be a planar curve, has a tangent which is also tangent to the curved surface and none of the others. Let us take the 1st option, the 1st option reads "it is always a straight line" that is obviously wrong, it is not correct. Cannot be a planar curve, planar curve means that a curve which can ultimately be a put into a plane, contained in a plane, it seems that it will always be a planar curve because it is essentially going to belong to the plane and essentially going to belong to the surface, so if it belongs to the plane it must be planar so obviously this is also wrong.

Has a tangent which is also tangent to the curved surface, since the curve is belonging to the surface that means is a subset, all the points on the curve they are belonging to a set of points belonging to the surface, it is a subset. Therefore, the tangent to the curve will eventually be a tangent to the surface so this is correct, the 3rd one is correct.

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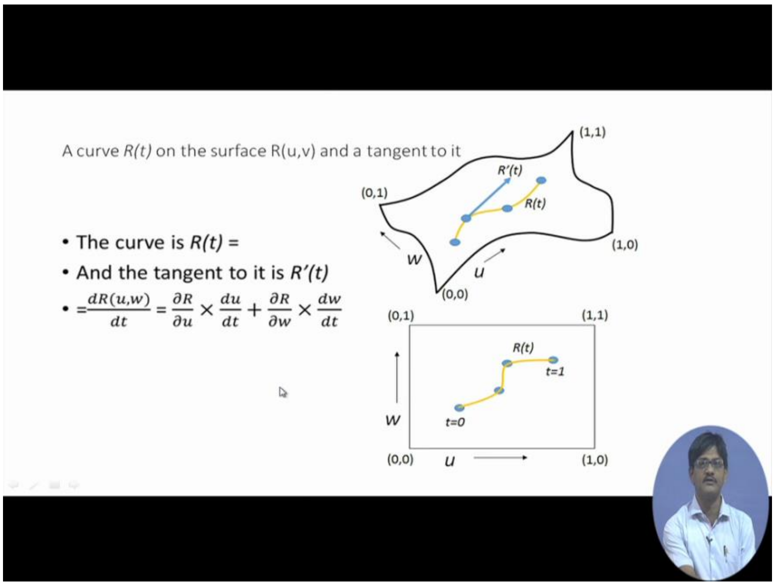

Coming back to the surface calculation, so we if we are given the problem of finding out the tangents to the surface with respect to the parametric values parameters U and W, we can easily find it out. Next and also we know that the cross product of these 2 tangents will also give me the normal. Now we are talking about a curve which is lying on the surface, so just think of it, there is a surface which is say R U W and on that there is a curve which is lying which is which we are referring to as R a function of a parameter t, so we are bringing in the 3rd parameter t but is this something which is faced in the real world that is, is this a problem really or is it something abstract of the importance?

This will definitely be a problem when we are finding out cutter paths which are moving along curves on a parametric surface that time we will definitely face this particular problem. Now if we are dealing with such a curve, how would its tangent appearing to us? So in this case we have to essentially find out the derivative of this particular curve with respect to a parameter t therefore, first of all we understand that R dash t = dR dt, but R is a function of U and W therefore, we can definitely apply the chain rule so that we will have Del R Del U into du dt this Del R Del W into dw dt, so this is the full form of the derivative of the curve R(t) lying on the surface R(u, w) okay.

So dR Del R Del U at a particular point on the surface is an invariant that is does not depend upon the direction whether we are considering our curve in this direction or that direction it is unique to a particular point on the surface and further therefore, these 2 terms Del R Del W and Del R Del U, these points are not depending upon the direction in which we are considering this particular tangent, but du dt and dw dt, they are concerned with the direction, they are basically the of sort of X and Y components of this particular tangent okay, so we understand that this particular tangent expression can also be draw. Now we will quickly go through, before ending this particular discussion we will quickly go through some expressions which will be useful to us in the later calculations.

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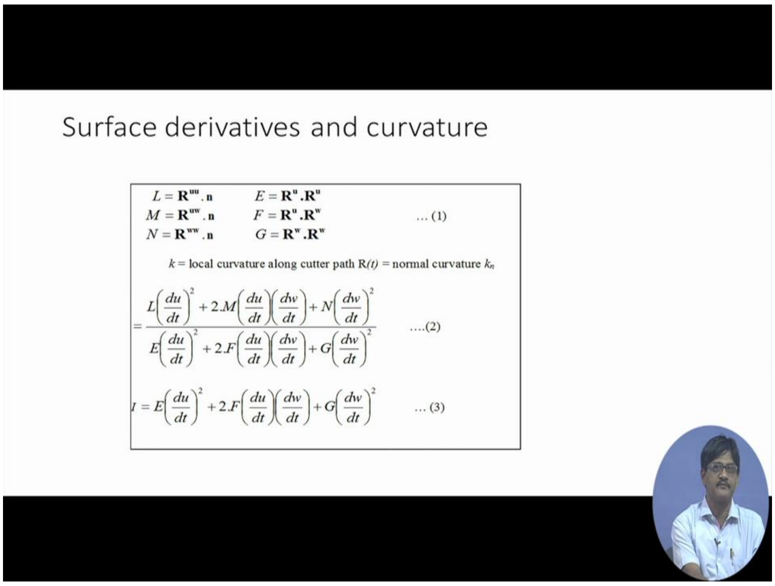

Let us have a look for example, we are defining L as the 2nd derivative of the surface with respect to U, 2nd derivative dot producted with the normal, so dot producted with the normal vector. We are defining M as the mixed derivative of the surface with respect to U and then with respect to W and dot producting with the normal, so this way they are having these 3 derivatives involving double derivatives. And the 1st derivatives are also multiplied like R dot R u dot R u that means the derivative of the surface with respect to U dot producted with itself make equal to a term called E. F = R u dot R w and G = R w dot R w, these are required for finding out the curvature of the surface, the curvature of the surface has this formidable expression.

We will not be using this much but this will, this one and this 1st fundamental form these will be appearing on and off in our later formulations, these are not very difficult and I will try to avoid as much as possible mathematical number crunching and instead we will try to concentrate upon the concepts, thank you very much.