Biomechanics of Joints and Orthopaedic Implants
Professor. Sanjay Gupta
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur
Lecture No. 25
Stress Transformation

Good morning everybody, welcome to the lecture on stress transformation.

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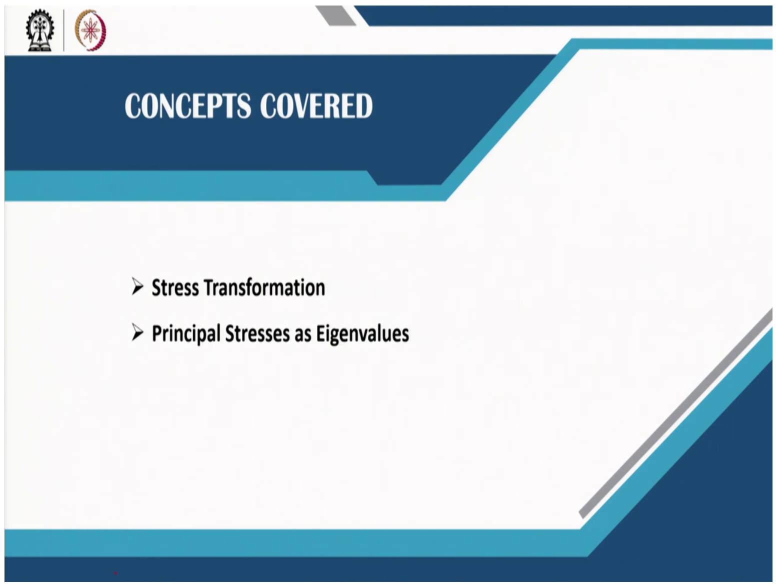

In this lecture, we will be discussing about the following topics the stress transformation and principal stresses as Eigenvalues.

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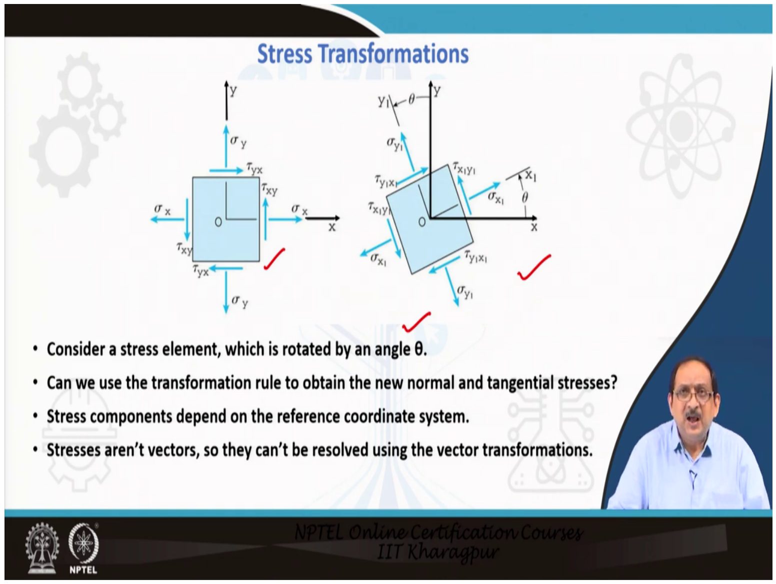

Based on the concept discussed in the earlier lecture, we restate that stress components acting at a point will depend on the orientation of the area on which it acts. So, let us consider a square stress element as shown in the figure on the left, which represents the state of stress acting at a point.

Considering this stress element, we apply a rotation through an angle θ as shown in the figure on the right. Can we use the transformation rule to obtain the new stress components or the new normal and tangential stresses? The answer is yes. The stress components depend on the reference coordinate system.

Please note that stresses are not vectors. So, they cannot be resolved using vector transformations, and we need to multiply the stress components with the corresponding area on which it is acting to convert it into force vectors, which can be eventually resolved and transformed using the vector transformation.

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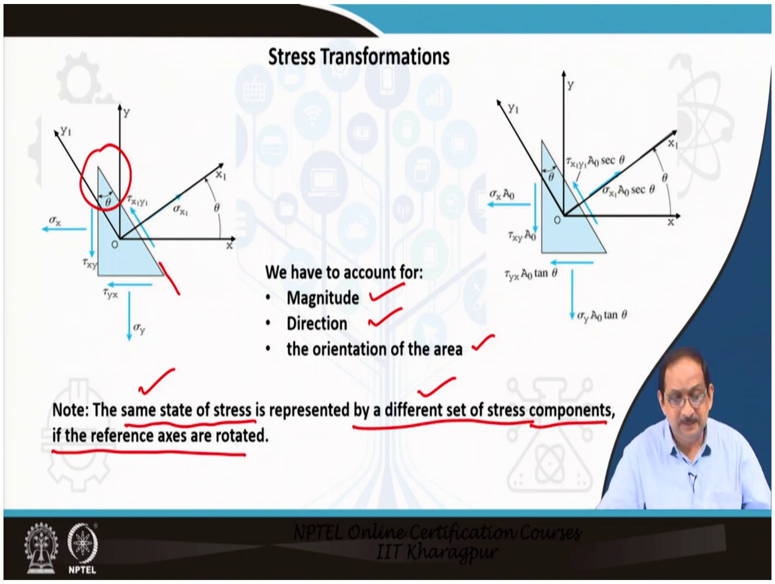

Now, to find out the stress components corresponding to a reference system, we have to account for the magnitude of the stress, the direction of the stress components and the orientation of the area on which it is acting. Now, we define a cutting plane as shown here inclined at an angle θ equal to the stress element's angle of rotation or the coordinate system's angle of rotation. So, please note that a different set of stress components represents the same state of stress if the reference axis are rotated.

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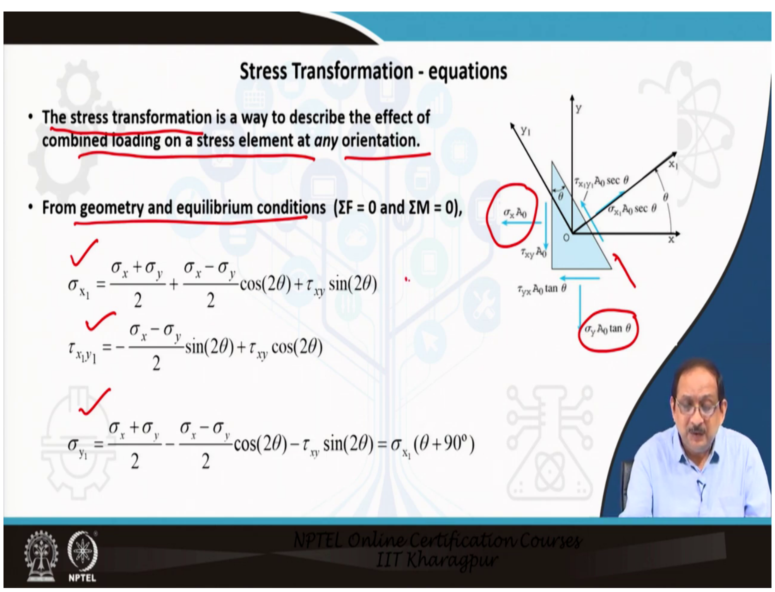

Now, let us find out the equations of the set of stress components corresponding to a new orientation or transformation. Now, the stress transformation is a way to describe the effect of combined loading on a stress element at any orientation. So, from geometry and equilibrium conditions, we can find the equations of the normal stresses σ_{x1} , σ_{y1} corresponding to the new coordinate system x1, y1 and the shear stress acting on the plane.

To do that, we need to multiply the various stresses by the areas of the faces.

So, we multiply the various stresses by the areas of the phases on which they are acting, and then the forces on the triangular element can be identified after that, considering force equilibrium along x1, y1, we can find out the new stress components corresponding to the new coordinate system.

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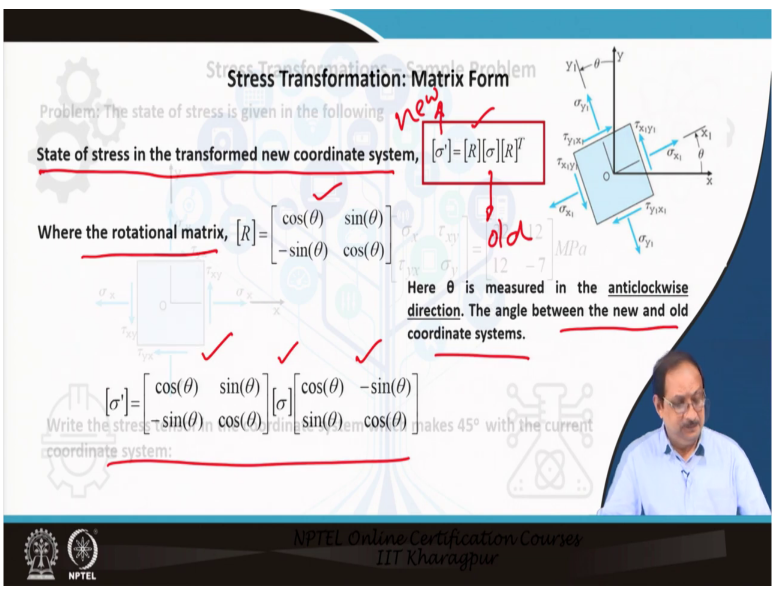

The stress transformation can also be expressed in matrix form as indicated here a state of stress in the transform new coordinate system can be found out by using the rotational matrix R and employing the transformation rule. So, R, the rotational matrix, is given here in the slide, where the θ is measured in the anti-clockwise direction, and it is the angle between the new and the old coordinate system.

So, the transformation rule for obtaining the new stress components is that we need to pre multiply the old stress components by the rotational matrix and post multiply by the transpose of the rotational matrix to obtain a new stress component. So, the same thing is expressed here: σ' is equal to the rotational matrix multiplied by the old stress tensor, and the rotational matrix transpose.

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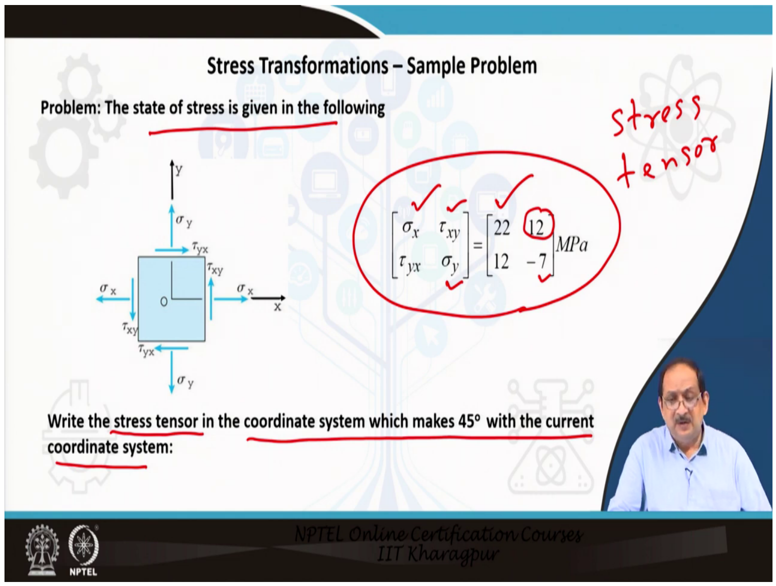

Now, let us discuss a problem, a state of stress is given.

So, we formed the matrix. So, write the stress tensor in the coordinate system, making 45° with the current coordinate system. So, we have the current stress tensor, and we would like to transform it to a new stress tensor when the coordinate system is rotated

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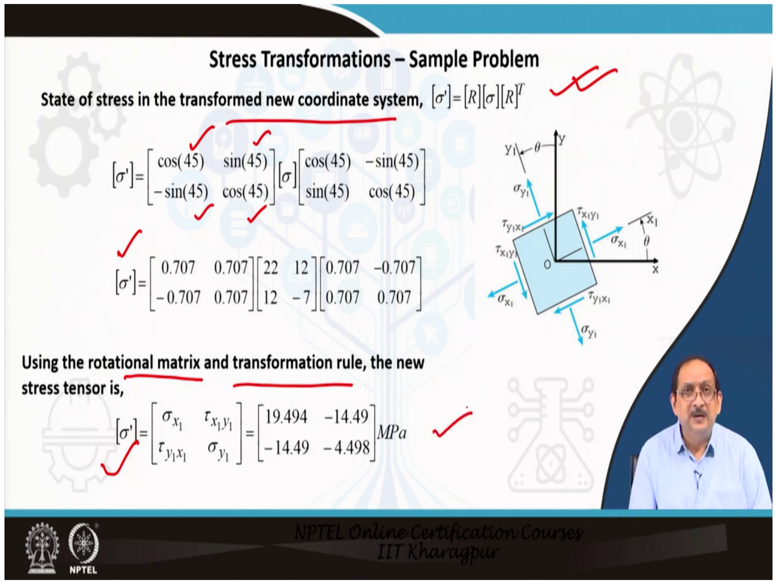

So, we substitute the value of θ in this equation and then find out the value of the new stress tensor using the rotational matrix and the transformation rule. So, the new stress tensor is given by σ' and the values are presented in this slide.

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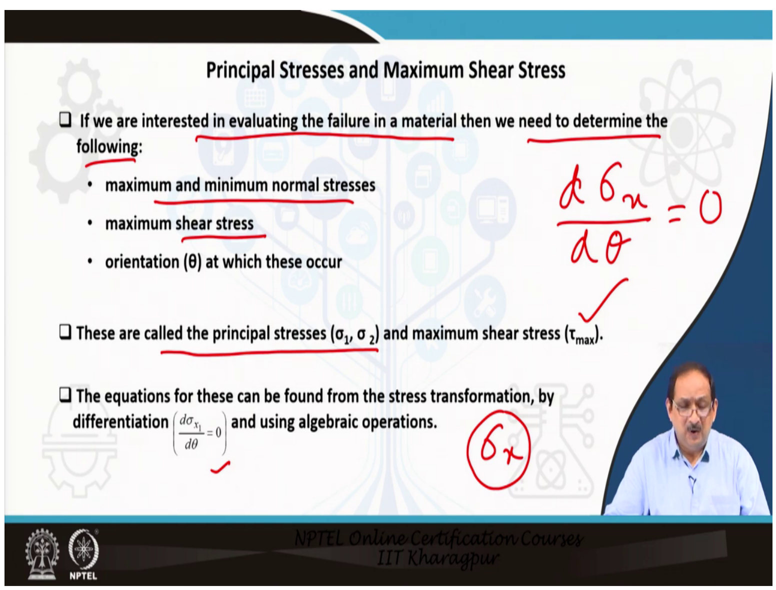

Let us discuss principal stresses and maximum shear stress. If we are interested in evaluating failure in a material, we need to determine the maximum and minimum values of normal stresses the maximum shear stress and the orientation or the direction or the plane at which these stresses are acting.

Now, the maximum and minimum normal stresses are known as the principal stresses; σ_1 and σ_2 and the maximum shear stress is known as τ_{max}. The equations for this can be found out using stress transformation as discussed earlier. So, we take the equations of the σ_x and σ_y and $\tau_{xy,}$ and we differentiate the terms to determine the conditions for which these stresses can be maximum or minimum.

And after that, we use that condition through algebraic operations. We find the expressions for the principal stresses σ_1 and σ_2 and the maximum shear stresses. Now, to find out the location of the plane of maximum normal stress say for example, σ_x we actually need to take the derivative of the normal stress and equated to zero and then from the algebraic equation, we can actually determine the condition for which the normal stress is maximum or minimum.

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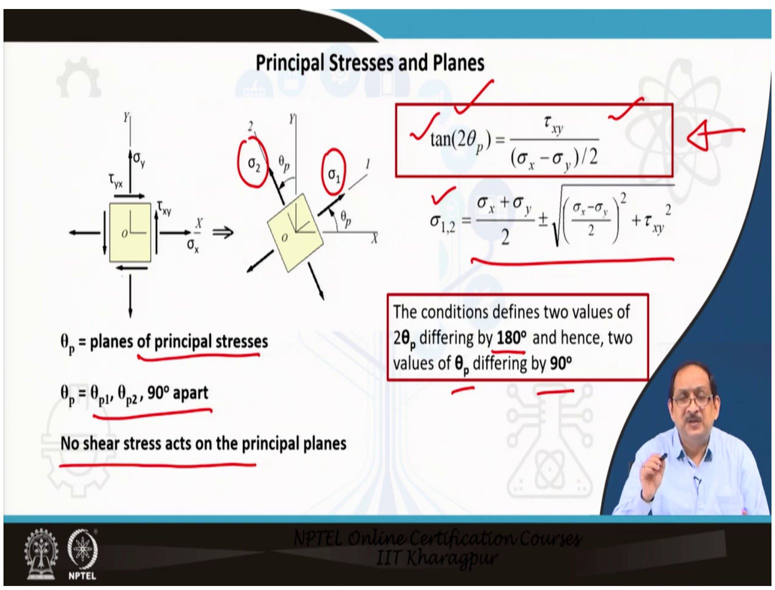

So, following that procedure, we can actually find the angle in terms of tan 2θ at which the normal stresses can be maximum or minimum so, once we have determined that, we can put the value back in the equation and find out the maximum and minimum normal stresses, which corresponds to the principal stresses σ_1 and σ_2 and this is given by the following expression as indicated in the slide.

The planes of the principal stresses are 90^0 apart. So, you can see σ_1 and σ_2 is 90 degree apart.

Now, if I substitute this value, the plane's orientation in the equation of the shear stresses, we see that shear stress at this plane orientation of the plane comes out to be zero. So, no shear stress acts on the principal planes, which are the planes on which the principal stress is act. So, principal planes are the orientation on which the principal stresses are acting, but no shear stresses are acting on the principal planes.

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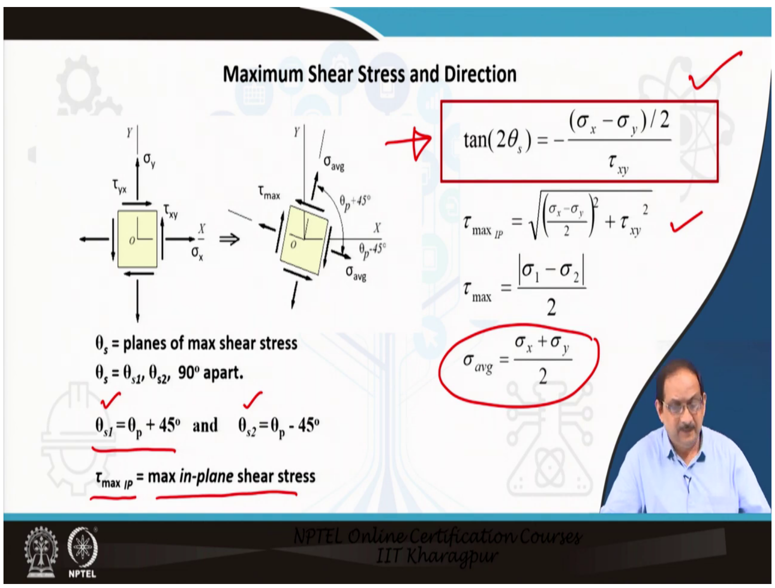

Let us, now find out maximum shear stress and its direction following the similar procedure as done in case of the normal stress, we take a derivative of the shear stress and equate it to 0 to find the condition for which the shear stress can be maximum. So, it comes out to be tan 2θ equal to an expression consisting of σ_x σ_y and τ_{xy} and if we substitute this expression in the original equation of the shear stress.

We can actually find the maximum shear stress given by the expression in the slide. So, the maximum shear stress actually can be related to the principal stresses as well.

So, $\tau_{max} = (\sigma_1 - \sigma_2) / 2$.

Now, in the case of the maximum shear stress, we have obtained the directions given by the expression here. Now, when we compare this orientation of the maximum shear stress with the maximum normal stress used to earlier, it is observed that the maximum shear stresses occur on orthogonal planes, bisecting the angle between the principal planes at 45 degrees to the planes of principal stresses.

So, you can see here that the planes on which the shear stresses are maximum are basically at 45 degrees with the principal planes and the $\tau_{max\ IP}$ as given here, is the maximum in-plane shear stress.

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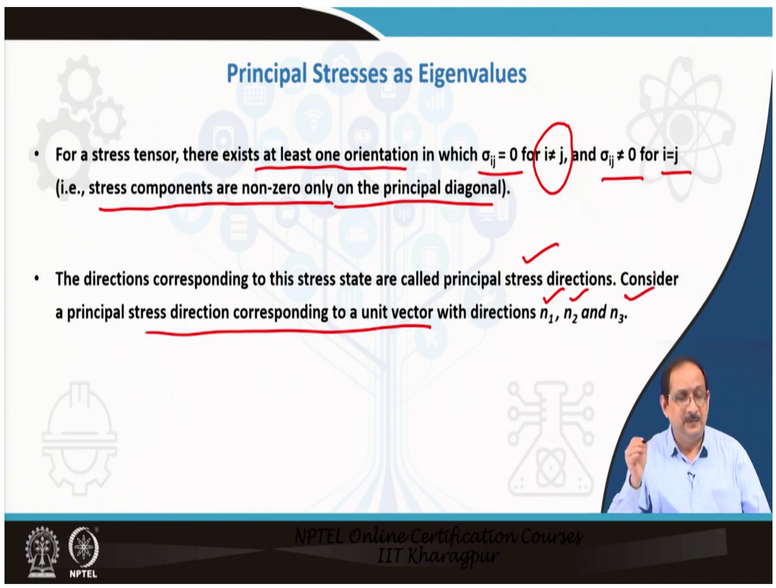

Let us, now come to the second topic that is principal stresses as Eigenvalues. Now, for a stress tensor, there exists at least one orientation in which $\sigma_{ij} = 0$, for which i is not equal to j and σ_{ij} is not equal to 0 for i= j, that means, that stress components are nonzero only on the principal diagonal if we consider the stress tensor the directions corresponding to this stress state are called principal directions. So, we can consider a principal stress direction corresponding to a unit vector with directions n1, n2 and n3.

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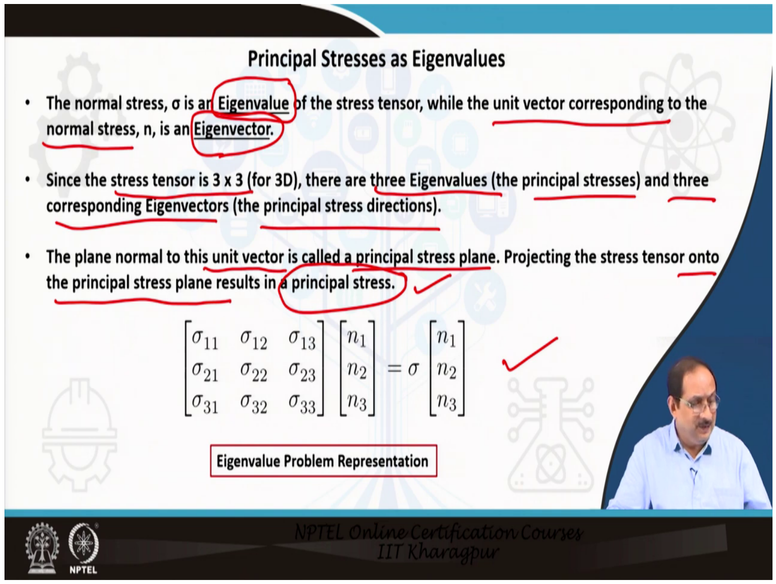

The normal stress σ is an Eigenvalue of the stress tensor while the unit vector corresponding to the normal stress n is the Eigenvector very important the normal stress σ is an eigenvalue and the unit vector or directions corresponding to the normal stress n is an Eigenvector. But, if we consider a 3D problem, then the stress tensor is expressed as 3 by 3 matrix.

So, there are 3 Eigenvalues: the principal stresses and 3 corresponding Eigenvectors: the directions of the principal stresses. The plane normal to this unit vector is called the principal stress plane projecting the stress tensor onto the principal stress plane results in principal stresses. So, eigenvalue problem can be represented as shown in this slide.

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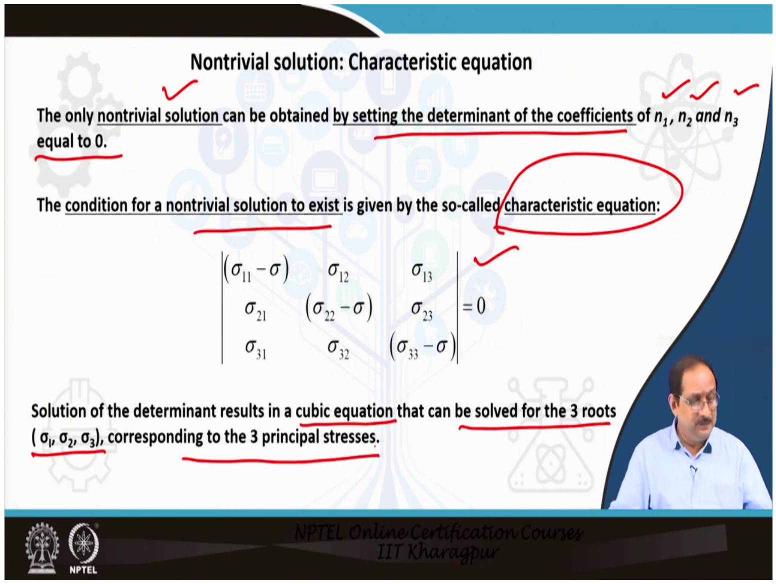

Now, let us discuss about the non-trivial solution and the characteristic equation. The only non-trivial solution can be obtained by setting the determinant of the coefficients of n1, n2, n3 equal to 0. So, we need to set the determinants of the coefficients of n1, n2 and n3 = 0; the condition of a non-trivial solution to exist is given by the so, called characteristic equation, which is represented in the slide, solution of the determinant results in a cubic equation that can be solved for the 3 roots: σ_1 σ_2 σ_3 corresponding to the 3 principles stresses.

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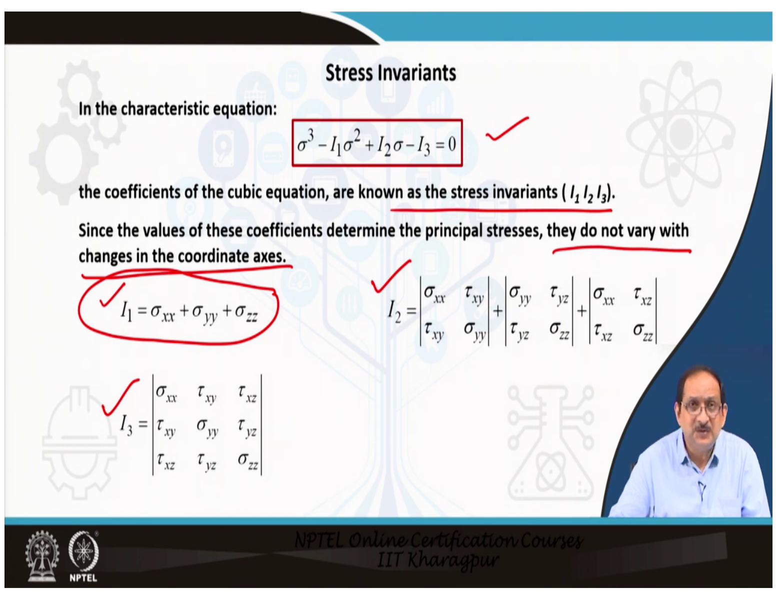

Now, we define stress invariants by considering the characteristic equation that is formed by taking the determinant. So, the characteristic equation is represented by a cubic equation of the

$$\sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0$$

were the coefficients of the cubic equation I_1, I_2, I_3 are known as the stress invariants. Since the values of this coefficient determine the principal stresses, they do not vary with the changes in the coordinate axis.

So, the stress and invariants do not vary with the changes in the coordinate axis. Now, first invariant I_1, second invariant, I_2 and the third invariant, I_3 is represented as shown in the slide. The first invariant I_1 is shown here as some of the normal stresses. Now, it defines a useful relationship that the sum of the normal stresses for any orientation in the coordinate system is equal to the sum of the normal stresses for any other orientation.

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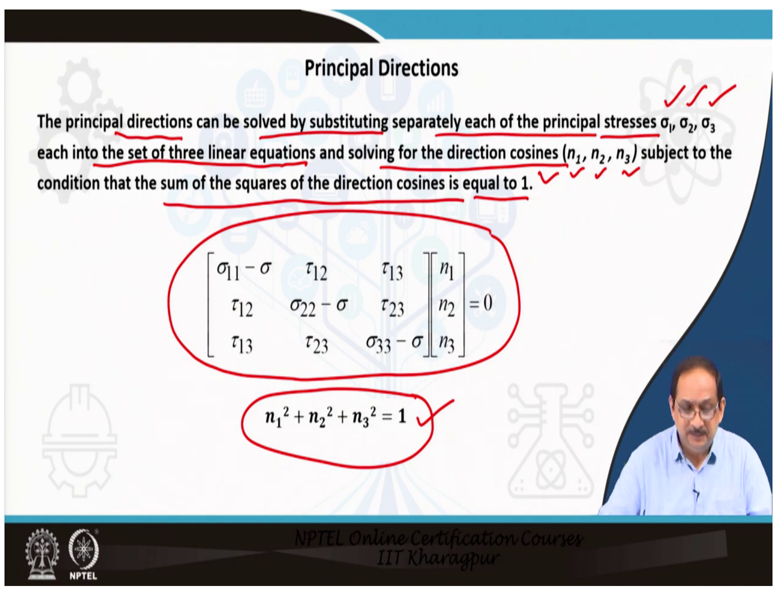

The principal directions can be solved by substituting the principal stresses separately: σ_1 σ_2 σ_3 into the set of 3 linear equations formed from this eigenvalue problem and solving for the direction cosines n_1, n_2 and n_3 subject to the condition that the sum of the squares of the direction cosines is equal to 1 as stated here in the slide. There are several methods to solve the system of linear equations like Gauss elimination, which could solve n1, n2 and n3.

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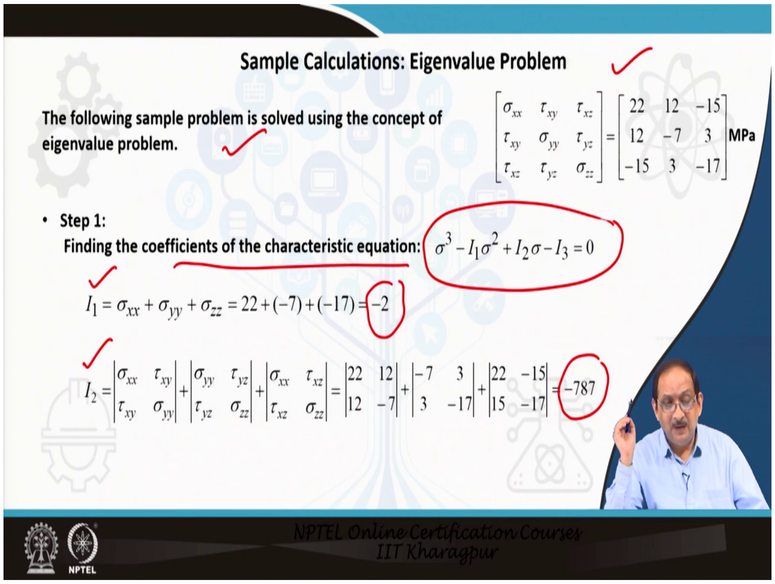

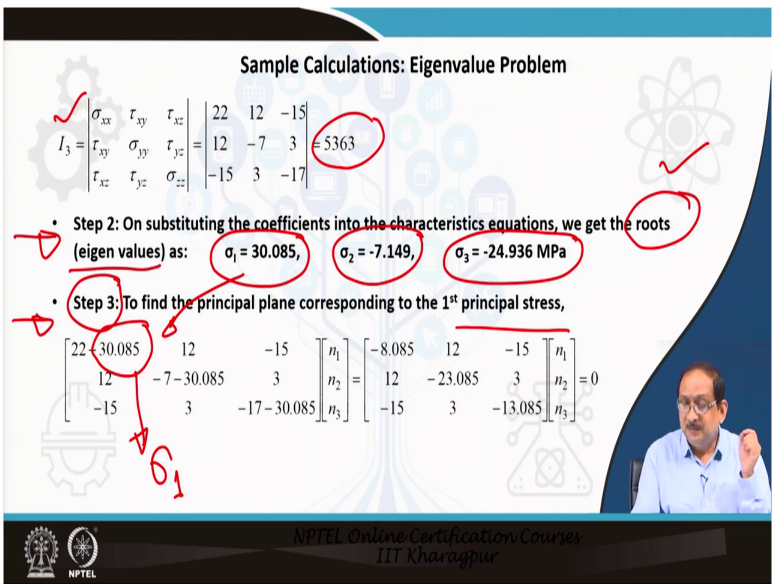

Let us, now undertake some sample calculations on an eigenvalue problem. The following sample problem is solved using the concept of the eigenvalue problem. So, a 3D state of stress is given here. So, the first step is finding the coefficients of the characteristic equation which is stated here.

So, from this characteristic equation, we can identify I_1, I_2 and I_3, and we can actually go back to the earliest slide. So, I can find the values of I_1, I_2 by substituting the values from the givens stress tensor. I can also find out the value of I_3 by substituting the values of the stress components.

Now, let us come to the second step on substituting the coefficients in the characteristic equations, we can obtain the roots or the Eigenvalues, we can obtain σ_1 we can obtain σ_2 and we can obtain σ_3. So, this is an important step where we determine the principal stress values.

The next step, number 3 is to find out the directions to find out the principal plane corresponding to the first principal stress, say σ_1. So, in the eigenvalue problem, we substitute the σ_1 value and find out the orientation of the principal plane. So, we set the left-hand side of the eigenvalue problem equal to 0.

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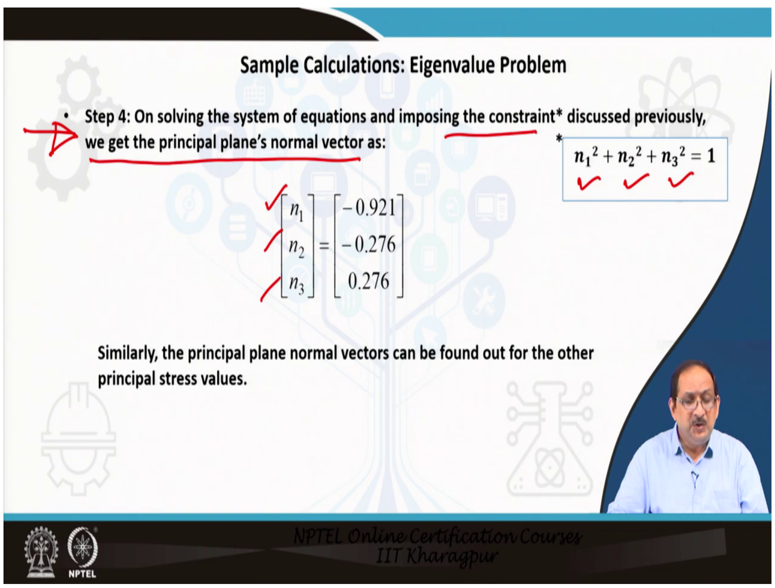

So, in step number 4 we are solving the system of equations imposing the constraint that is the sum of the squares of the direction cosines is equal to 1.

$$\text{So, } n_1^2 + n_2^2 + n_3^2 = 1$$

that has been discussed previously. So, solving the system of equations with these constraints allows us to get the principle planes normal vector as a set of n_1, n_2, n_3.

Similarly, if we substitute σ_2 in the eigenvalue problem, we will get another set of n_1, n_2, n_3.and we can also get another set of n_1, n_2, n_3 corresponding to σ_3. So, there are several methods to solve this system of linear equations like the Gauss elimination method, which could be used to solve for n_1, n_2 and n_3.

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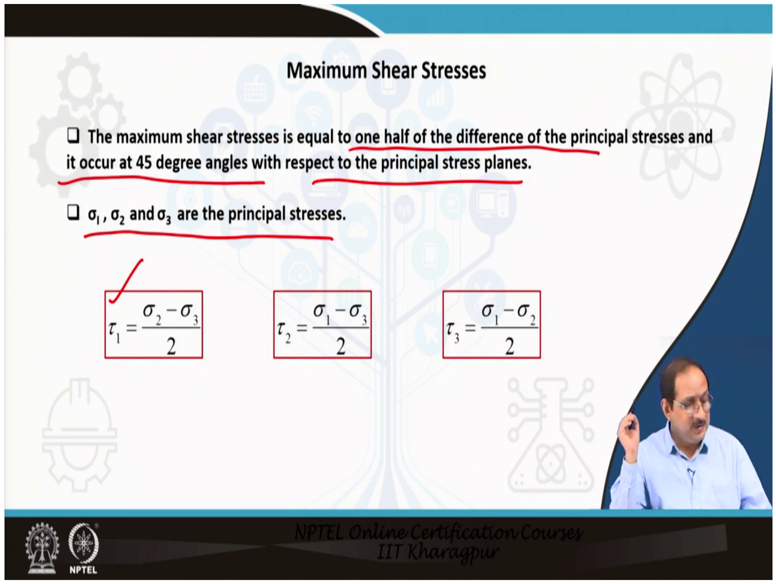

Now, the maximum shear stress is equal to 1 half of the difference of the principal stresses and we have found this out earlier and it occur at 45° angle with respect to the principal stress plane. So, τ_1, τ_2 and τ_3 can be determined using the principal stresses as shown here in the slide. So, at σ_1 and σ_2 and σ_3 are the principal stresses.

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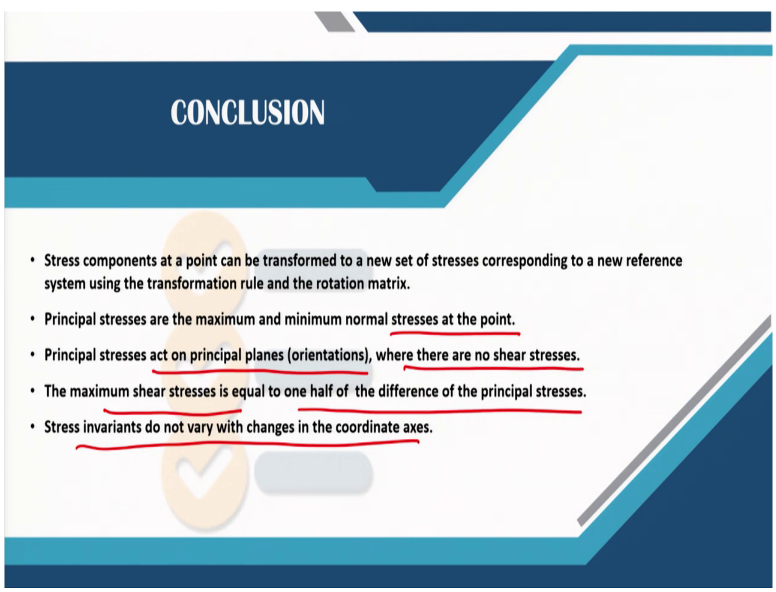

Let us conclude this lecture that stress components at a point can be transformed to a new set of stresses corresponding to a new reference system using the transformation rule and the rotation matrix. Principal stresses are maximum and minimum normal stresses acting at a point the principal stresses act on principal planes orientation where there are no shear stresses. The maximum shear stress is equal to 1 half of the difference of the principal stresses, and these stress invariants do not vary with changes in the coordinate system or coordinate axis.

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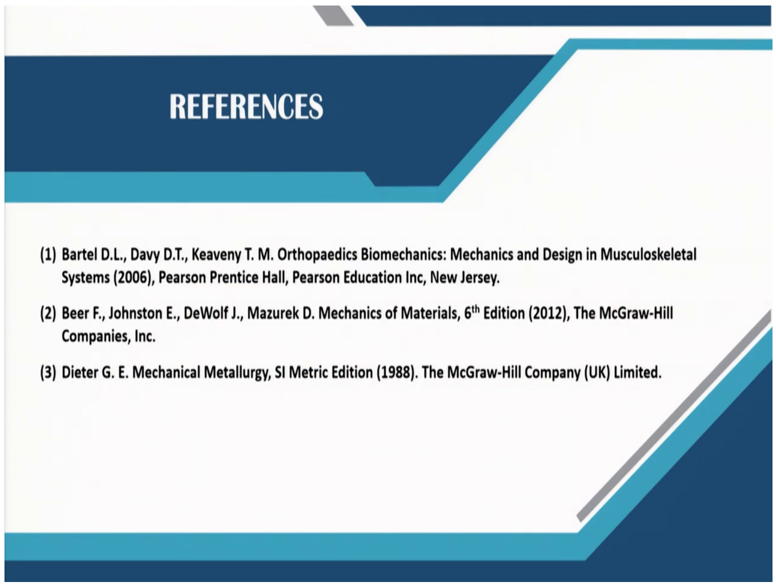

The references are listed here based on which the lecture has been prepared, and thank you for listening.