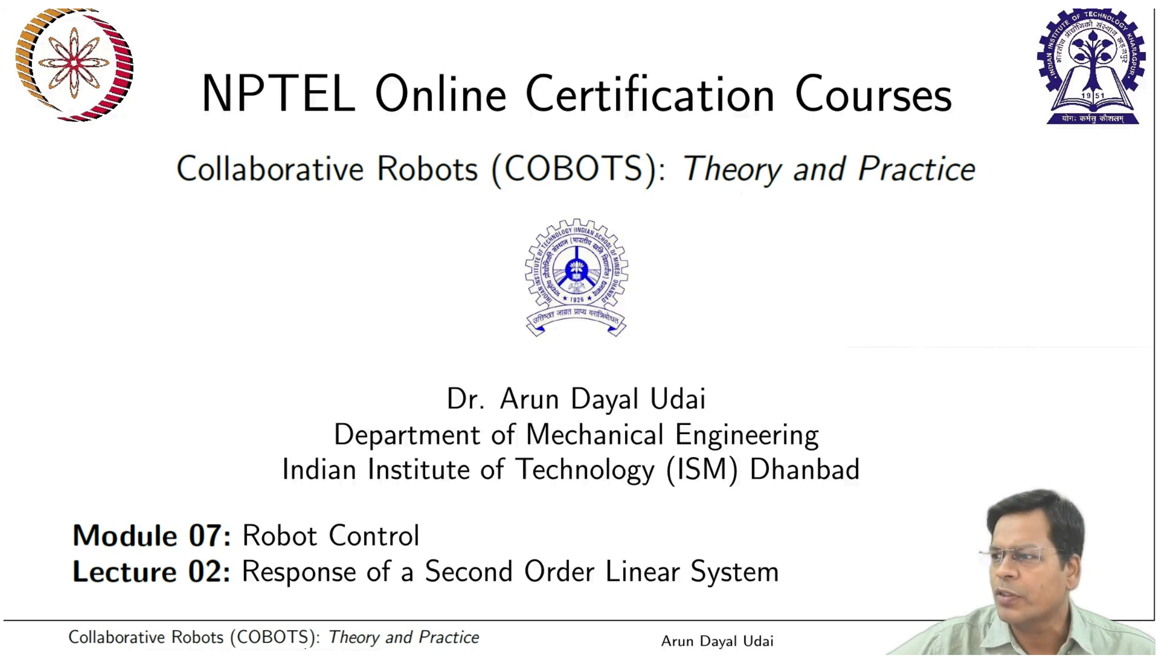
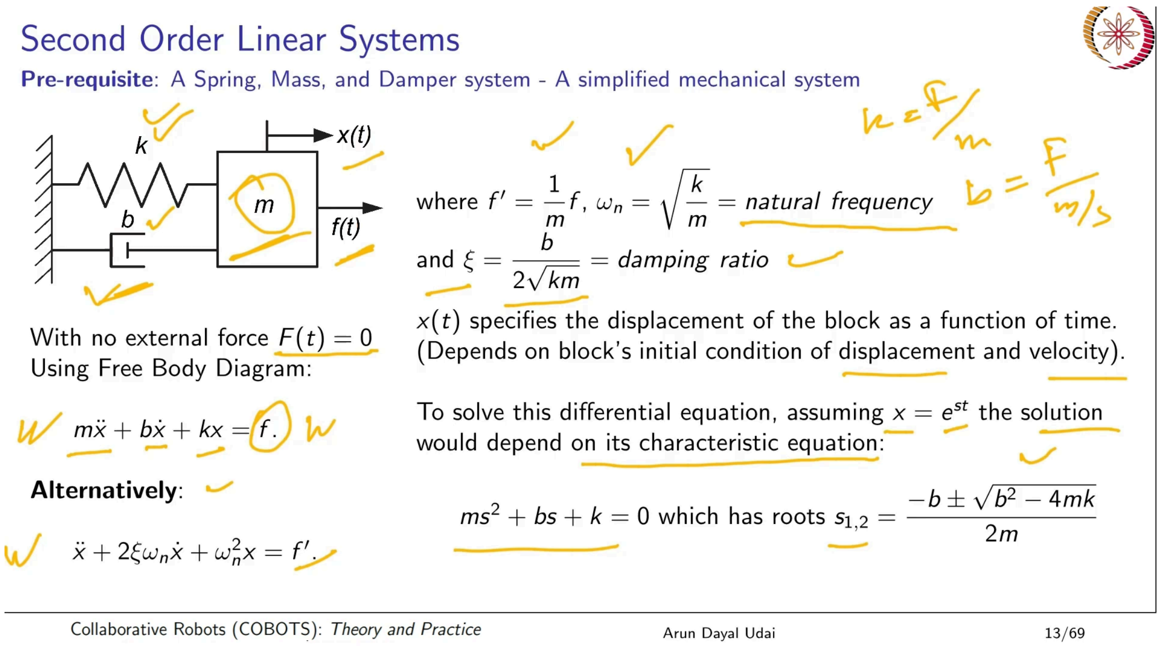
NPTEL Online Certification Courses
COLLABORATIVE ROBOTS (COBOTS): THEORY AND PRACTICE
Dr Arun Dayal Udai
Department of Mechanical Engineering
Indian Institute of Technology (ISM) Dhanbad
Week: 07
Lecture: 29

Response of a Second Order Linear System



Welcome back. So, in the last class, you learned what a controller is and why a robot needs a controller. You also saw an example of a Spring-Mass Damper System, which is a Second-Order Linear System. So today, we'll move further with that and understand the behaviour of such systems with different parameters, and their behaviour. So today, we'll begin with the response of a second-order linear system. So, let us continue.



So, this was the slide that you saw even in the last class. So, you saw this is a typical spring-mass damper system with here is your mass, this is your damper with damping constant as b, and spring coefficient that is k. So, k is nothing but force generated per unit displacement.

$$k = \frac{F}{m}$$

$$b = \frac{F}{\frac{m}{s}}$$

So, the unit could be Newton per meter. Similarly, you have b, which is given by the force generated per unit velocity of the system. So it should be a meter per second. So, the units are accordingly.

So, you see, these are the two constants that are there that basically determine the behavior of the system. Basically, that also determines the characteristics of the system. So that is also defined by the mass which is there. It is applied by an external force F(t). You see it is here, and let us say if I have a displacement which is given by X(t) at any instant of time. So, with no external force that is with F(t) is equal to zero. The system can just be written as:

$$F(t) = 0$$

So, this goes zero. So, it is mx double dot plus bx dot plus kx.

$$m\ddot{x} + b\dot{x} + kx = f$$

So that becomes your time domain equation dynamic equation of this spring mass damper system. This is nothing but a differential equation of the second order. Alternatively, the same can be written as you saw it can also be written as x double dot plus 2 psi omega n x dot plus omega n square x is equal to f dash.

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = f'$$

So, I have divided the whole of the equation by m. So effectively, the f dash is nothing but f divided by m. So that is here. So, it is proportional to f. So whatever is the behaviour of f that is fed as an input. So, f-dash also goes in a similar way.

$$f' = \frac{1}{m}f,\ \omega_n = \sqrt{\frac{k}{m}} = natural\ frequency$$

So, omega n is your square root of k by m, which is known as the natural frequency of the system.

$$\xi = \frac{b}{2\sqrt{km}} = damping\ ratio$$

Xi (ξ) is given by b by root over km. So, that is nothing but a damping ratio. X(t) specifies the displacement of the block as a function of time. That depends on the block's initial condition, also that is displacement and velocity.

$$x = e^{st}$$

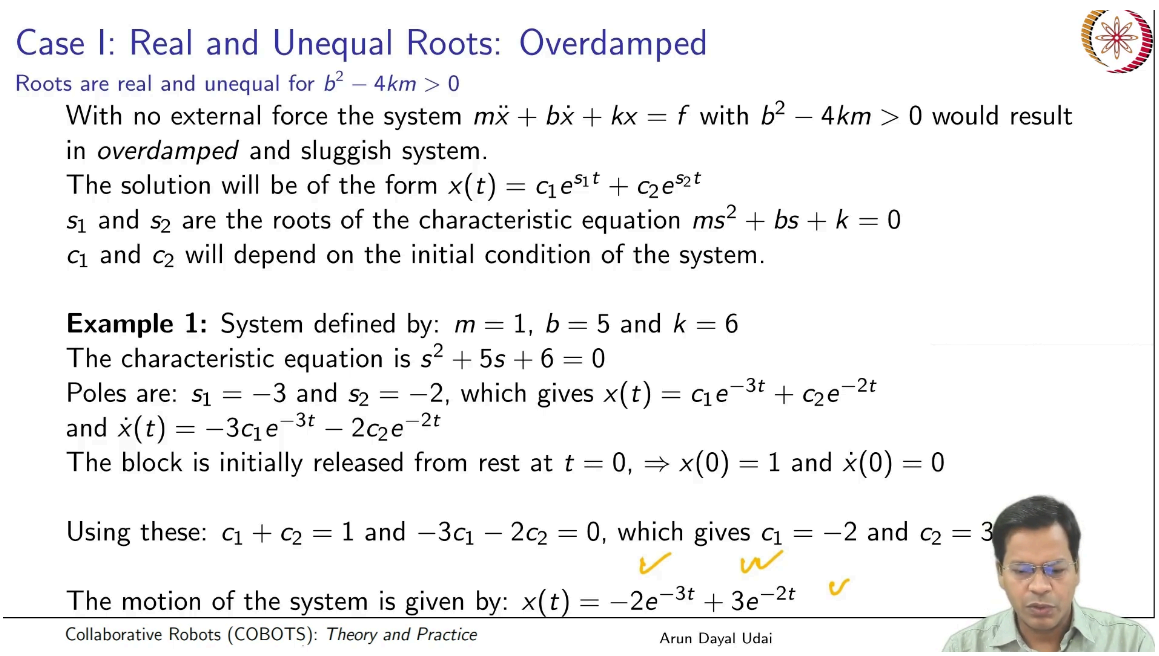
To solve this differential equation, we assumed in the last class that x is equal to e to the power st, and the solution, you know now, depends on its characteristic equation that is given by ms square plus bs plus k, which has the roots S1 and S2.

$$ms^2 + bs + k = 0$$

So, that is S1 and S2. So, it is minus b plus minus root over. This is the discriminant which is here, that is b square minus 4mk root over divided by 2m.

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

So, it has something which is under the square root that can be positive. It can be a real value. In case it is negative, it will have a complex value. So, in that case, it becomes minus b plus minus a complex number out here divided by 2m. So, whole of these S1 and S2 are known as roots of this equation, the characteristic equation, and this decides the behaviour of the system.



Let us begin with different kinds of such systems. One such system could be an Overdamped system in which b is dominant; that is, damping is dominant over the spring stiffness. So, the roots are real and unequal. So, in that case, we have no external force. So, f becomes equal to zero,

$$m\ddot{x} + b\dot{x} + kx = f$$

This is your time-domain solution of your system, and with b square minus 4km going to be greater than 0, which would result in an overdamped system, and it behaves in a sluggish way.

$$b^2 - 4km > 0$$

The solution would be of the form x(t) is equal to c1 e to the power s1 t plus c2 e to the power s2 t, where s1 and s2 are roots of the characteristic equation.

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

c1 and c2 are the constants that can be found out by initial conditions, the boundary conditions, that is, initial velocity and displacement, which were given while you just pulled it at a distance and just released it so that it can oscillate or it can dampen.

So, s1 and s2 are the roots of this characteristic equation. c1 and c2 would depend on the initial conditions.

So yes, let us just take one such example and see the behaviour of this system. So, for this system, I have assumed m is equal to 1. Units are accordingly in the SI system. So, if it is 1, it is 1 kg. Similarly, b, that is, the damping constant, is 5. Stiffness is 6. So, the characteristics equation can be written as ms square, that is s square, plus bs, that is 5s, and plus k, so it is 6. It is equal to 0. So that goes here.

So, in the last class you saw why this is called as a characteristic equation. Basically, this relates to the output by input. So, output and input are related by this, and in the denominator, you saw this goes. So, if that is real, that is zero, that is a constant which are having some imaginary numbers, so that will basically govern the behaviour of the system, and that is the reason it is known as the characteristics equation also.

Let us start analysing this. So, what are the roots here which will be known as the poles, that is, s1 is minus 3, and s2 is minus 2? So now, the equation which I told you should be like this. So, the time domain solution would look like X(t) is equal to c1 e to the power minus 3t; one of these will go here, and similarly, s2, that is minus 2, will go here. So, effectively, your system's time domain solution should look like this.

$$s_1 = -3 \text{ and } s_2 = -2, \text{ which gives } x(t) = c_1 e^{-3t} + c_2 e^{-2t}$$

Now, c1 and c2 are to be determined. We know the system has started from rest. The initial displacement was given. So, first of all, let me just take the derivative of this. Taking the derivative would give me, in time, x dot t, which is equal to minus 3 c1 e to the power minus 3t and minus 2 c2 e to the power minus 2t. Got it?

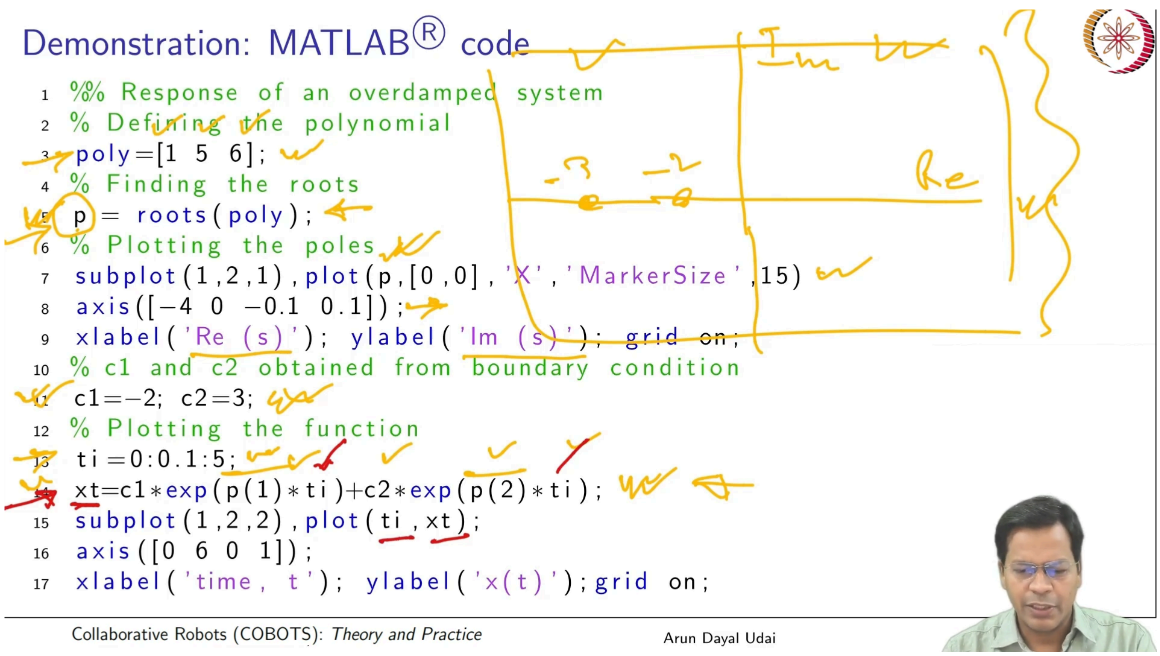
$$\dot{x}(t) = -3c_1 e^{-3t} - 2c_2 e^{-2t}$$

So, I now know the time domain solution looks like this. The velocity looks like this. So, displacement and velocity. Substituting the initial condition. I know t is equal to 0 at time t is equal to 0. When the block was released, it had a displacement of some unit meter. So, 1 meter, and then x dot 0 is equal to 0. That is initial velocity is 0. so this is how these two conditions can be put over here and here. So, at t is equal to 0. I just put t is equal to 0 in this and put x t is equal to 1 here. So, I'll get one equation and again, x dot 0 is equal to 0. So, I'll put again that t is equal to 0 in this equation. Okay, and I will get one equation for x dot t is equal to 0 and put it here.

So, these two will give me two equations.

$$c_1 + c_2 = 1 \ and \ -3c_1 - 2c_2 = 0$$

This one and this one. Solving these two simultaneous equations, I could extract c1 and c2. So you see, I got c1 as minus 2 and c2 as 3. These values make this time domain solution of this complete this system complete. So, I'll put them here, and I got to the motion of the system, which is given by x(t) is equal to minus 2 e to the power minus 3t and plus 3 e to the power minus 2t. So this is the time domain solution of my system. So, now I can plot this with varying times, and I can see how it is moving over. So it may oscillate, it may dampen. So I'll see the behaviour. Also, I know the poles of this system. So those are given by minus 3 and minus 2. So that can also be plotted somewhere. So, let us see how to go about it.



So, I'll start with a simple MATLAB code. Matlab, I assume quite a few of you must be having this Matlab also with you. If you even if you don't have one, so octave is free software that can run any Matlab code of this type. So, you know displacement is to be plotted against time. So, you have the equation which is given.

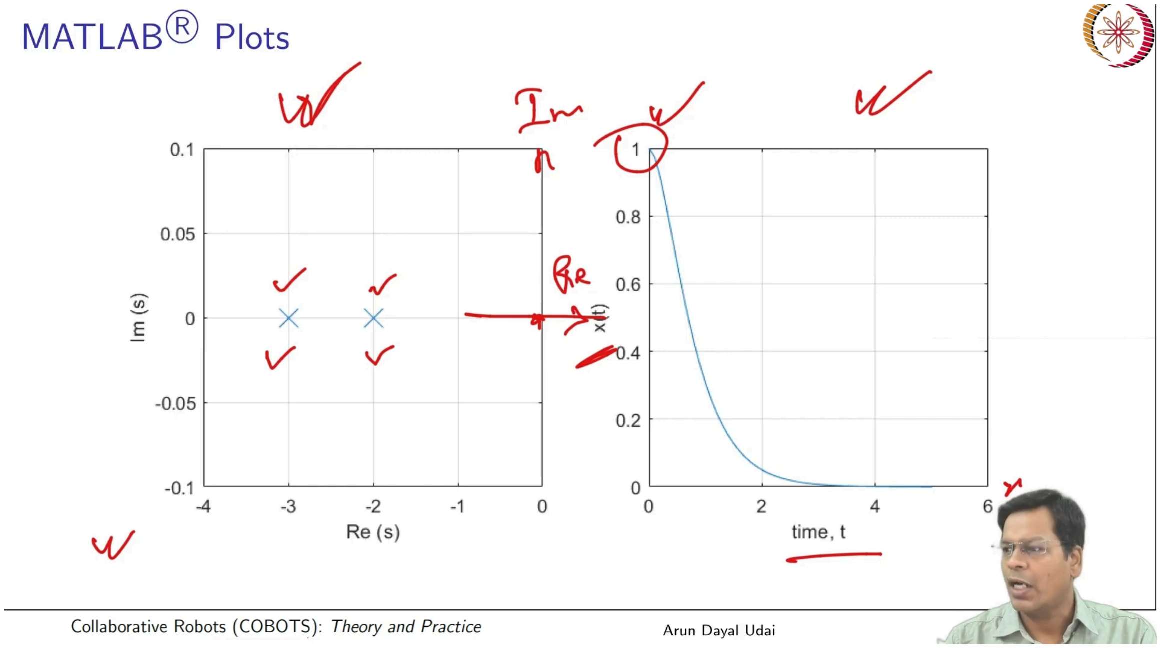
$$x(t) =- 2e^{-3t} + 3e^{-2t}$$

So, this can even be coded with simple Matlab code and you can just plot the results. You just have to plot x with time, with the same variables that I will be taking now.

So, yes, this is your system that is the polynomial of the characteristics equation. It was given as ms square plus bs plus k. Three things were there, so that is put here, as I have just created a polynomial with the coefficients which are here. So, the roots of the polynomial will be stored in the variable which is p over here. So, p will have the value. So, what value it will have? It will have minus 3 and minus 2. Got it? So, that goes to p, and I have created two plots. So, one of them will plot p. So, that is the first one. That is nothing but p. p means you have two values. So, one of them will plot p in the complex plane. So you have a real number that goes to one axis, and you also have an imaginary number. So, the imaginary is plotted along here. The real number is plotted over here. So, you know you have both the roots with minus 3 and minus 2.

So, both the roots will be somewhere over here in the real number line. So, minus 2 and minus 3 should be visible somewhere over here. So, all the poles are drawn in a complex plane. So as to analyse your system. This has a special significance you will come to know now. So, why are we plotting the roots of the characteristics equation in an imaginary plane somewhere like this, and what does this signify and how can this dictate the behaviour of your system you will come to know now?

So yes, so with the extents, I have just put the extents so that you can see the range. So, the whole of the extent is now set so that you can see the range of your values clearly, and I have labelled it along the x-axis along the y-axis like this. So, these are the two constants. With this, the time domain equation of your displacement varies. So, xt is equal to c1 e to the power s1 t is here. Similarly plus c2 e to the power s2 t. so you have all the values of constant c1 and c2 here. Whereas s1 and s2 are nothing but that you have taken you have stored your polynomial roots in p. so p1 and p2 will tell that goes here. So, effectively I am writing the time domain solution of my displacement over here, and this is to be plotted. So, what I have done I have simply made ti vary from 0 to 5 seconds in a step of 0.1, which is told here. Similar is the way in Python, similar is in octave, so you can just key in like that. So, ti goes like this. So, the whole of the ti is inserted here and here, and the xt array is created. So you get for each value of ti, the system has calculated xt and xt is stored over here. Now, this is to be plotted against Ti. So Xt against Ti. So, I have created another plot. So, what do I get? There are two plots.



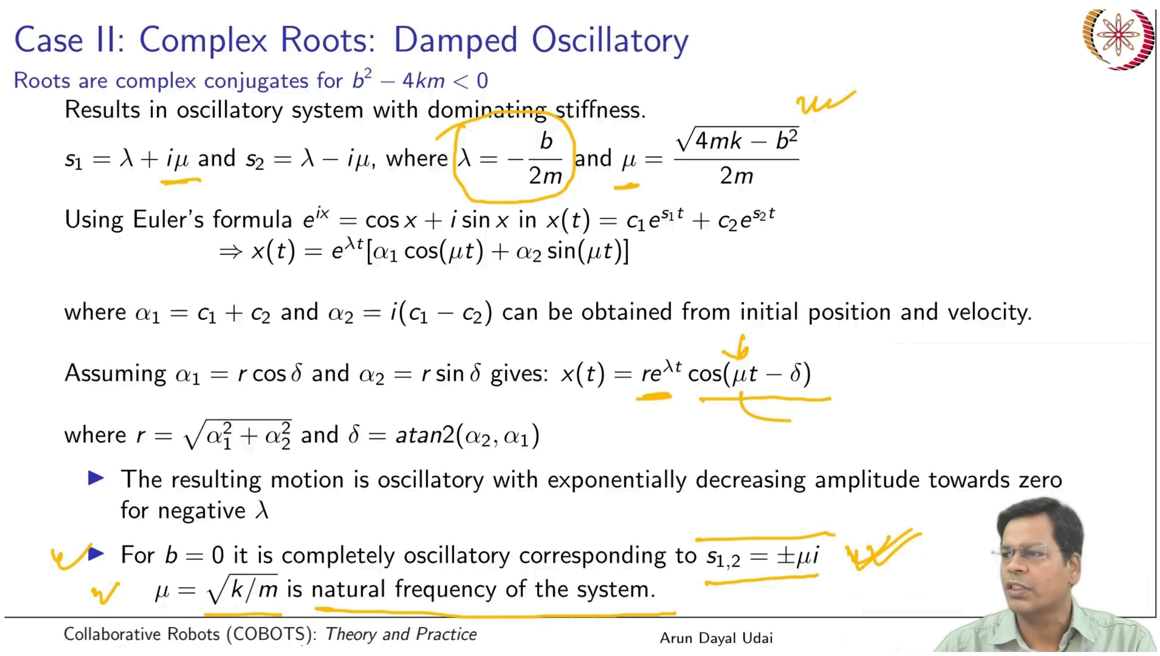
So, this is the one which is the plot of roots that are minus 2 and minus 3. The origin is here. So, this is your imaginary axis. This is your real number line. So, you see, it has some values on the real number line. So, you have some values which are here. So, these two are the poles, and both poles are on the negative side in the complex plane. So, both the poles are on the real number line. They are not imaginary.

So, this is your system, which was plotted against time. So, you have time which goes here, and x(t) that comes here. So, you see, you have released it from a displacement, which is 1, and then it gradually comes to 0.

So, without any oscillation, you see it is a system that is given by roots where real and unequal.

$$b^2 - 4km > 0$$

So, b was dominant. So, it was an overdamped system. So, it is an overdamped system. It quickly comes to rest. So, that is the zero thing. So this is the behavior of a system for real roots which are on the negative side in the complex plane. So, this is the system. This should govern the system, and this is your actual system, how it is behaving. So, just remember this, and we'll compare this with other results that we'll be getting in the next system example that I am going to give. So this is the first.



Now, the second one. So this is a system where you have b square minus 4km less than 0.

$$b^2 - 4km < 0$$

That means I should see complex conjugates. Complex number roots always appear in conjugates. You know that already. If it is a quadratic equation like your characteristics equation. So, those two numbers, those two poles, those are the roots may be written as lambda plus i mu and lambda minus i mu.

$$s_1 = \lambda + i\mu \text{ and } s_2 = \lambda - i\mu, \text{ where } \lambda = -\frac{b}{2m} \text{ and } \mu = \frac{\sqrt{4mk - b^2}}{2m}$$

Where lambda is given by minus b by 2m and similarly mu may be written as this. So, I have just taken i square is equal to minus one okay? So, that is a complex number thing that you are already familiar with. So, I'll just put it here, and you substituting the values s1 and s2 to this equation.

$$e^{ix} = \cos x + i \sin x \text{ in } x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

So, what do you get? You can quickly write it as X(t) is equal to C1 e to the power lambda plus i mu plus C2 e to the power lambda minus i mu into t, and again here, you also have t.

$$x(t) = c_1 e^{(\lambda+i\mu)t} + c_2 e^{(\lambda-i\mu)t}$$

So, this can be taken out as e to the power lambda t that can be taken out, and inside, you will have c1 e to the power i mu t plus c2 e to the power minus i mu t. So that goes inside.

$$x(t) = e^{\lambda t}[c_1 e^{i\mu t} + c_2 e^{-i\mu t}]$$

So now, I can substitute the second part of it. This one e to the power i x is equal to cos x plus i sin x. So if you substitute that once again here, so you can get e to the power lambda t that is still outside and c1 here, I will write it as cos mu t plus i sin mu t. So, that comes here, and similarly, over here, I can write it as plus c2 cos mu t. This time, it should be with a negative sign here cos minus mu t, and again, you have plus i sin minus mu t. So, this can be written like this. So that comes here minus mu t, minus mu t. And finally whole of the bracket will come here. And that is it.

So, if you take all of these commons, you can write it as e to the power lambda t c1 plus c2 cos mu t that will come here and plus i c1 minus c2, and you can write sin mu t got it. So, this is how you can write your system x(t). So, that goes exactly like this.

So, if i again take this as alpha 1 and this as alpha 2. So, again, you can proceed it like this. So, you can write it as X(t) is equal to exactly like this.

$$x(t) = e^{\lambda t}[\alpha_1 \cos(\mu t) + \alpha_2 \sin(\mu t)]$$

So, everything comes here, whereas alpha 1 and alpha 2 are nothing but C1 plus C2, and i C1 minus C2.

$$\alpha_1 = c_1 + c_2 \text{ and } \alpha_2 = i(c_1 - c_2)$$

They can be obtained from initial position and velocity. So, you already know when you have released your block and at what displacement you have released, what velocity you have imparted. So, those initial conditions can be put the way we did just now in the earlier example, and I can evaluate alpha 1 and alpha 2.

So yes, again, I will assume alpha 1 and alpha 2 as r cos delta and r sin delta.

$$\alpha_1 = r\cos\delta \text{ and } \alpha_2 = r\sin\delta$$

That can lead me to this.

$$x(t) = re^{\lambda t}\cos(\mu t - \delta)$$

So how? I will tell you once again. So, what was that? So you just got x(t) is equal to e to the power lambda t. Here, you see you have r cos delta and cosine mu t plus r sine delta and sine mu t. So, that is here. So you see you can bring out your r that goes as e to the power lambda t. Inside what you see, cos delta cos mu t, sine delta sine mu t. So that can be written as cos mu t minus delta. Those are simple trigonometrical substitutions. So you see. Your system behaviour will now be given by this.

$$x(t) = re^{\lambda t}\cos(\mu t - \delta)$$

So this is the time-domain solution of this. So if you just. Square and add your substitution which you made here. So squaring and adding will give you alpha 1 square plus alpha 2 square. Root over becomes r, and by dividing alpha 2 by alpha 1, you can get to this. That is, delta is equal to tan inverse alpha 2 by alpha 1. So that is there.

$$r = \sqrt{\alpha_1^2 + \alpha_2^2} \text{ and } \delta = atan2(\alpha_2, \alpha_1)$$

So quickly, you have obtained your equation for the time dependency of your system. so you can plot your system now with time. The displacement with time you can obtain from this. That will tell you how your system is behaving. Is it oscillating, or what is it doing? You already know what your roots are. Your roots are nothing but complex conjugates.

The motion is oscillatory with exponentially decreasing amplitude. You see, this is a constant. For any value of lambda, this is a constant but multiplied by e to the power lambda t. Lambda is negative. You see, lambda is negative. So, it is exponentially decreasing amplitude. So, this basically creates the amplitude. So, you see, it is exponentially decreasing, and this is creating an oscillatory behaviour.

With time, if you plot, your displacement will behave in a cosine manner with some phase over here. So. it won't start with the exact position where you see the peak, but you see here you have some delta, which is given by this.

$$\delta = atan2(\alpha_2, \alpha_1)$$

So, the resulting motion should be oscillatory and exponentially decreasing amplitude towards 0 for the negative value of lambda. Lambda is negative; you know that. For b equal to 0, that is, if you substitute b equal to 0, so this becomes 0.

$$\lambda = -\frac{b}{2m}$$

So, lambda is 0. So, in that case, this is purely a constant value, and you are left with just cosine oscillations.

$$re^{\lambda t}\cos(\mu t - \delta)$$

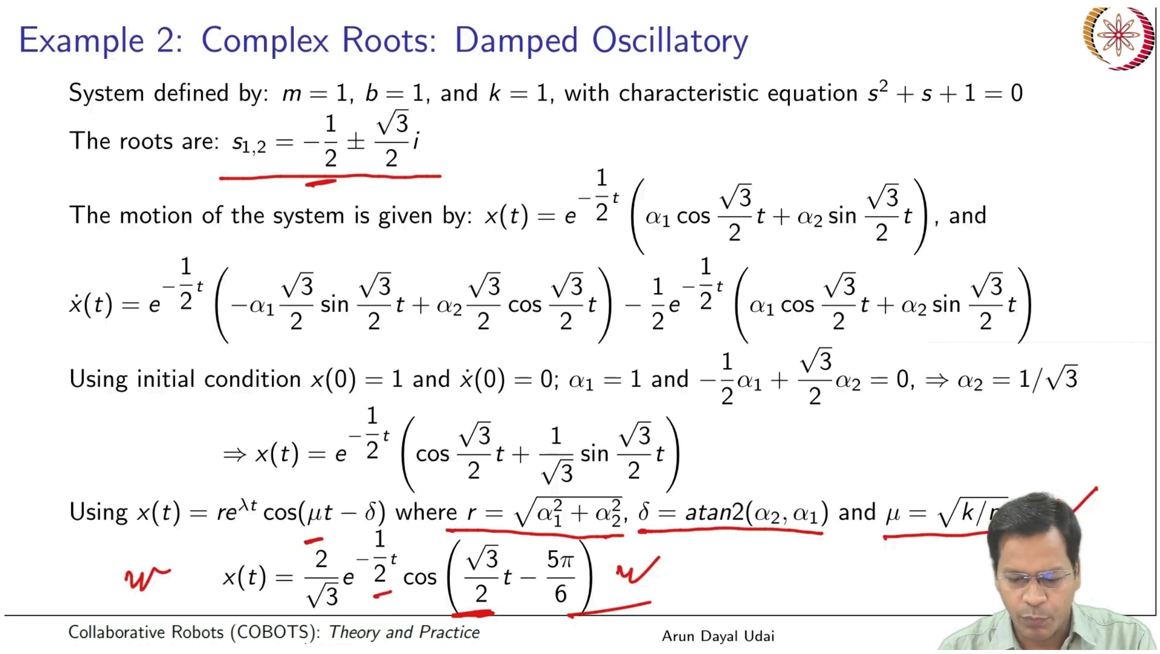
So, your system now is completely oscillatory, corresponding to s1,2, which is just plus or minus I mu.

$$s1, 2 = \pm \ \mu i$$

So, what is this? It is just pure complex numbers. So, it does not have any real value. So, it is imaginary. So, both the roots are imaginary, and they are conjugates, and you do not have anything. So, it is exactly on the real-imaginary number line. You have mu, which is given by the square root of k by m, which is basically the natural frequency of the system.

$$\mu = \sqrt{k/m}$$

So, you have mu. So, that is the angular frequency of cosine. So, you see, mu was here, and what was mu? So, you just substitute b is equal to 0 here, and you get mu. So, mu is equal to the square root of k by m, which will show the natural frequency of the system with which it will oscillate.



So, now let us just see how to plot this with the system, which is similar to this. So, you have your system given by m is equal to 1, b is equal to 1, and k is equal to 1. The characteristic equation is given by s square plus s plus 1. Roots are complex conjugates with a real value here and an imaginary value here.

$$s_{1,2}=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$$

The motion of the system will be given as x(t).

$$x(t)=e^{-\frac{1}{2}t}\left(\alpha_1\cos\frac{\sqrt{3}}{2}t+\alpha_2\sin\frac{\sqrt{3}}{2}t\right)$$

x(t) is e to the power lamda t. r e to the power λt. You see, r is 1 here. So e to the power λt. So λ was minus 1 by 2, which comes here, and plus-minus mu. So, mu will go here mu t. So, alpha 1 cos mu t plus alpha 2 sin mu t. So mu comes here.

So again, moving further. So, if I take the derivative of this. So, I should be getting this.

$$\dot{x}(t) = e^{-\frac{1}{2}t}\left(-\alpha_1\frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t + \alpha_2\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t\right) - \frac{1}{2}e^{-\frac{1}{2}t}\left(\alpha_1\cos\frac{\sqrt{3}}{2}t + \alpha_2\sin\frac{\sqrt{3}}{2}t\right)$$

You can do it yourself. So, first into a derivative of second and second into a derivative of first will give you this. So, this I have done to just get the position and velocity equation of my system, and when I substitute the initial condition, that is at time t is equal to 0, x is equal to 1. Similarly, for velocity, the x dot is equal to 0. At time t is equal to 0, putting these two values in these two equations, I can get alpha 1 is equal to 1 and minus 1 by 2 alpha 1 plus root 3 by 2 alpha 2 is equal to 0.

$$x(0) = 1 \text{ and } \dot{x}(0) = 0;\ \alpha_1 = 1 \text{ and } -\frac{1}{2}\alpha_1 + \frac{\sqrt{3}}{2}\alpha_2 = 0, \Rightarrow \alpha_2 = 1/\sqrt{3}$$

So, these are two simultaneous equations I am getting out of these two input equations. Solving this I also will get alpha 2 as 1 by root 3. Alpha 1, you got here, and alpha 2 is here. That makes this system time domain solution complete.

So now, my system equation will tell x(t) is equal to e to the power lambda t alpha 1 and alpha 2 will come here, and you get the complete solution. So, this is how your system will behave.

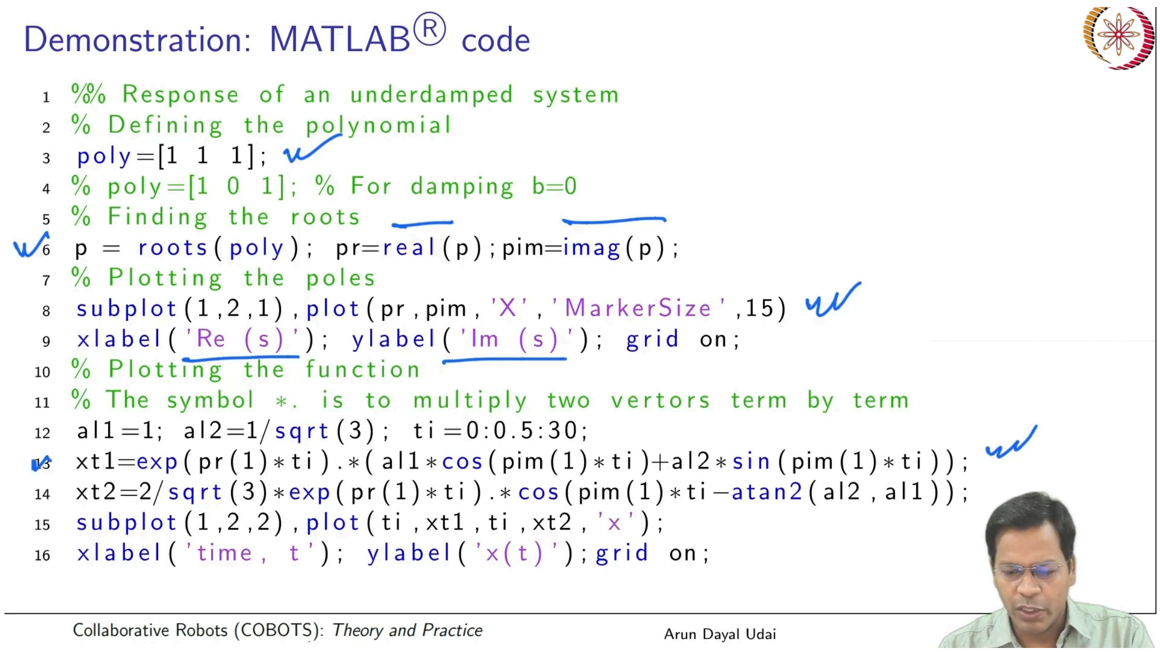
So, if I can again substitute for those constants you already know that, so you get to cosine variation of your equation. So, you can directly use those substitutions where r is equal to this, and delta is equal to this, mu is equal to this. So, you know you can write your equation in a new form, which looks like this.

$$x(t) = re^{\lambda t}\cos(\mu t - \delta) \text{ where } r = \sqrt{\alpha_1^2 + \alpha_2^2},\ \delta = atan2(\alpha_2, \alpha_1)$$

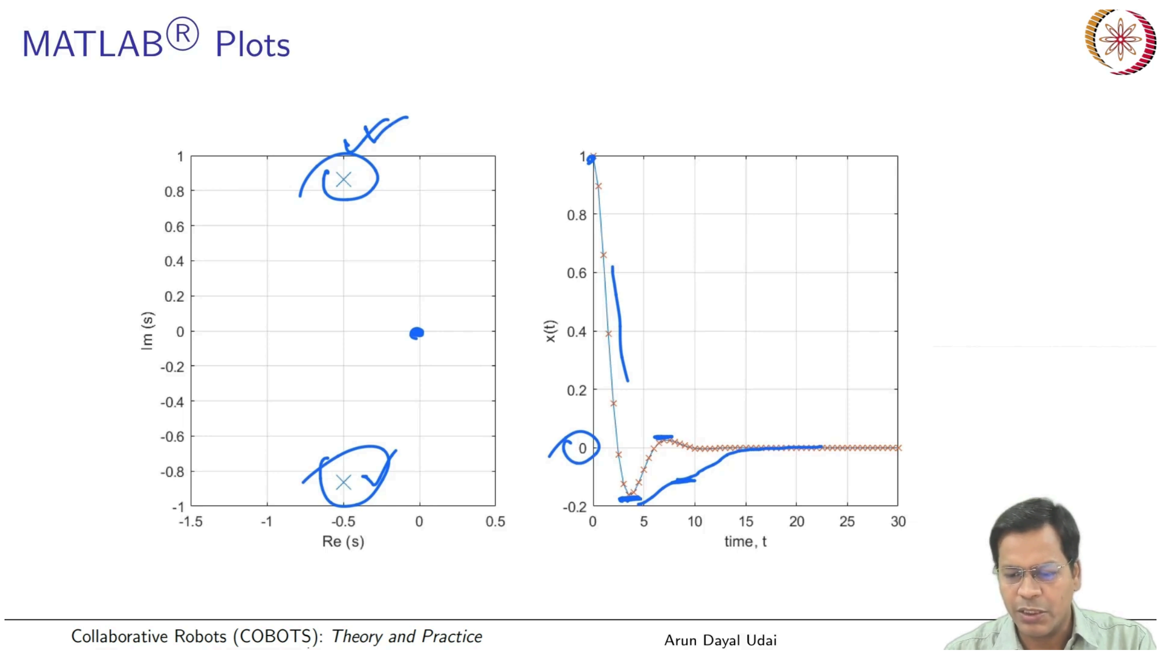
So, here you see this is your r-value that is going to come and e to the power minus lambda t. So you have e to the power lambda t, lambda was minus 1 by 2. So, r is calculated here and cosine mu t minus delta. So, the delta is 5 pi by 6. So, this is your system dynamic equation.

$$x(t) = \frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t - \frac{5\pi}{6}\right)$$

So, x(t), this is the time domain solution. You can quickly plot this and see its behaviour. You already know the roots. The roots are complex conjugates. So, let us plot both. You can plot in the complex plane. You can show your poles where they lie, and you can also plot the behaviour.



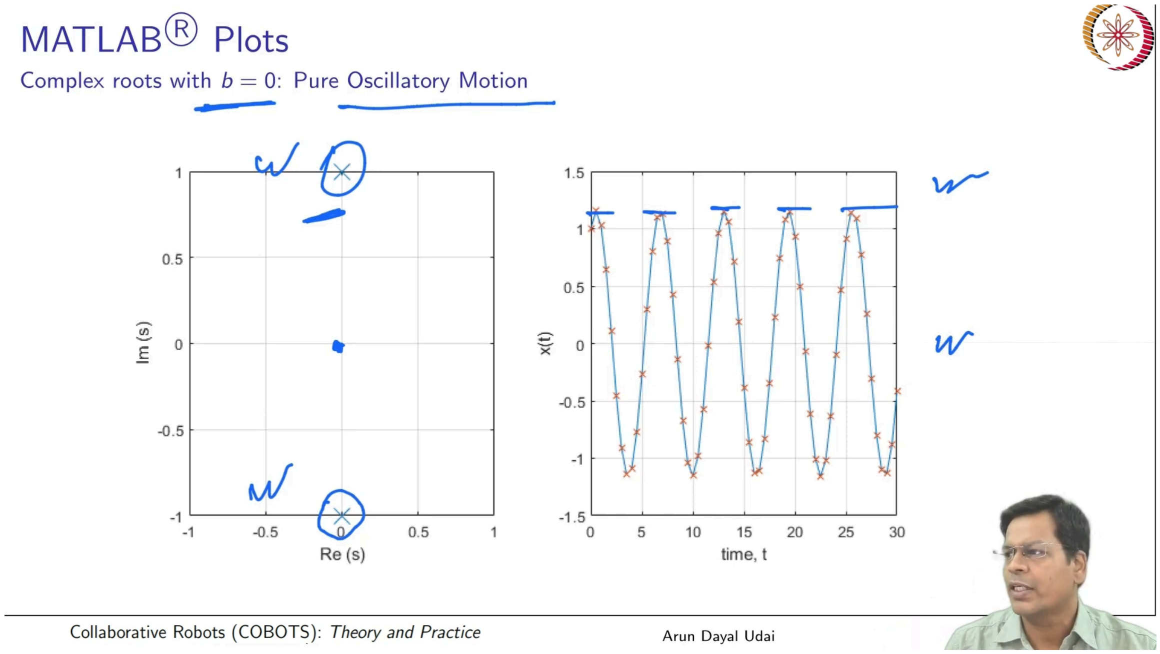
So now, I am doing that. So, I am using my MATLAB code once again. So, I am defining my system, storing the roots here, storing the real and the imaginary values here, and showing them in the complex plane. Now, I am labeling them also here. Again, I am defining my system over here for the time variation in how my xt will go, and I am putting them, plotting them once again here, and I am labelling it here.



So, see how my system is behaving. You see, I have zero and zero, so that is coming here; that is your origin. So, both the poles are on the negative half of your complex plane, and both are complex conjugates. So, whatever this is, it is the same value mirrored like this. And your system, you see, this time it started from here. It is overshooting to some value, overshooting on the other side, and coming back. It is still oscillating, which is not clearly visible here, but it is oscillating and finally reaches zero.

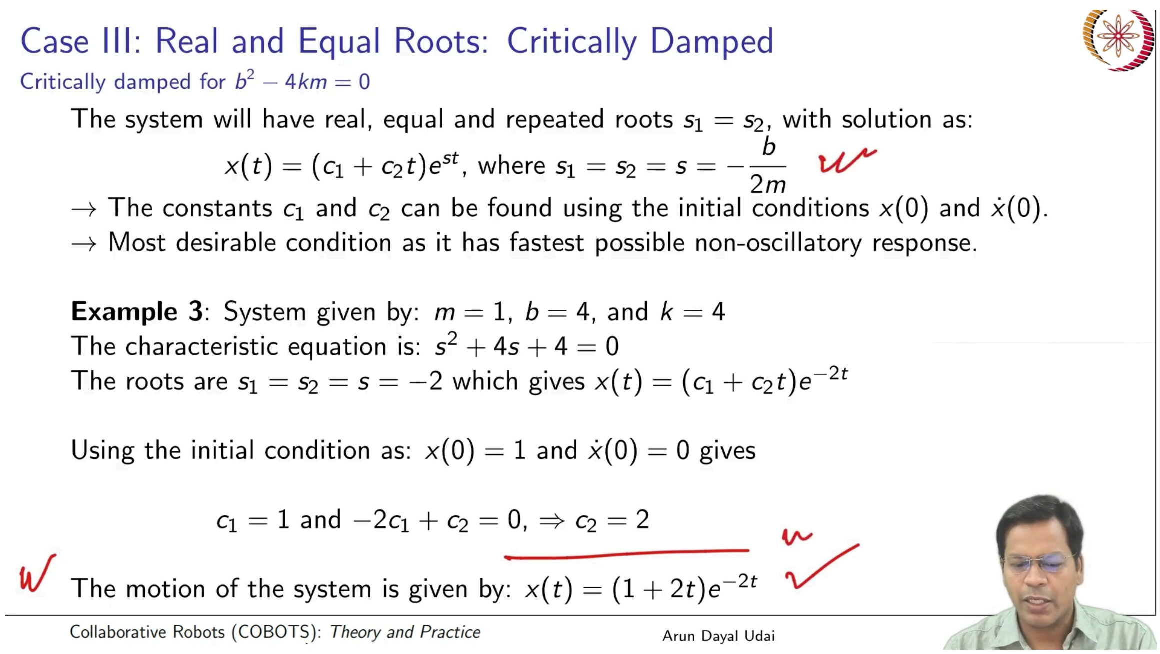
So, what kind of system did you expect here? It is a damped oscillatory nature. You see, your system is damped oscillatory, and the amplitude decreases exponentially every time. So, whatever the amplitude is here, the next amplitude at every cycle will be reduced exponentially, and finally, it comes to this. So, that is how it behaves.

So now, you see, again, you can compare it with your earlier results. So, this time, you have poles which are on the complex side of it. So, you have these two complex number values here, and they are on the negative side of the complex plane. So, negative and complex conjugate systems exhibit damped oscillatory behaviour.



Now, let us look at the system when it was an undamped system. So, it is purely oscillatory. You have seen this by analysis. So, by plot analysis, both the roots are shown here. You only have mu here; that is plus and minus in the complex plane mu i and minus mu i. So that comes here. Exactly on the imaginary line. You see. So, these are your roots. So, the roots are imaginary this time. It doesn't have any real value. The system is oscillatory. The system is plotted here, and you see it has the same amplitude every time. It is not an exponentially decreasing amplitude.

So, based on the roots that lie in the complex plane, you have the system behaviour. So, roots basically decide the behaviour of your system. So, as soon as you get your system, you quickly find out the roots of its characteristic equation. You can quickly say how my system is going to behave without further looking at the time-domain solution and the plots. So, you can directly tell. So, this is the beauty of having such plots.



Now, let us look at the third case when you have real and equal roots. That is a critically damped case. I will tell you what a critically damped thing is. So, this time, you have b squared minus 4km is equal to 0.

$$b^2 - 4km = 0$$

This time, you have repeated roots: s1 is equal to s2, and the solution by differential equation you already know. So, this is x(t) is given by c1 plus c2t e to the power st. s is nothing but s1, and s2 is s, and that is given by minus b by 2m.

$$x(t) = (c_1 + c_2 t)e^{st}, \text{ where } s_1 = s_2 = s = -\frac{b}{2m}$$

So, that quickly comes here. So, you can directly substitute s in this equation, and you already know that c1 and c2 can be obtained by taking the derivative of this. That is, x and x dot t and substituting the boundary condition you can obtain two equations. By solving those two equations, you can quickly get c1 and c2. So, c1 and c2 are obtained by the boundary conditions and the initial conditions. So, that is this. At time t is equal to 0.

So, this is the most desirable condition, as it is the fastest non-oscillatory response. So, the system is not at all oscillatory in any case, and it quickly comes to 0. So, this is the behaviour you want. You know what you are doing with all these systems. You have commanded your robot to go to a place. Your robot may be a second-order system. In this case, you just command your robot to go to a place, and it can simply oscillate and come back to a location, or it can just go to a place and stop there, or it can go to that place very fast and still stop there without oscillating. So, this is the case of the critically damped case, in which it quickly goes to that place and stops there. So, this is the behaviour you want. Most of the time, you should be like this.

Let us say your system is oscillating. What will happen? You reach to a place and oscillate. So, that is very much undesirable because what you will see, you have been commanded to pick a ball from a table and what it will do. It will simply hit the table because it will overshoot and come back. It will overshoot on the other side again overshoot. So, it can hit the table. Without actually going to the precise location on top of the ball and holding it, it will hit the table. So, that is what is strictly undesirable in robotics. So, you should either be overdamped, or you can be critically damped. Critically damped is not always possible. So yes, you can be near to critically damped. So, this is the most desirable condition. This is what should be obtained through various parameter tuning. We'll see tuning later on also. So, this is a non-oscillatory response.

Let us just start with one example once again. So, this is m is equal to 1, b is equal to 4, and k is equal to 4. Your characteristic equation is like this. Now, see the roots. Roots are equal, that is -2, and I have made my time domain solution. Taking the derivative of this and substituting the initial boundary condition. I can get two equations: that is, c1 is equal to 1 and -2c1 plus c2 is equal to 0.

$$c_1 = 1 \text{ and } -2c_1 + c_2 = 0, \Rightarrow c_2 = 2$$

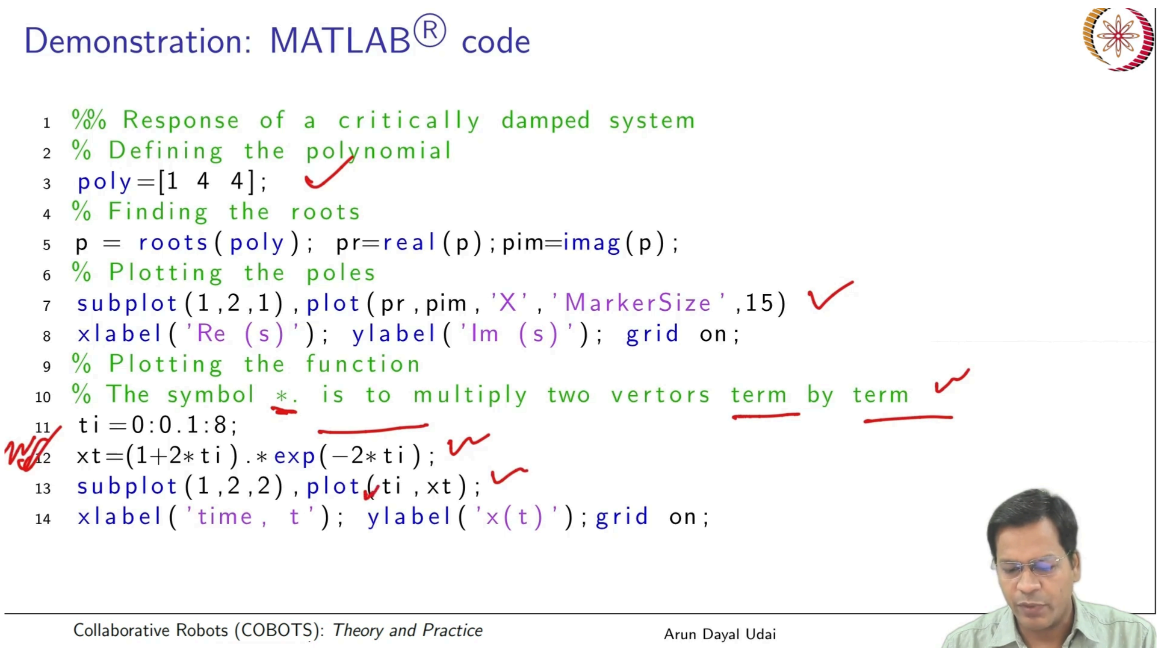
So, these two are the simultaneous equations. Solving this, I got c2 is equal to 2. So, c1 and c2 both are obtained. So, that can now be put here,

$$x(t) = (c_1 + c_2 t)e^{-2t}$$

And you get the complete time domain solution.

$$x(t) = (1 + 2t)e^{-2t}$$

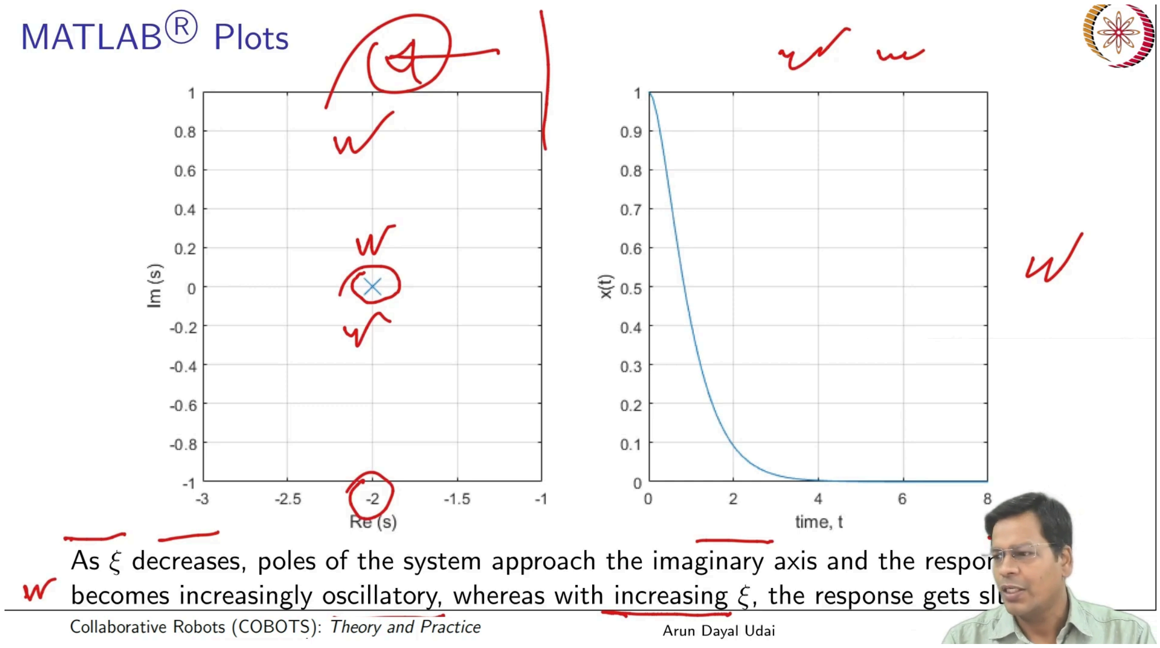
So, it is your displacement given over time. It will vary over time like this. So, this is how your system will behave. So, you can plot this. You can plot your roots also. Roots are nothing but real and equal roots.



So again, I am using a similar MATLAB code. So, this is your system polynomial. Roots are stored in p. So, you have taken real and imaginary here. You know already. You should not get any imaginary value here and plot your markers. So, that is nothing but your roots that can be plotted in the complex plane. So this is the first plot, and time this time, I am varying my time exactly like this. So your time exactly goes like this. It is from zero to this.

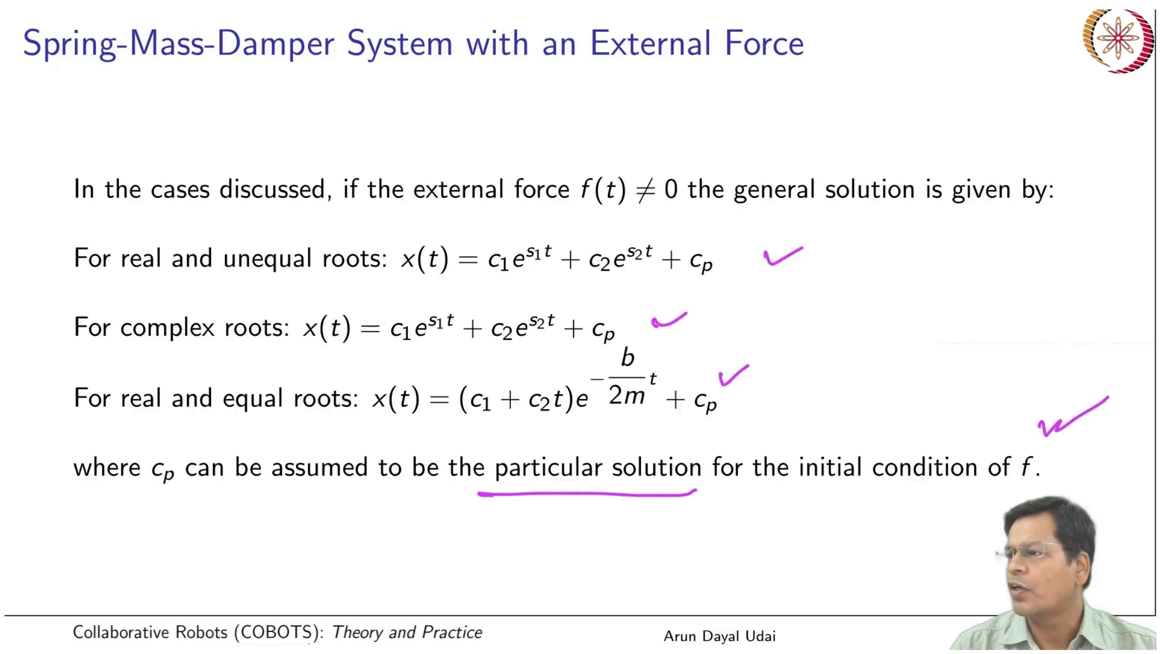
So, yes, you see, you have xt, which is defined here. It is 1 plus 2t, and you have exponential variation that is defined here. So, your xt is defined like this. So, the symbol star dot especially says to multiply the two vector terms by term. So, that is the way to program it in MATLAB.

So now, I'll plot xt versus ti. So, ti will vary along the x-axis, and xt is plotted. So, this will show the behaviour of my system with time, and those are labelled here.



So, let me just see the system. How does it look? So, you know, this time, my roots were exactly lying at the same location, and that is real. So, both the values were in the imaginary plane. It was zero. It was purely real, and both were minus two, as shown here. So again, it is on the left-half side of your complex plane. So, it is on the left side. Both are equal and real, and your system is critically damped. So again, remember this. So, in this case, both are real, the left side in the Complex plane, and your system is critically Damped. Real and equal gives you this case. So, this is the fastest Non-oscillatory response.

So you see, As your damping ratio Xi decreases, the poles of the system approach the imaginary axis, and the response becomes increasingly oscillatory, whereas, with increasing damping, the response gets sluggish. So, there is a time in between in which you should see a critically damped case. After that, it becomes overdamped. So, this is the set of behaviours that you should see. In all these cases, I have given some examples to make you understand. So, you just have to remember how and where in a second-order system you should see those roots. Where does it lie? So, it should lie on the left half side of it, and preferably, we prefer to get this. But if not this, you should be happy with a critically damped case, or you should be better having an overdamped case, but definitely not an oscillatory nature and pure oscillatory is definitely very, very bad. So, it will infinitely keep on oscillating. So, those are the undesirable cases.



So yes, let us move ahead now and see how your system behaves with an External Force. This is what your system actually is. It is not that it is naturally oscillating in some manner with its natural characteristics. So, that is what we have analysed now. So, how does that affect any external force? So, you get to see similar results. So, in the cases as discussed, if the external force is not equal to zero, the general solution in the first case, when real and unequal roots, you should see a similar result.

$$\text{For real and unequal roots: } x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + c_p$$

The only thing that will change here is. You will see the term which is here (c_p). So, apart from the constant which is c1 and c2. This is the additional constant. So, you should know x0, you should also know to x dot 0, and at that, t is equal to 0; you should also know how much is your f what is the force value. So, if you also know the third condition, you can get to the third constant. So, these two will create c1 and c2 variations. Whereas this one will directly affect this one, and all three constants will be calculated using three boundary conditions. So, that is for real and unequal roots.

Similarly, for complex roots, you should see a similar one, a similar equation with c_p coming here.

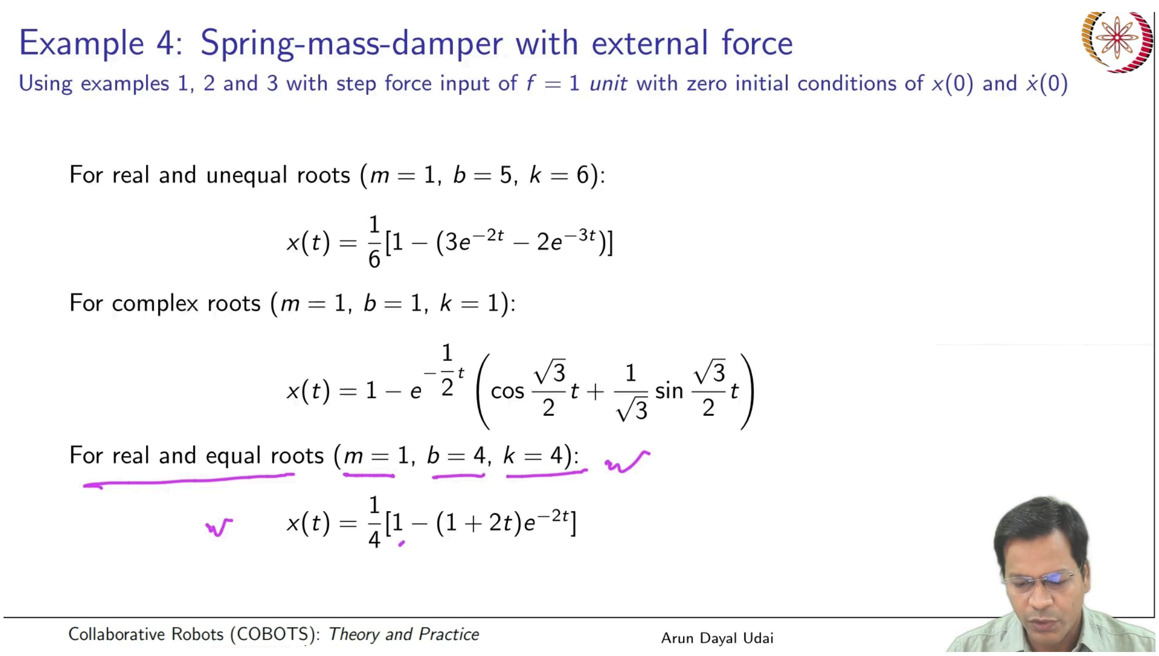
$$\text{For complex roots: } x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + c_p$$

Again, the same parameters are required to find out these constants. s1 and s2 are the roots of the characteristics equation. So, your system characteristics equation remains the same. So, that is not affected by the external force. So, that is why it is external.

In the case of real and equal roots, you see this equation again, once again, and this is your c_p.

$$\text{For real and equal roots: } x(t) = (c_1 + c_2 t)e^{-\frac{b}{2m}t} + c_p$$

So, c_p can be calculated using all those initial parameters. Yes, c_p can be assumed to be a particular solution for the initial condition of f. So, with the initial condition of f, you can find out the value of c_p in all these.



Let us see a few examples for real and unequal roots. The equation that we have taken we started with the same one. Example 1, I have put it once again here. m is equal to 1, b is equal to 5, k is equal to 6. The characteristic equation remains the same. So, the roots go here. s1 and s2 will come here. The only thing that has changed is the c_p value. If you relate it with the previous one, c_p can directly come to the other side of the equation. I can write it like this.

$$x(t) = \frac{1}{6}[1 - (3e^{-2t} - 2e^{-3t})]$$

So, this is your c_p value that has come here. So, this becomes your time-domain solution.

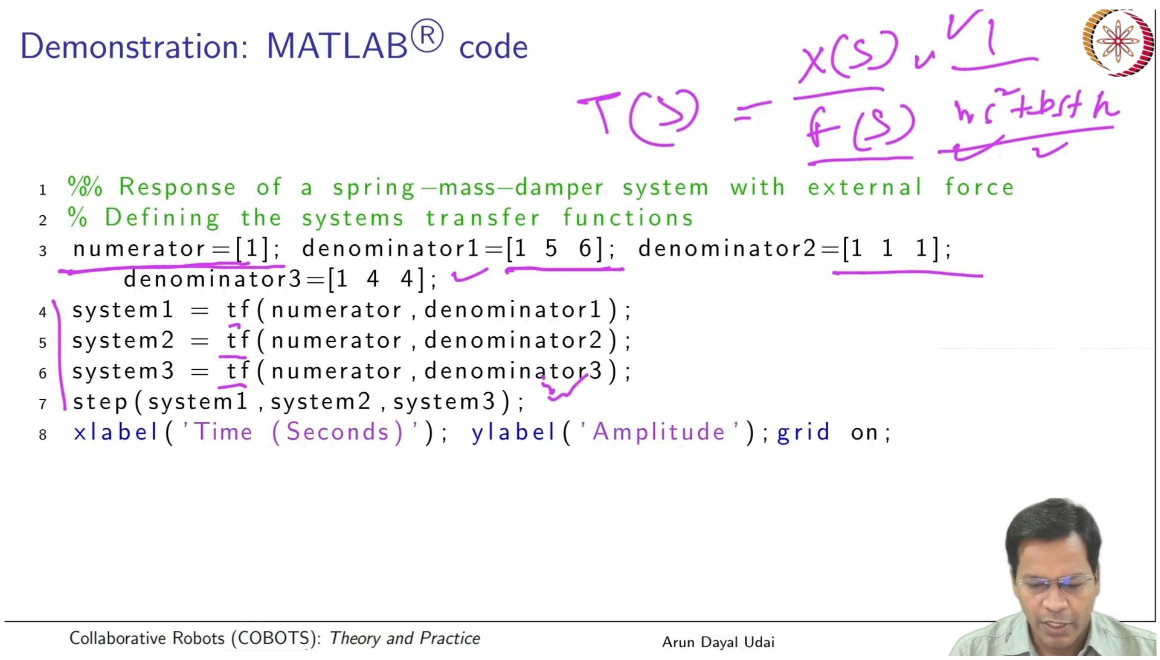
Again, in the case of complex roots. Again, you can write your equation like this. You have lambda plus minus i mu. So, the mu term will go here, and the lambda term will go here, and c_p comes here, and the time-domain solution will be like this.

$$x(t) = 1 - e^{-\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right)$$

So, by plotting these, you can see the influence of the external force and how your system behaves with that external force.

Similarly, for real and equal roots, m is equal to 1, b is equal to 4, and k is equal to 4. Roots are equal, and they are b square minus 4ac is equal to 0. In that case, you get to see this equation. This should be a critically damped case once again, and then this is your c_p value that comes here, and everything is like this.

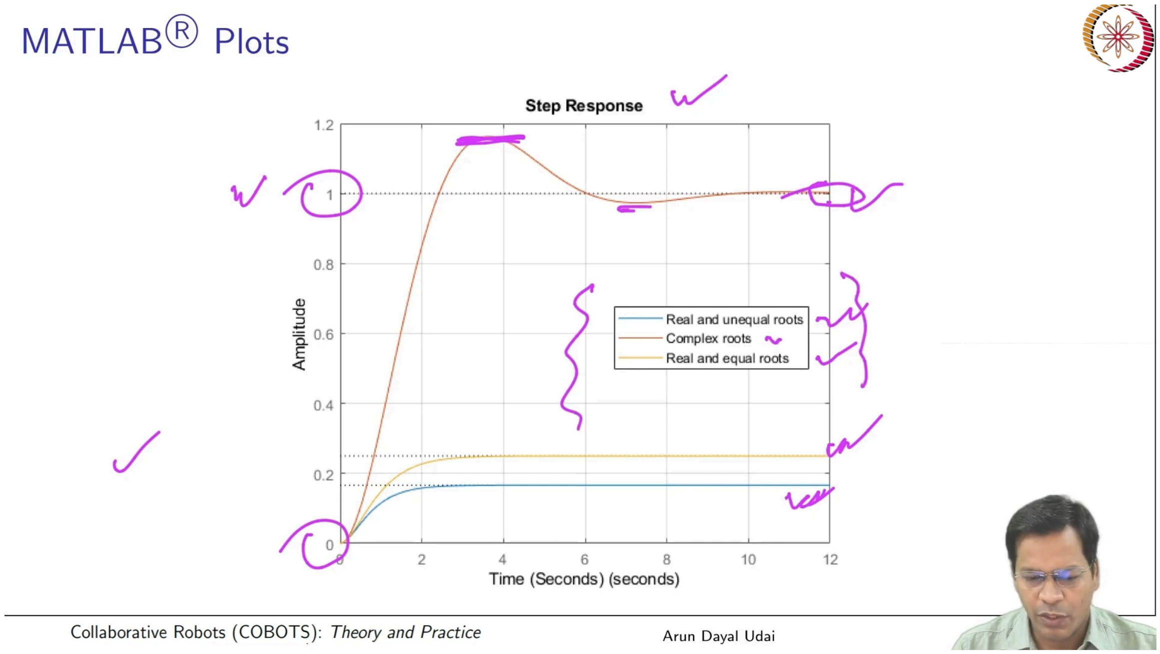
$$x(t) = \frac{1}{4}[1 - (1 + 2t)e^{-2t}]$$



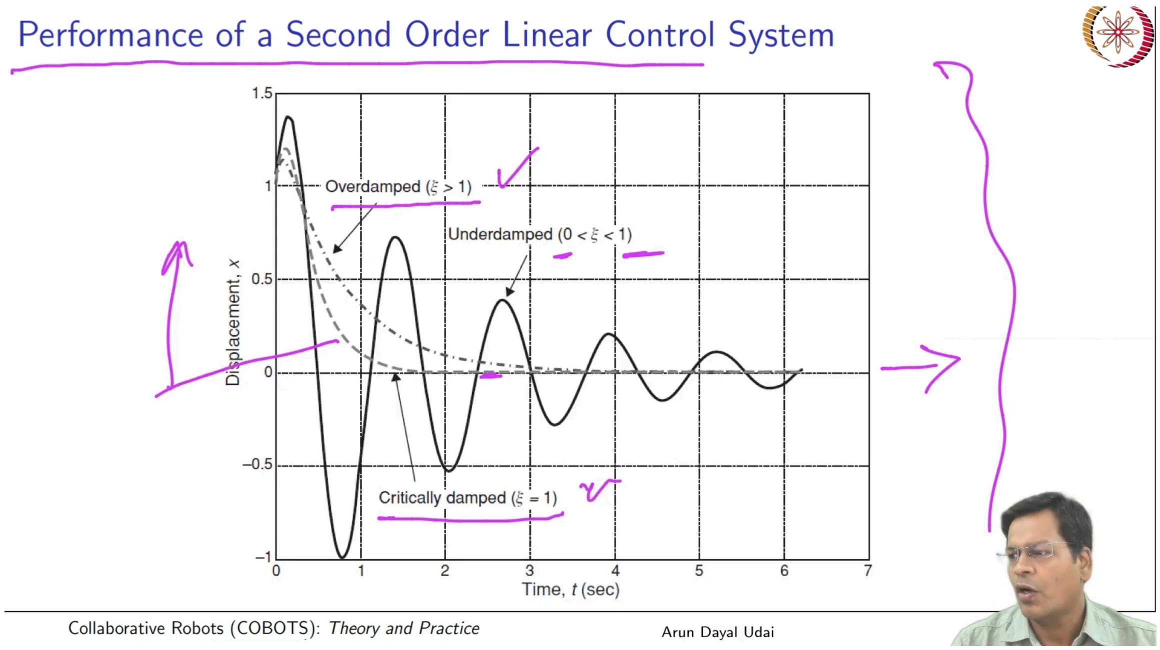
So now, I'll plot all of them. So, this time, I am using a different way to plot it. I am defining it as a transfer function, the way we discussed it in the first class. So, this is your first denominator, which is basically x(s), which is the output by F(s).

$$T(s) = \frac{x(s)}{F(s)} = \frac{1}{ms^2+bs+k}$$

So, this becomes your transfer function. So, x(s) is the numerator. So in the numerator, you have one only. In the denominator, you see you have F(s) output by input. So, that is the transfer function. So, you see x(s) by F(s). So, that should tell you 1 by ms square plus bs plus k. So, that basically gives you the transfer function. So now this becomes your numerator that, is your denominator. For the numerator you have the same values for all the cases, whereas this will change. So, that is the denominator for the first case, the second case, and the third case. So, you have three systems. All of these can be plotted together in MATLAB. I'm labelling it like this (Time and Amplitude).



So, I am plotting all the systems together here. So, this was a system with real and unequal roots. This is the one. You have complex roots. It is oscillatory. It has an overshot. It has come back, and it is like this, and the third one, which has real and equal roots. So, this is the critical condition in which it settles down very fast. So, you see, you have an external force, and with time, it settles to a value. This time, it won't settle for a value that is zero. This time, it gradually reaches rest, which is here. This is a step-input response. So, the step was just one unit. So, you quickly reach the one that is here, and you just overshoot it, come back, and ultimately settle somewhere over 1. For different roots, this is the thing that will basically define your system, and this is the behaviour of your system. So, these are the MATLAB plots. You can do that yourself also in Octave or maybe in Python.



This is what the performance of your control system is. So, you can quickly write it as Displacement is plotted here, and your time varies like this. You see, you have an overdamped case for xi greater than 1, an underdamped case for xi lying between 0 to 1, and this is the case that was critically damped, where xi is equal to 1. So, in that case, it is a critical time, and in this case, the system settles to the desired value very quickly.

So, this is the fastest non-oscillatory response. So, everything can be seen here. So, this is the plot of the system. So, you already know where your roots lie. So, you can, based on the roots and their location in the complex plane, decide how your system is going to behave. So, you know how to get the characteristic equation of your system, and you can find out the roots. You can say, how is my system?

So, that's all for today. In the next class, we will see the Transfer Function Approach and State Space Approach to analysing your system. We will also look at the DC Motor Model of a robot joint and discuss it further.

So, yes, that's all for today. Thanks a lot.