Engineering Fracture Mechanics

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Module No. # 02

Lecture No. # 08

Elastic Strain Energy

We will now move on to the chapter on energy release rate. In fact, many concepts in fracture mechanics can be comfortably understood by looking at the energy. Although we have seen in the earlier classes, the solution by Inglis, for the sake of continuity we will start with that. And what I will learn in an Inglis solution you have got? For an infinite plate with an elliptical hole, the maximum stress is given as sigma into 1 plus 2 a by b.

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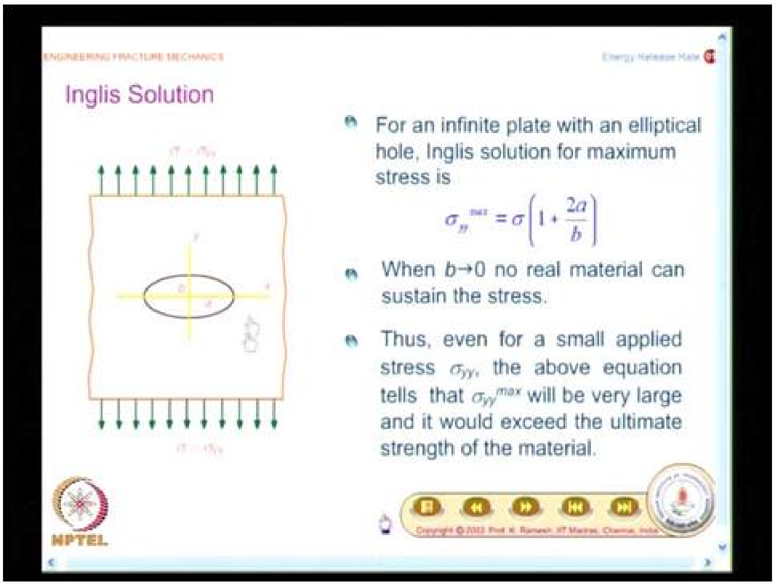

What you will have to look at is many aspects in fracture mechanics come from your problem of elliptical hole. You know when you take an ellipse you take the major axis as 2 a, and you also label the crack length as 2 b. When I have a center crack, I will label it as 2 a, and the minor axis given as b and what Inglis allotted was when the elliptical hole reduces to a crack where b tends to 0, the stresses are very high. And another issue you have to look at is, I have an infinite plate subject to a uniaxial loading, you have to watch it carefully; in the case of a plate with a circular hole we have taken infinite plate with uniaxial loading, in the case of an elliptical hole also you have applied a uniaxial loading.

When we develop the stress field we have to look at very carefully for the problem of a crack, are we looking at a uniaxial loading or a biaxial loading, which we would look when we take up the chapter on crack-tip stress and displacement fields? In this chapter, we would look at the energy approach and immediate consequence of Inglis solution is even for a small load the crack may grow and break the component into pieces.

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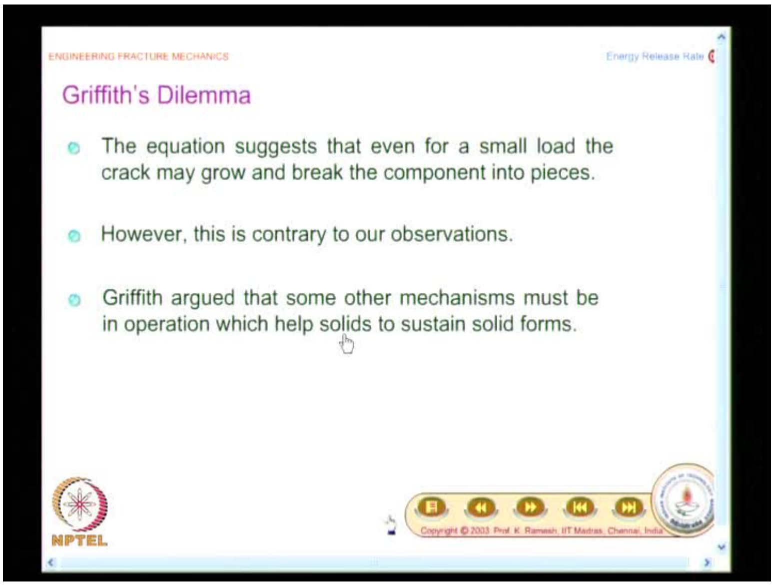

This is labeled as Griffith's Dilemma, it is not only the dilemma of Griffith's, any one who looks at the solution of Inglis would only wonder how the solid remains a solid after looking at a solution. Our practical observation is you find solids contain crack and they remain. So, how is this possible? This is possible only when you have some other mechanisms that operate, which help solids to sustain solid forms, this is key to Griffith's analysis.

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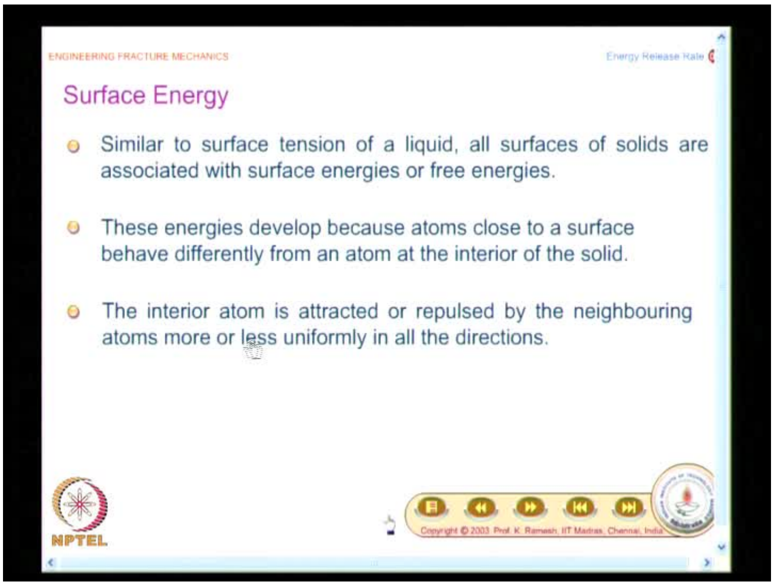

Similar to surface tension of a liquid, all surfaces of solids are associated with surface energies or free energies. See, this is a very important step. See, in those days Griffith was working on glass and surface energy in glass was very small. So, you are really looking at second order effects, unless you have a conceptual step forward, one would not have thought about the role of surface energy playing a role. Why do the surface energies come out? The surface energies develop because atoms close to a surface behave differently from an atom at the interior of the solid; see, in the interior of the solid the atom is surrounded by atoms on all the sides, so it remains in equilibrium.

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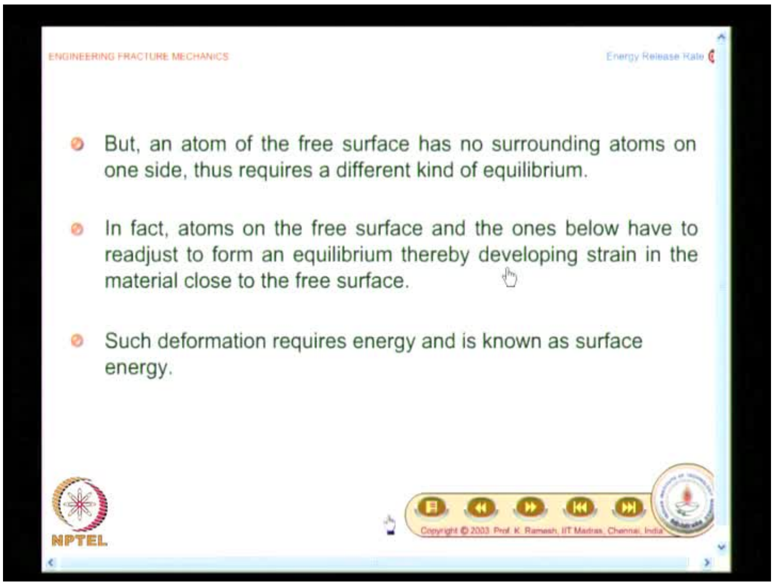

So, what you find is the interior atom is attracted or repulsed by the neighboring atoms more or less uniformly in all the directions, but that is not the case when I have the atom on the surface. This leads to surface energies which is very similar to surface tension of a liquid, and I am going to give you a conceptual appreciation how this is possible? On the surface what happens? There are no surrounding atoms on one side, thus requires a different kind of equilibrium.

In fact, atoms on the free surface and the ones below have to readjust to form an equilibrium thereby developing strain in the material close to the free surface. Such deformation requires energy and is known as surface energy. In fact, looking at the role of surface energy was a conceptual step put forward by Griffith and we will look at the values of surface energies for a variety of material.

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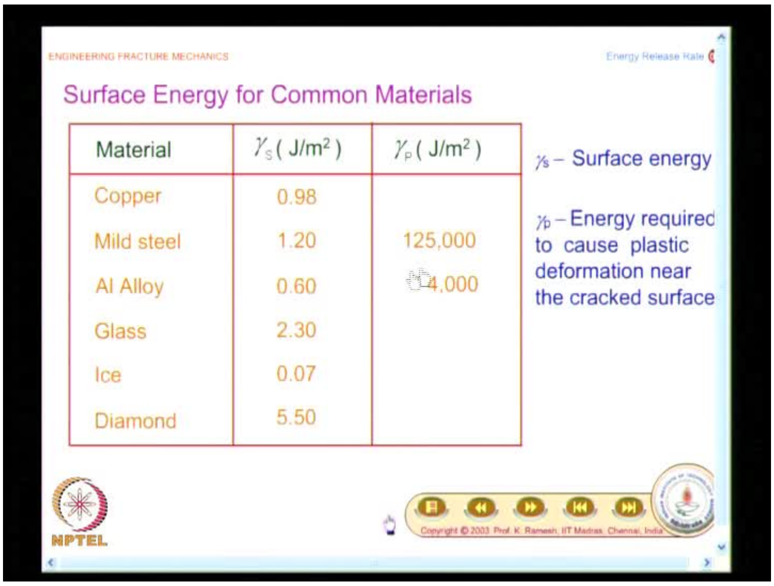

I have a list of material here and I have the surface energy which is given as joules per meter square, and for copper it is 0.98, mild steel about 1.20, aluminum alloy it is 0.6, glass is 2.30, ice is 0.07, and diamond is 5.50.

Griffith could not have worked on diamond because it is so expensive. So, he worked on glass and he find glass has the highest surface energy compared to other materials excluding diamond. And this table also gives another energy which is given as gamma P, which is the energy, required to cause plastic deformation near the cracked surface and what is it is value? It is very high for the case of mild steel gamma s is only 1.20 whereas, gamma P is 125000 and for the case of aluminum alloy it is 4000.

In fact, we would look up later in the chapter, how Irwin extended the analysis of Griffith which was applicable to brittle solids to ductile solids? With the simple extrapolation of Griffith’s approach by using gamma P, he could apply concepts of fracture mechanics for ductile solids. That is the reason why I am showing right away the value of gamma P, although in this chapter, in the major portion we would focus on brittle solids.

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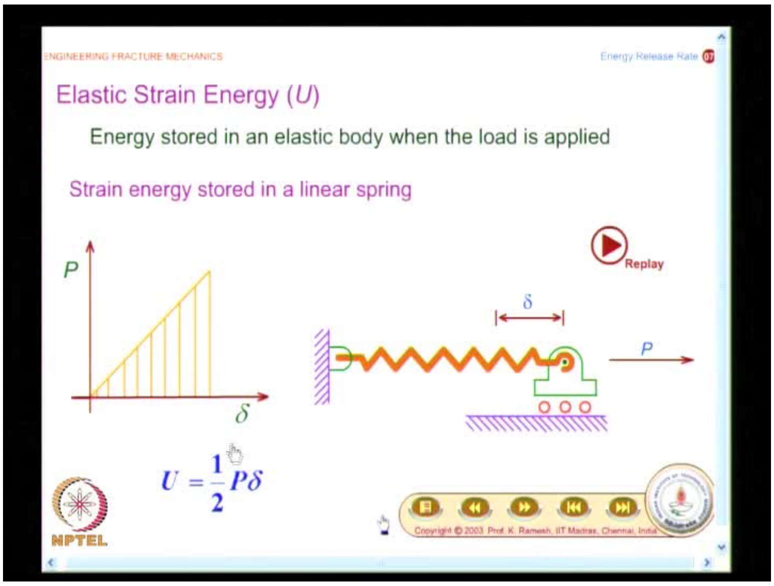

In this chapter, we will have to look up the energies. So, it is better to review the concepts on elastic strain energy, we will look this up for various simple cases and also reassure as on the methodology that we adopt. And some of these would help you to solve problems in this chapter and first let us take the case of a simple spring; it is a linear spring and watch the animation very carefully. And also, you have the graph which gives you P versus displacement. I will redo the animation, the points which I want to look at is, how P has been shown. It is very carefully shown that P varies gradually from 0 to the final value, this is very important.

The deflection of the spring is delta when P is supplied gradually. So, the area under the curve is what is the energy that is stored within the spring, you all know about it because P is applied gradually, you find elastic strain energy U equal to 1 half of P into delta. suppose I have a constant load acting then the energy would be P into delta a corresponding displacement is delta. In many problems in solid machines, when you show a final load we implicitly assume that the final load was reach through a gradual process from 0 to the final value. This is very important, all these certainties we will have to keep in mind.

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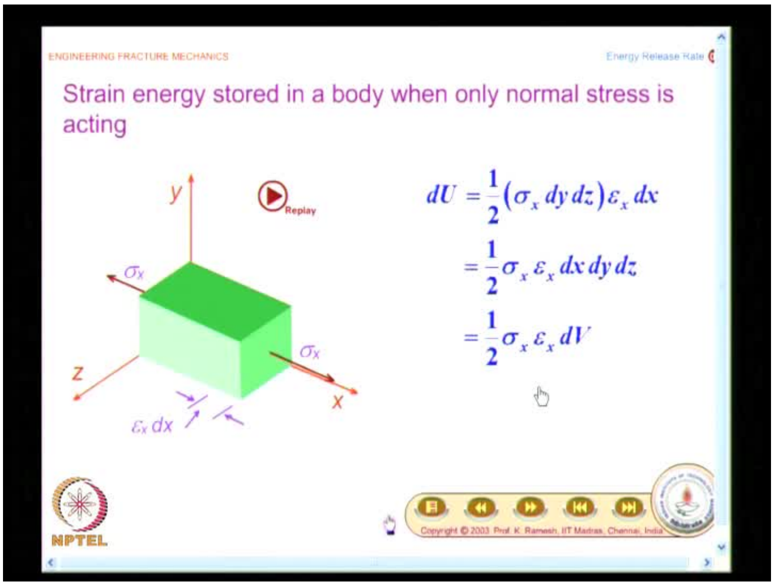

So, in the case of a spring also we have carefully put the animation in such a way that P varies from 0 gradually, and you know what is the strain energy stored? And we will extend this to a body subjected to normal stretch, a body subjected to shear stress, and a combination of all the stress components, etcetera. All these expressions will come in handy, when you want to solve problems in this chapter.

So, what you have here is, a small element is taken and you are having only normal stress, because of that the element has elongated and you can find out the force which is acting on it, you can find out the corresponding displacement and it is fairly straight forward. So, what I have is, I have the incremental strain energies given as 1 half of sigma x dy by dz, this is the area on which the stress is acting.

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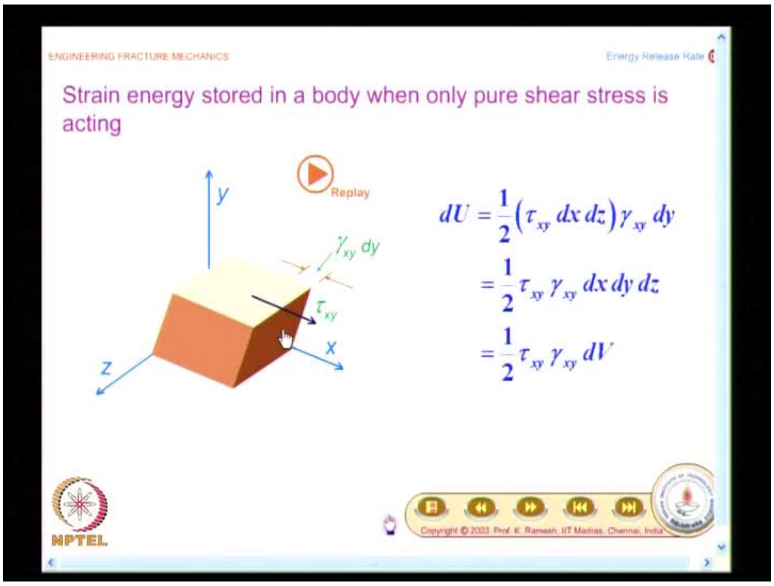

So, this gives you the component of force and epsilon x dx give the corresponding displacement and if you look at carefully, the incremental strain energy is one half of sigma x epsilon x into dV , and you would develop a similar expression when you have shear stress acting on it. You have an incremental element infinities of an element, you have with dx dy dz and this is subjected to shear stress. Here, again you can find out what is the force which is acting and what is the corresponding displacement? The force is tau x y dx dz and the displacement is gamma x y dy and we also assume, you have the factor one half comes, because the stress magnitude is applied, gradually varies from 0 to the final magnitude tau x y gradually.

Here, again you find the incremental strain energy stored in the element is one half of tau x y gamma x y dV. So, this is all you get for volume, per volume you are actually calculating it, and also we will look at for a unit volume. In this chapter, we would look at for unit thickness; we would also look at for finite thickness and so on. I have deliberately used those equations one after another, because you should know in which context we are talking about, and fracture mechanics is also uses a very funny symbol for thickness, it uses capital B as represented in the thickness of the specimen.

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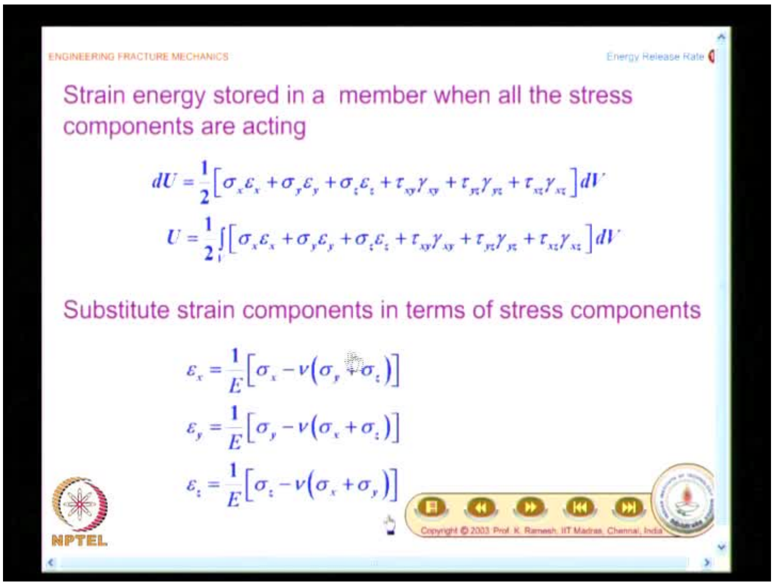

If you have to refer some of the older books which are undoubtedly classics, there you would find B as the thickness of the specimen. So, you would find our discussion, we would use the same symbolism to denote the thickness which will come a few slides later. Now, we have seen it for individual stress magnitudes, suppose I have combination of the stresses, then I can easily write the final expression as one half of sigma x epsilon x sigma y epsilon y sigma z epsilon z plus tau x y gamma x y plus tau y z gamma y z plus tau x z gamma x z dV. And If I want to find out the total energy stored in the system, you do the integration over the volume, here you find product of stress and strain, it is also desirable to look at these expressions in terms of stresses alone.

So, if I have to do that, I have to use the stress strain relations and one of the dangerous people use is, when they use a tension test, strain is only a function of the axial stress. Because the other components of stresses are 0, but if you write it for a generic situation, strain will have component from other components of stresses also, the axial strain you have to be very careful in writing it. So, you write epsilon x as 1 by E of sigma x minus nu times sigma y plus sigma z, never forget include this, this is very important. This is one of the common mistakes students do, when they move from a tension test to a generic situation, they never come out of the tension test, in a tension test sigma y and sigma z are 0.

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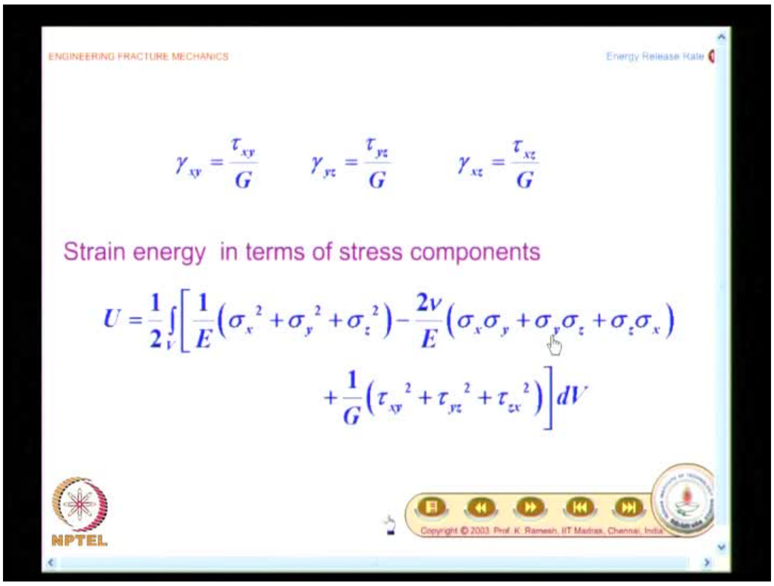

So, strain is related to stress by epsilon equal to sigma by e you cannot use that in a generic situation. So, the normal strains are related to normal stress in this fashion, shears strain there is no problem, gamma x y is related to tau x y, gamma y z is related to tau y z, gamma x z is related to tau x z.

So, when I use these, I can get the strain energy in terms of stress components and that reads like this. So, if I want to get the total strain energy, I would integrate over the volume and within it, I have the terms like 1 by E sigma x squared plus sigma y squared plus sigma z squared minus 2 nu by E into sigma x sigma y sigma y sigma z plus sigma z sigma x plus 1 by G tau x y whole squared plus tau y z whole squared plus tau z x whole squared. And you know very well that E is the young’s modulus, G is the shear modulus and they are related by the Poisson’s ratio.

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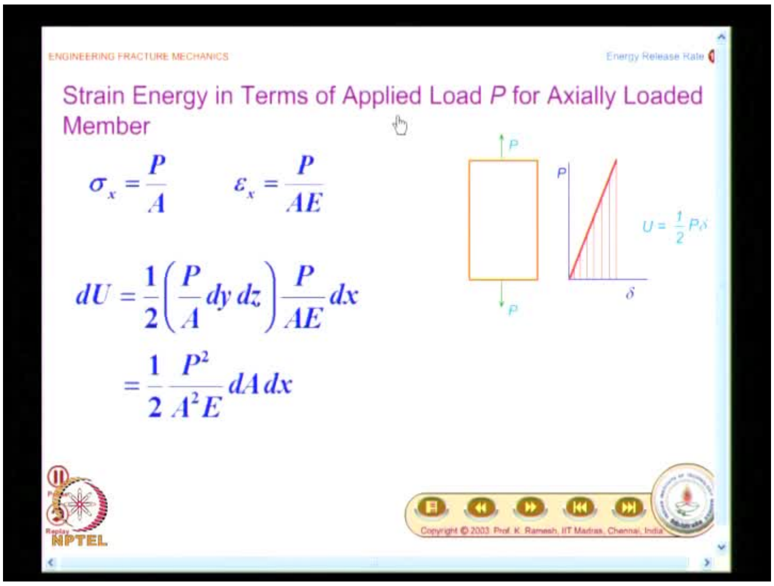

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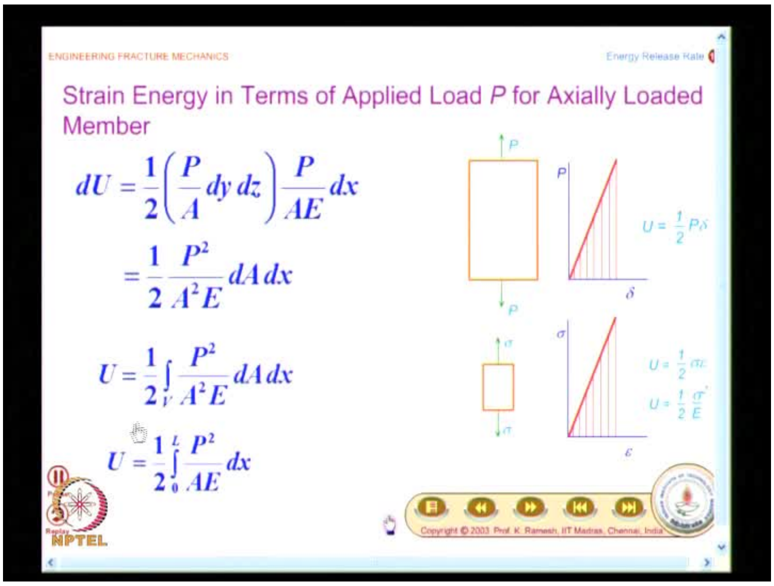

We will also now look at strain energy in terms of externally applied load, we will now look at for axially loaded member, this is what we look at now. These are basic expressions which would come in handy, when we want to look at how to apply the concept of energy release rate for fracture calculation, for simple geometries.

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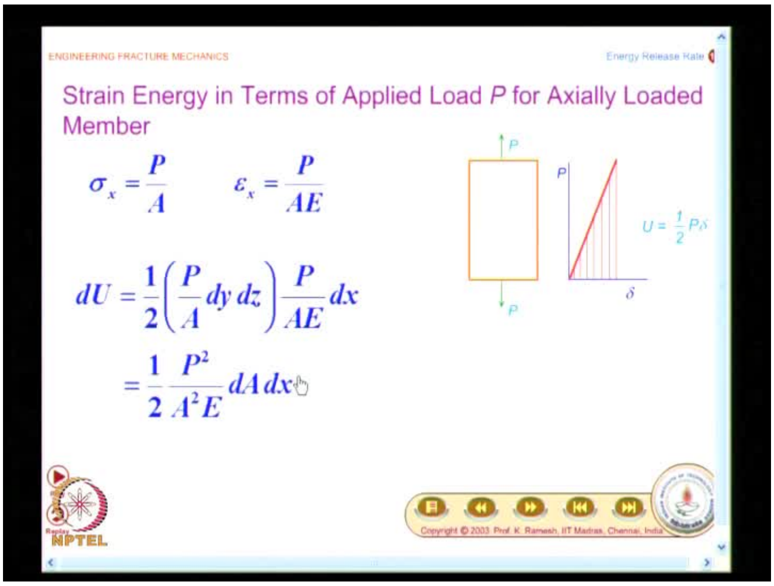

So, these preliminaries are required to do that, although you should have got this knowledge from your earlier course, we are reviewing them for continuity. And we will see for a axially loaded member, our interest is to get the final expression for strain energy in terms of the external load applied P, the area of cross section and young's modulus, this is what we are looking at.

One way of writing is U equal to one half of P into delta, we have written sigma x is nothing but P by A, epsilon x is equal to P by A E, and d U becomes, we have already seen sigma x into epsilon x is the way that we have looked at.

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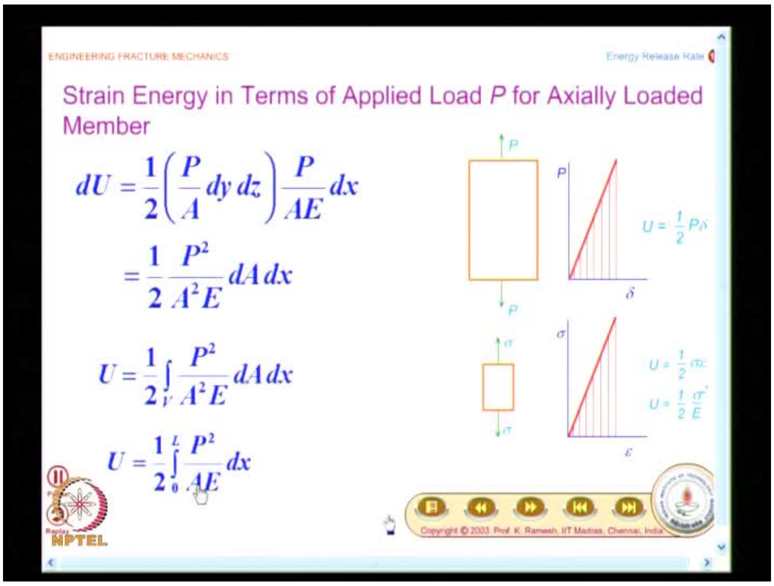

So, if you look at in this fashion, you get this as one half of P squared by A squared E into d A into dx, that gives the volume, and this we usually do it for the slender member. When I do it for the complete energy, this d A becomes when I integrate over the volume, you will have d A becomes A into dx.

So, you will get this as strain energy as one half of 0 to l P squared by A E into dx. So, when you are really looking at slender members, you want to find out energy on that member. So, the distance is accommodated in your dx parameter and this is a very famous expression, you will get similar expressions for bending as well as for torsion. Nevertheless, we will look at them, it is better that you have these equations in your notes and you should also look at one more aspect, what is focused here is, the form of this strain energy. When I have strain energy, it is like one half of sigma squared by E into unit volume, here the unit volume is taken, and so that is why you do not have anything.

So, this is the form of the strain energy expression, because our final interest is to see, what is the kind of strain energy in the presence of a crack in a solid, this is what we want to arrive at. In fact, in this chapter we would go by dimensional analysis or a relaxation analogy, after we develop crack-tip stress and displacement fields, we will come back and derive them based on stress and displacements. In fact, we can avoid much of the mathematics by looking at a relaxation approach.

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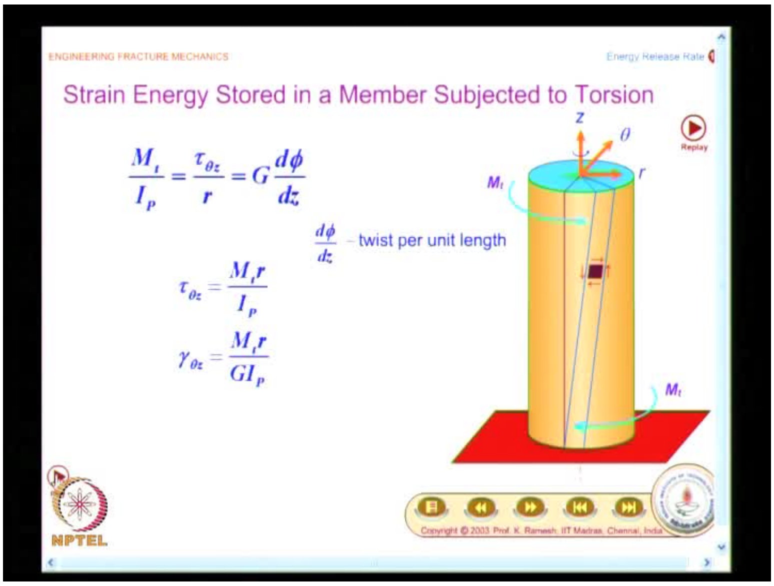

So, that is what we are going to do, so before we go into that, keep in your mind the form of the strain energy stored because of the loading applied to the component, when I have the crack also it will have a similar form. So, that is one way of verification, indeed we are in the right direction. Now, we take up what is the strain energy stored in a member subjected to torsion. We have already developed, when you have stresses and strains, simply a product one half of that into volume gives you the strain energy stored.

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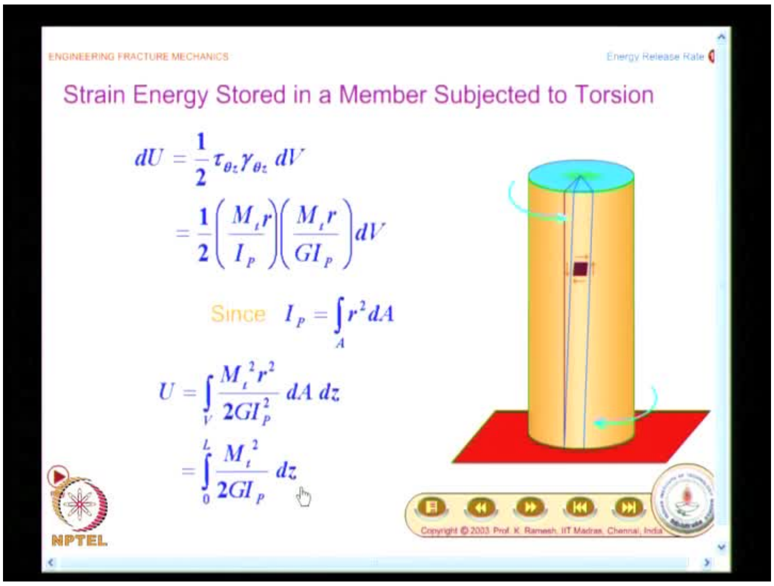

We will adopt the similar approach here, you have a torsion formula. So, you have the expression for shear stress as well as the shear strain, and you get the incremental energy as this, one half of tau theta z gamma theta z dV. And this could be replaced in terms of the torsional moment, apply and you also know the polar moment of inertia is given as integral r square d A. So, this will become I p after integration. So, you get the final expression as integral 0 to l M t squared divided by 2 G I P dz.

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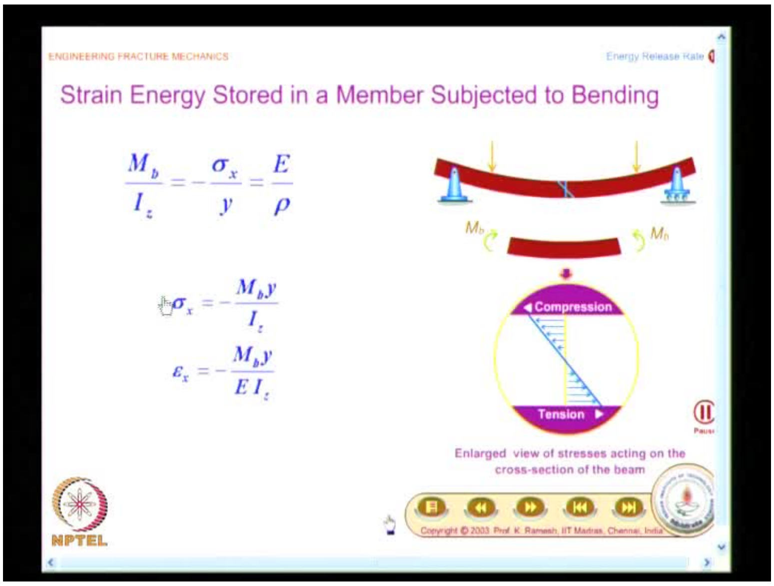

In fact, if you recall the lectures on Castigliano’s theorem to find out deflection, you would have developed these equations and also use them to find out the deflection. It is only recapitulation and this is desirable, now you have the expression for torsion, you will find a very similar expression for bending. Here again, you have the fracture formula, you have the expression for normal stress and normal strain, and it is easy to write the strain energy stored, and here we have M b which denotes the bending moment.

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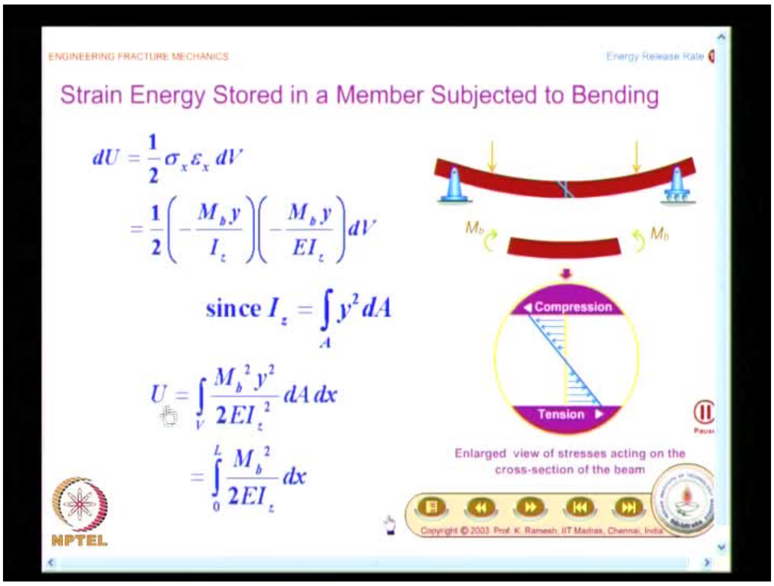

So, what you finally get is, strain energy stored in the member due to bending is integral 0 to l M b squared 2 E I z into dx. We are really talking about slender members, when you say slender member, the cross sectional dimensions are much smaller than the length. In fact, a variety of practical problems could be modeled as slender members to make your life simple, so it is very important from that point of view.

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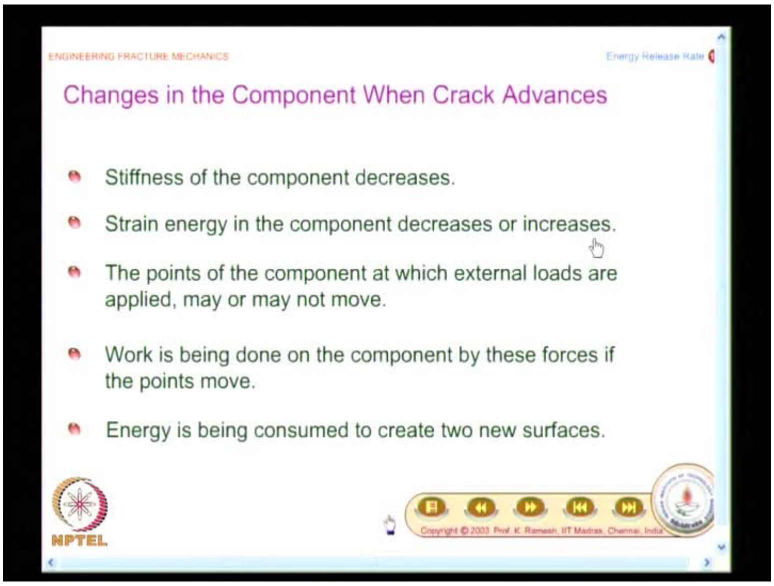

Now, what we will look at is, what are the changes in the component when crack advances? Stiffness of the component decreases, very easy to visualize the moment you have a crack; obviously, the stiffness will change, there is no dispute in that. We will see the next statement, the next statement says, strain energy in the component decreases or increases, this statement needs to be qualified unless you take up an example, look at it. What happens when the crack advances? We would take up situations in a manner that, in one case strain energy increases, in another case strain energy decreases.

So, it is possible, although you may not understand right now, we can take up an example and see. And another minor detail is the points of the component at which external loads are applied may or may not move. The moment you come to fracture mechanics literature, you always talk about fixed grips; that means, we have constant displacement applied to the specimen, this is one possibility, another possibility is constant load, why do we do that? It helps in our mathematical development, if you look at independently constant load and constant displacement, we would develop certain kind of understanding in your theoretical development.

Later on, we will show whatever the energy available for crack formation, in constant displacement or constant loading is one and the same. And finally, we would also show a general loading can be thought of as smaller steps of constant load and constant displacement. This way, we will convince our self, spending time on constant displacement or constant load is good enough. So, we will develop certain concepts and simplify our equations by choosing one of the two, that is the reason why we look at it.

On the one hand, it helps you to simplify the development of mathematics; on the other hand, it also provides the via media to calculate the energy release rate experimental, so it serves both the purposes. So, when you say strain energy in component decreases or increases that are actually dictated by are we having a fixed grip situation or a constant load situation? So, if I have a fixed grip, work done by the external force is 0, work is done only when the forces move, that is what happens in the case of a constant load. And in all this discussions, we have to keep in mind, energy is being consumed to create two new surfaces, this was the key observation of Griffith.

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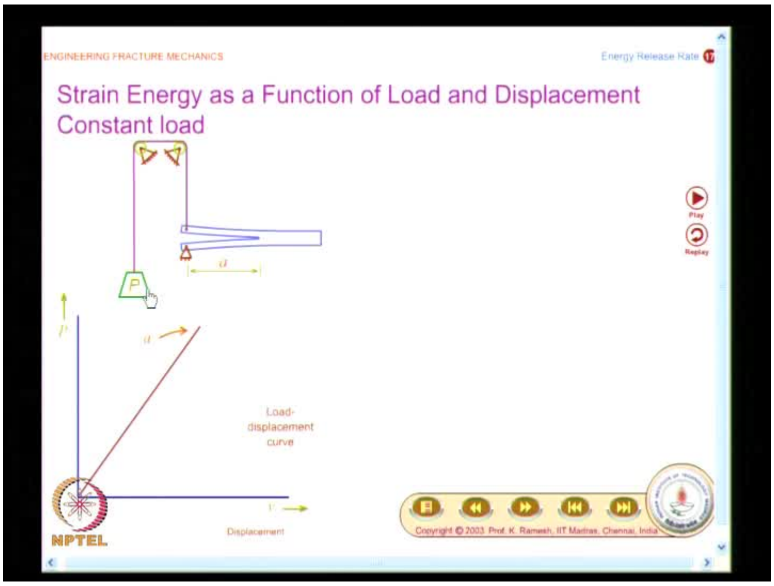

If you directly look at Inglis solution, you will only get alarm and it does not satisfy your actual observation. To bring in this understanding, Griffith was intelligent enough to appreciate that energy is being consumed to create two new surfaces. And we will take up one after another, what happens when I have a constant load.

I have taken a simple example, I would like you to make a neat sketch of it and also plot this graph. What this sketch shows is, you have a double cantle ever beam specimen, you have a long crack, the crack link is given as a, and to make your visualization easier, it clearly shows through a cable system, you are hanging a load P. So, it is very clear that you are having a constant load applied to the component, and what you could do? You could go from 0 to P gradually, and you can record the value of P as well as the displacement.

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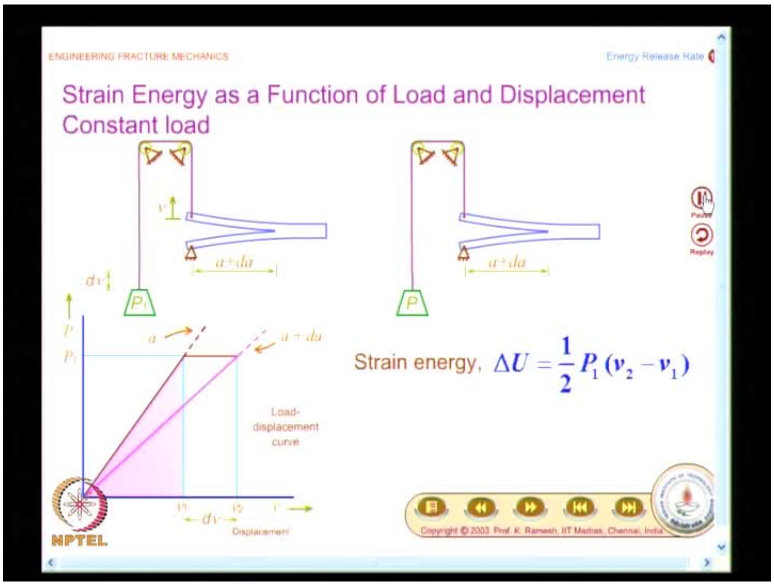

So, you can get a graph like this, and this graph is obtained for the component having a crack of length a. Now, by some means, I have another component which is very similar, where in the crack length is not a, but it is incrementally increased to d a. So, I have a crack of a plus b, and suppose I want to plot the load versus displacement, how do you anticipate the graph to be? I have a longer crack now, so definitely stiffness comes down.

So, the graph will be below the earlier one, the slope will be different, and this also can be recorded, and you have a graph which is for a plus d. Mind you, in this chapter, we want to find out what is the energy required for a crack to grow incrementally, and we are proceeding towards that. For that, we develop so much of background information, before we finally take up the energy balance equation.

So, now I have taken two similar components: one, you have a crack of length a, another you have a crack of length a plus d a and this figures are also drawn very carefully. This load has come down, because the stiffness is lower. Now, what I will do is, I will keep on increasing the load in this gradually, so what will happen is, at a particular load, it so happens that the crack advances by the distance d a.

We would look at it graphically and what our interest is, to find out what is the strain energy stored in the first case as far as the second case. Then we will find out and comment up on what happens to the strain energy, when the crack has advanced by d a, that is what our final aim is. So, what I will do now is, I will increase the load and observe the crack has started to advance when the load has increased to a value P1, and it has moved by a distance dv. Animation is very nicely done, it is as if I am doing an experiment, the animation is shown, so this will go and touch this graph.

Now, I have sufficient data to analyze what is the final strain energy of the system and how the strain energy has changed because of an incremental extension of the crack? You have already seen for the case of a spring, you have seen the load deflection graph and area below the graph is the strain energy stored here again the loads are gradually applied. So, I can use the similar approach and when I want to find out the difference, I will find out the final strain energy minus initial strain energy, we will look at now.

So, what is the change in strain energy is what we want to look at, the final strain energy is the one below this line and you have this triangular area, and that is given by half of P1 into v2. Mind you, we are looking at a case of constant loading, so in the case of a constant load, I have P1 remains constant, so the final strain energy is half of P 1 into v2. What is the initial strain energy? That is below this graph, below this line. It is below this line and you have this triangular area, mathematically it is half of P 1 into v 1. So, what is the change in strain energy? That is this small triangle, what you see here.

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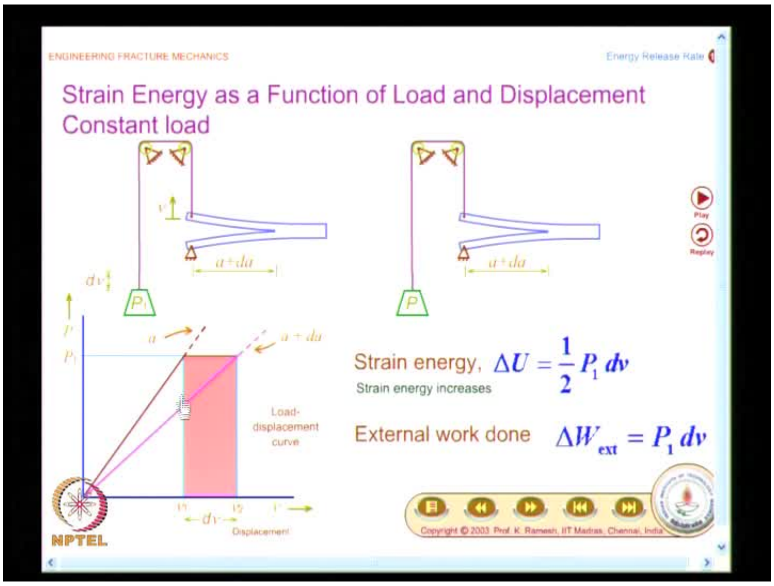

You should be able to identify this very carefully, and this is how experiment is also done. So, you can find out experimentally, what is the energy release rate and what you have here is, I get this as half of P 1 dv, and in this case strain energy eventually increases. It is very obvious, because you have this triangle; this triangle area is much larger than the initial triangle. So, by advancement of crack in a constant load situation, strain energy of the system increases, not only this, you have an external load and the external load has moved by a finite distance, it has moved by a finite distance dv and what is the work done in the process? He said half of w into dv or w into dv, there is a certain difference, all along we have been seeing the load is gradually applied.

But what happens in this is slightly different, what you find is when the load P 1 is reached, suddenly the crack has jump to a plus delta a, but the lower has remains, and same it has had a displacement of dv, so the external work done is P 1 into dv. So, you will have to be very careful about that, this is the rectangular area. We should not make a mistake, everywhere we use half of P 1, you should not say that this is also half of P1; it is actually P1 into dv. And what we have learnt here is, under constant load when there is an advancement of crack from a to a plus d a, the strain energy of the system increases.

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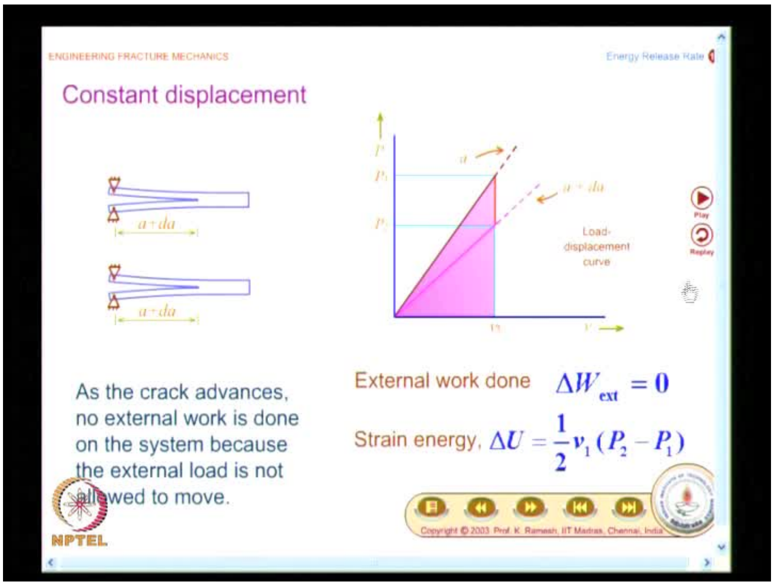

Now, we go and look at what happens in the case of a constant displacement. So, here again I take a problem, which was very similar to the earlier one, here it is put under fixed grips and when you get the P versus v, I will get a graph like this. Suppose, the crack is longer than that, I would have a graph which is below this, this is the load displacement graph for a plus d a.

Now, what you observe is, the load is sufficient for the crack to grow from a to a plus d a, when this happens, what will happen? The load will eventually decrease when the crack has advanced, because it is under constant displacement, it is fixed grip, you have understand this. So, what you will have is, I will move down from this graph to this vertically, just observe the animation, the crack is going to a plus d a, when crack is going to a plus d a, I will have to move from this graph to this. And that is why this is put as dotted line, why this is put as dotted line is, if I take a specimen and then have this load as P, the moment I reach the load P, the crack would advance and I have to interrupt only from the next graph.

Make a sketch of this also, here again we will find out what way the strain energy changes. So, we will look at what is the final strain energy, we look at the initial strain energy, and comment what happens to the strain energy in this case. And before that, you will have to make one important observation: as the crack advances, no external work is done on the system because the external load is not allowed to move. It is a very important observation, we are looking things in detail, so there is no external work done in this case.

Obviously, you can look at the final triangle is this, initial triangle is much larger, so what you find is that the strain energy decreases in this case. First one is external work done is 0, strain energy changes, half of v 1 into P2 is the final one minus initial one, half of v 1 into P1 in this case, it is happening at a constant displacement.

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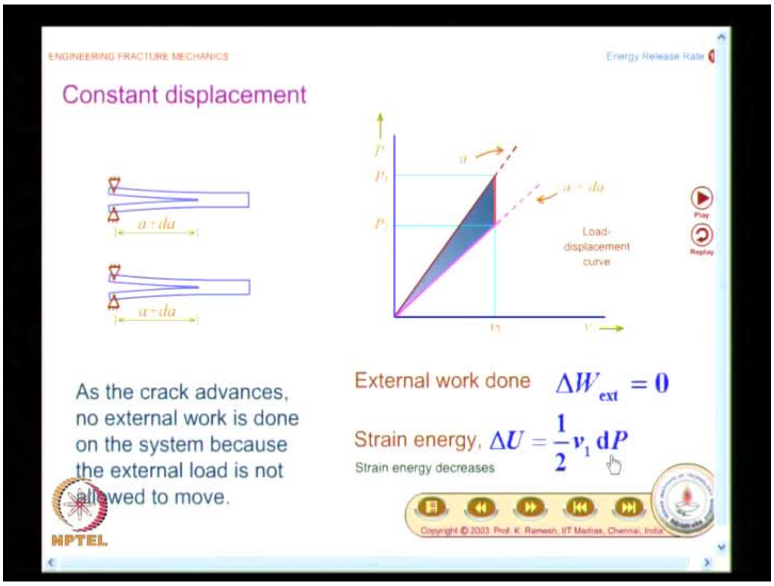

So, v remains same for the final as well as initial cases and what is the difference in strain energy? That is nothing, but this strain, and you have to make a note that strain energy decreases and the change in strain energy is nothing, but half of v 1 into d P. See, in the first case where we saw constant load, there was a change in the displacement. In the second case, we were looking at constant displacement; there was a change in the load. In one case, strain energy increased because of crack advancement, in another case strain energy has decreased because of crack advancement.

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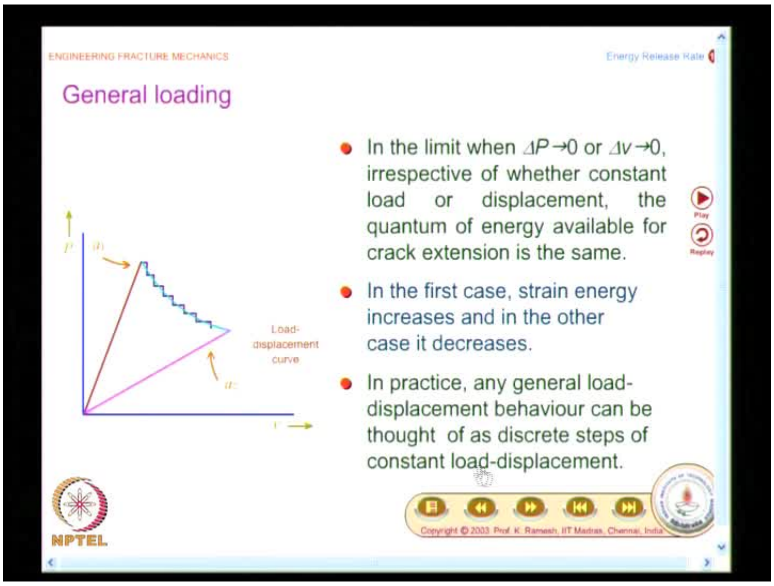

So, this is a statement I made initially, at that time you do not have a background to appreciate whether such a thing can happen. Now, you have seen when you look at the situation under constant load and constant displacement, the strain energy changes or differ. And now, we will look at for a general loading, we are not fixing whether the load change is happening in a constant displacement or constant load. So, what you will find is, the difference is, in one case you have the complete triangle as that strain energy difference in another case, it is a truncated triangle like this.

So, I will have an extra red portion like this and here only mathematics comes to your rescue in appreciating what kind of processors that we are looking at, and the animation is also very well done to give you this visual experience. For understanding, we show these incremental changes as large, for us to draw the diagram comfortably, in the limit what we want, delta P tends to 0 or delta v tends to 0 irrespective of whether constant load or displacement, the quantum of energy available for crack extension is the same from a mathematics point of view, what will happen?

You will find dv is small, d P is tending to 0 and d a is also tending to 0, what will happen to this triangle? This triangle will keep on shrinking. So, from this observation it is possible to make a statement, the quantum of energy available for crack extension is the same for both constant load and constant displacement, have a look at it, the animation will give you that physical observation.

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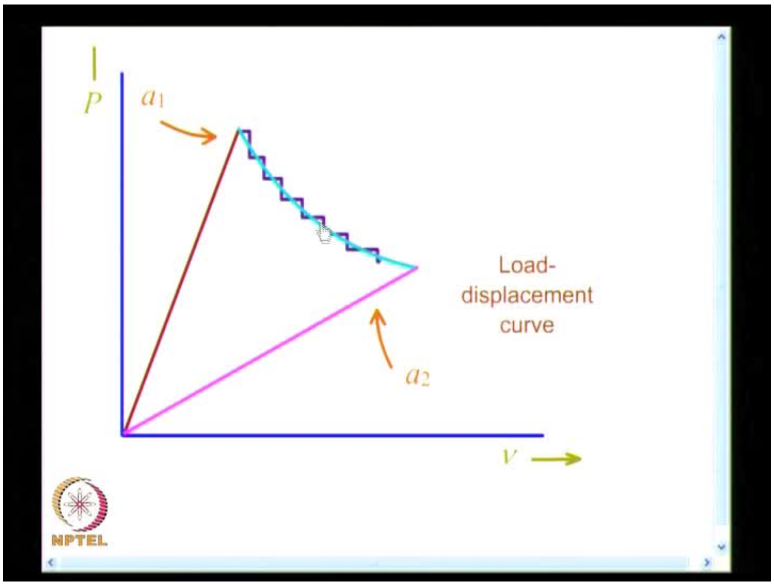

So, you could see, do you see the triangle at all in this case? You do not see the triangle at all in this case. So, you have to visualize it that way that is the only way you can satisfy yourself. And this animation is nicely done to give you that visual appreciation and that is what is summarized here. In practice, any general load displacement behavior can be thought of as discrete steps of constant load and displacement, that is what is shown here, may be I could magnify it and then show.

So, what I have is, the load and displacement may vary in a generic fashion like this. This could be thought of as steps of constant load, constant displacement, constant load, and constant displacement so on and so forth. So, what we have seen in this exercise is, the quantum of energy available is same in both the cases of constant load and constant displacement. In one case, the strain energy increases by crack advancement and in another case, strain energy decreases by crack advancement.

So, in this class, we started of with Inglis solution, just for continuity say, which reminded as stresses become very high when you have a crack and in order to explain physical observation, Griffith came out with the brilliant idea, that you need energy for the formation of new surfaces. It is a very key step that Griffith has taken and this is termed as a surface energy, but the surface energy was very small. If you do not conceptually look at the problem, you would normally take that as a second order effect and ignore it, this is what engineers will do. When you compare the plastic energy that is required, it is very high, several orders of magnitude than the surface energy.

Griffith developed the theory based on brittle materials, so he was very intelligent to identify the role of surface energy and this was a very important conceptual step. So, the goal in this chapter is to find out, what is the kind of energy required for advancement of a crack. In order to do that, we have reviewed various expressions that we have learned in a general course and mechanics of solids to find out the elastic strain energy.

Then we moved on to analyzing a crack body under constant load and constant displacement. In one case, we found strain energy increased, in another case, strain energy decreased. However, we were able to show when the incremental changes are small, the energy availability is same in both the cases. We will see further advancements in next class.