Advanced Operations Research

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Lecture No. # 39

Benders Partitioning Algorithm

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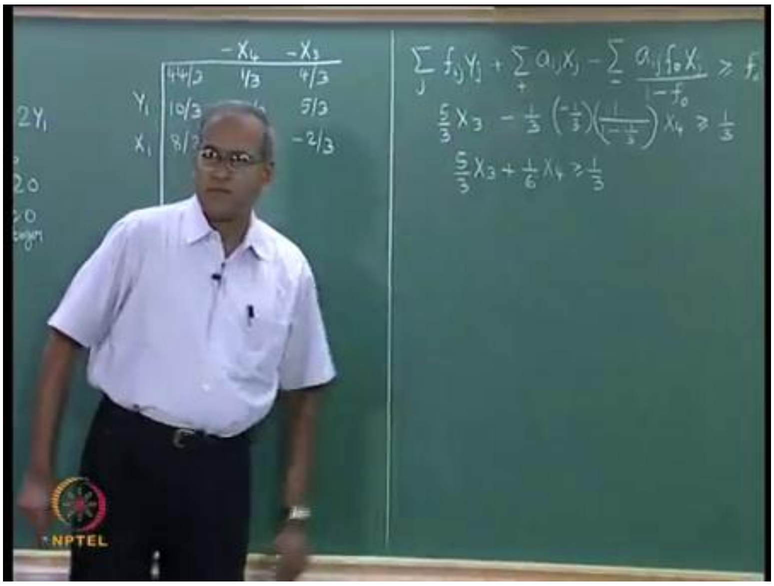

We continue the discussion on the MILP cut. In the last lecture, we derived the expression for the MILP cut, and the MILP cut looks like this.

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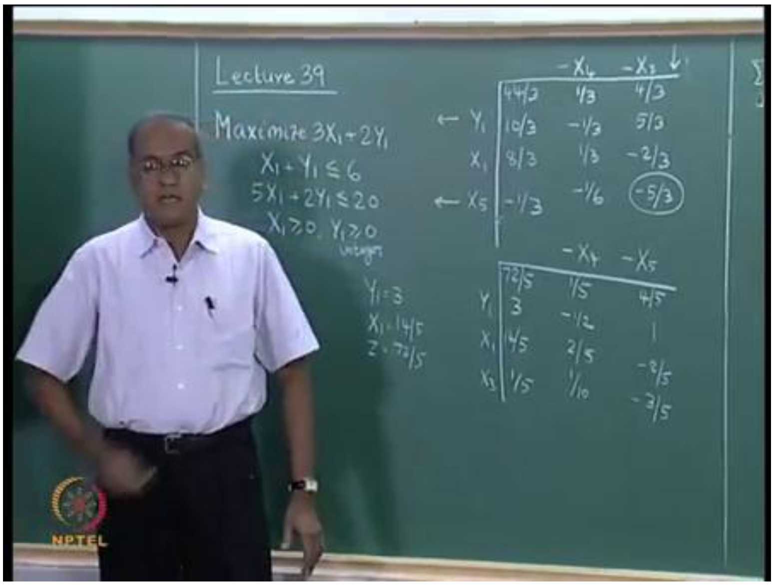

So, we go to the familiar example. The example is written here again, and the LP optimum is shown here. In the example problem Y 1 is greater than or equal to 0 and integer, so LP optimum is giving as Y 1 equal to 10 by 3 which is not an integer value. So, the Y 1 row will act as a source row to generate the MILP cut, and from here we use this MILP cut equation; f naught is the fractional portion of the right hand side value. So, 10 by 3 is 3 plus 1 by 3; so, f naught takes value 1 by 3. Now, we realize that there are no non basic Y j's in this table. Therefore, this term does not figure in our MILP cut for this example.

So, the f i j Y j terms do not exist. So, we have look at only these two terms; now this is the term that has a positive coefficient for X j, the other one is the term that has negative coefficient for X j. So, the one with a positive coefficient remains as 5 by 3 X 3, one with a negative coefficient becomes minus a i j f naught is 1 by 3, a i j is minus 1 by 3 by 1 minus f naught 1 minus 1 by 3 which is 2 by 3. So, this on simplification would give as 5 by 3 X 3 plus 1 by 6 X 4 is greater than or equal to 1 by 3. So, this is the MILP cut; the presence of this negative number negative coefficient here, has now change this 1 by 3 to 1 by 6, because you have f naught by 1 minus f naught which is a multiplication factor.

So, we include this into the LP optimum as a MILP cut; so, we introduce a new variable X 5 which comes as a result of the MILP cut. This X 5 is the surplus variable which is added to convert the greater than or equal to inequality to an equation. So, this is like minus X 5 is equal to 1 by 3. So, take X 5 to one side, and 1 by 3 to the other side; so, X 5 will become minus 1 by 3 plus 5 by 3 X 3. So, plus 5 by 3 X 3 there is a minus already there, so plus 5 by 3 X 3 plus 1 by 6 X 4 would give as minus 1 by 6 X 4. So, now we have to do a dual simplex iteration.

So, this is the variable that will go out, which has a negative value for the right hand side. The entering variable is found based on a minimum ratio rule; so, leave out the negative portion 1 by 3 divided by 1 by 6 is 2, 4 by 3 divided by 5 by 3 is 4 by 5, 4 by 5 is smaller so variable X 3 enters the basis. So, we do one simplex iteration with this; so, X 5 becomes non basic. So, X 3 replaces X 5 in the basis. So, we have Y 1, X 1, and X 3, this is our pivot.

So, pivot element becomes 1 by pivot; so this becomes minus 3 by 5 divide every element of the pivot row by the pivot. So, you get plus 1 by 5, you get plus minus 1 by 6 divided by 5 by 3 is 1 by 6 into 3 by 5 which is 1 by 10 divide by the negative of the pivot. So, 4 by 5; 1 and minus 2 by 5 negative of the pivot minus 2 by 3 divided by 5 by 3 is minus 2 by 5. Now, we need to find out this value; so 44 by 3 minus 4 by 15.

So, this is 220 minus 4 is 216 by 15 which is 72 by 5; 10 by 3 minus 1 by 3 is 9 by 3 which is 3; 8 by 3 plus 2 by 3 into 1 by 5, 8 by 3 plus 2 by 15 is 42 by 15 which is 14 by 3. Now, this value is 1 by 3 minus 4 by 30 or 1 by 3 minus 2 by 15, 3 by 15 which is 1 by 5 minus 1 by 3 minus 1 by 6 which is minus half minus 1 by 3 minus 5 by 3 into 1 by 10. So, minus 1 by 3 minus 1 by 6 which is minus half. So, 1 by 3 plus 2 by 30 which is 12 by 30, which is 12 by 5.

So, now we look at this solution, and we realize that X 1, Y 1 which should take integer value has taken the value 3. So, this is optimal to the MILP with the solution Y 1 equal to 3, X 1 equal to 14 by 3, and Z is equal to… Let me just check this 14 by 3 again, so this value here will be 8 by 3 plus 2 by 15 which is 42 by 15; so, this is 14 by 5. So, that 3 X 1 plus 2 Y 1 is 42 by 5 plus 6 is 72 by 5. So, the optimum solution is given by Y 1 equal to 3, X 1 equal to 14 by 5, and Z equal to 72 by 5. This is the same solution that we obtain, when we solve the problem using the branch and bound algorithm.

Now, we can either use the MILP cut, the cutting plane algorithm or the branch and bound algorithm to solve this. Usually branch and bound is preferred, because when compare to a branch and bound for all integer; the branch and bound for MILP uses fewer nodes, because fewer variables alone or restricted to be integers. Whereas, in a all integer problem, we should have all variables take integer value at the optimum. Therefore, the number of nodes, when you compare with an equivalent all integer problem; the number of nodes will be lesser, when you solve the MILP. On the other hand, if we take a closer look at the MILP cut, at some point we realize that we had some quantity greater than equal to f naught. And some other quantity was also greater than or equal to f naught, and then we added both of them and created this cut. We see carefully, we had this portion which was greater than or equal to f naught, and this portion is greater than or equal to f naught. We added both of them, and set that we would have a cut greater or than or equal to f naught.

Now, we also realize the because of this the MILP cut is the little weak, and there can be instances where, we need more MILP cuts until we get to the optimum solution. In this particular example, just one MILP cut is enough to try and give as the optimum solution but, there can be situations where we need more cuts. So, keeping all those things in mind one would prefer using a branch and bound algorithm to solve an MILP rather than MILP cut; it also possible to look at better cuts then what we have seen right now. In the sense cuts which are much tighter, and much better than this cut which could help as get the optimum solution in fewer iterations.

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Now, let us look at another method to solve the MILP problem, where we essentially try to partition the variables into two sets; those variables that should take integer values, and those variables that can take continuous values. In fact our very idea if introducing different notation using X for those variables that take can take continuous values, and using Y for those variables that take integer values is to partition them at some point into two sets of variables.

So, we look at some kind of a partitioning algorithm, where we partition the variables into two sets, and then we try and solve this problem.

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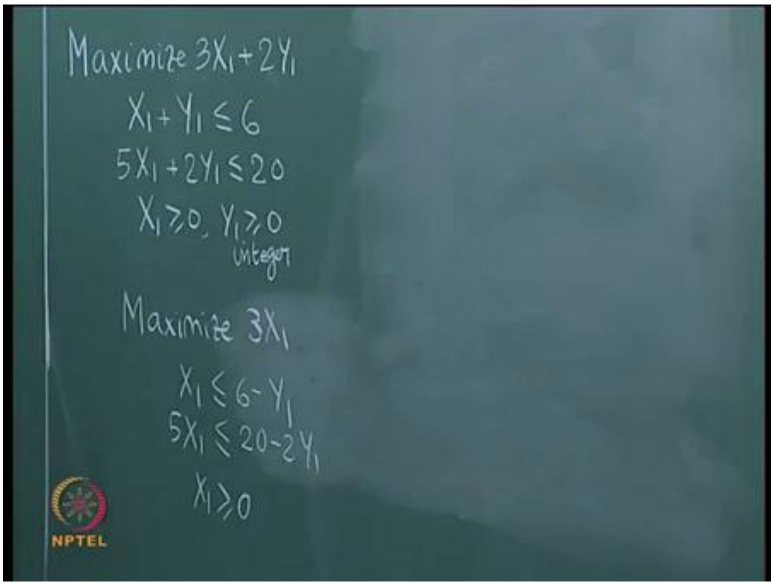

So, let us look at this problem again, and write this as maximize 3 X 1 subject to X 1 less than or equal to 6 minus Y 1, and 5 X 1 less than or equal to 20 minus 2 Y 1, X 1 greater than or equal to 0. So, what we have done here is, we have now written the problem in terms of X assuming that X are unknown; so maximize 3 X 1, I have right now left out the 2 Y 1, we will see Y; X 1 is written as 6 minus Y 1, and 5 X 1 is written as 20 minus Y. Now, X 1 greater than or equal to 0.

So, if you assume that we know the value of Y, there is only one variable Y which you can call as Y or Y 1 in this example. If you know the value of this Y 1, then X 1 is 6 minus Y 1, 5 X 1 is less than equal to 20 minus 2 Y 1, and if Y 1 is known, this 2 Y 1 becomes a constant. And therefore, this is 3 X 1 plus constant. And write now, we can ignore a constant in a linear programming problem. So, for known values of Y 1 this becomes a LP.

So, if the values of Y 1 are known, then for known values of Y 1 we can solve this to get the value of X 1. In our example, there is only one X variable, and one Y variable; in another example involving more number of X, and Y variables. Then once the values of all the Y variables are known, the X variables can be found out by solving this linear programming problem, this is the LP.

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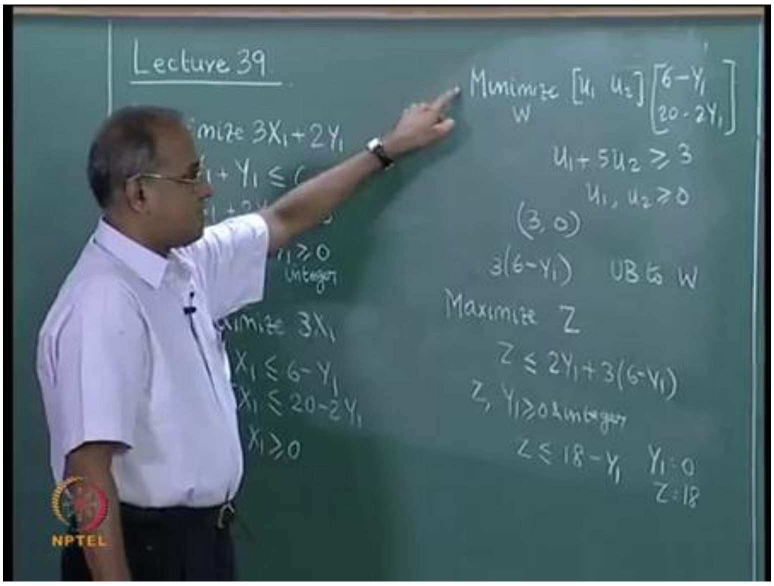

Now, let us write the dual of this LP. So, the dual of this LP will be to minimize, the dual will be to minimize; so, we introduce two variables which we call as u 1 and u 2. So, the dual will be to minimize u 1, u 2 into 6 minus Y 1, and 20 minus 2 Y 1; such that u 1 plus 5 u 2 is greater than or equal to 3; u 1, u 2 greater than or equal to 0.

So, this is the dual, at present we may tend to think that there is some non-linearity, but actually there is no non-linearity, because this dual is written under the assumption that Y 1 values are known. So, the moment Y 1 values are known this becomes a linear term involving u 1 and u 2; the constraints are anyway linear, there is only one constraint, because a primal has only one variable. There is only one constraint that we have. Now, at the moment we do not know the values of Y 1; therefore, we let us say we are unable to find the optimum solution to this dual. If the values of Y 1 are known, then we can solve the LP to get the optimum solution to this dual. Since, we do not know the Y 1 at the moment, we take only a feasible solution to the dual - and a feasible solution to the dual is given by (3,0), because this constraint would give as (3,0) as a feasible solution to the dual.

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So, now since (3,0) is a feasible solution to the dual, then 3 into 6 minus Y 1 plus 0 into 20 minus 2 Y 1 is a value of the objective function of this for a known feasible solution (3,0). I am not writing this 0 into 20 minus 2 Y 1, because it is 0; so 3 into 6 minus Y 1 is the value of the objective function corresponding to the solution (3,0), and (3,0) is feasible. Therefore, 3 into 6 minus Y 1 is the objective function value of a feasible solution to the minimization problem, and because it is an objective function value of a feasible solution to a minimization problem.

It is an upper bound to the optimum solution to this, because any minimization problem the objective function value of a feasible solution is greater than or equal to that of the optimum. Optimum is a smallest value that you can think of; therefore, 3 into 6 minus Y 1 is an upper bound to W, if we called this as W. Now, by weak duality theorem, we know that the objective function value of every feasible solution to the dual is greater than or equal to the objective function value of every feasible solution to the primal. So, if Z is the objective function value here, then Z is less than or equal to 3 into 6 minus Y 1 by weak duality theorem.

So, we go back and re-write this problem, and this problem now becomes maximize Z, now this Z has two components 3 X 1 plus 2 Y 1. The 3X 1 components has been written here, and this as gone. So, 3 into 6 minus Y 1 is an upper bound to 3 X 1; therefore, Z is less than or equal to 2 Y 1 plus 3 into 6 minus Y. Note that there are two components to this, is called as Z, now so there is a 3 X 1 component, there is a 2Y 1 component, when we wrote this we assume that Y 1 is known. And therefore, this becomes a constant.

So, we wrote only the 3 X 1, and then we wrote its do well, and we set that 3 into 6 minus Y 1 is an upper bound on this value, which imply it is on upper bound on this value. Therefore, Z should be less than or equal to 3 into 6 minus Y 1 plus 2 Y 1, because Y 1 greater than or equal to 0 and integer. What is Z? Z is the objective function value here 3 X 1 plus 2 Y 1, now Z need not be an integer, because 2Y 1 is an integer, 3X 1 need not be an integer.

So, in principle we should write Z greater than or equal to 0; Z is a continuous variable, but then if we want to find out the value of Y 1 which maximizes Z, such that Z is less than equal to 2 Y 1 plus 3 into 6 minus Y 1, that Z greater than or equal to 0, Y 1 greater than equal to 0 and integer. This problem is a MILP, where Y is integer valued, Z is continuous.

So, we start by solving on MILP, where X are continuous, Y are integer; and then somewhere inside we get a sub problem which is also an MILP. So, we solve that by saying that we are now going to force Z also to be integer, and then say that now this problem becomes on ILP. And this is not a MILP; so instead of solving an MILP with in an MILP, you actually end of solving an ILP integer linear programming problem within a mixed integer linear programming. So, let us try and solve this; so this is will reduce to Z less than or equal to 18 minus Y 1; Y Y is integer value, so the obvious optimum solution is Y 1 equal to 0, and Z equal to 18, because you have Z less than or equal to 18 minus Y 1. So, the optimum value will be at Y 1 equal to 0, and Z equal to 18.

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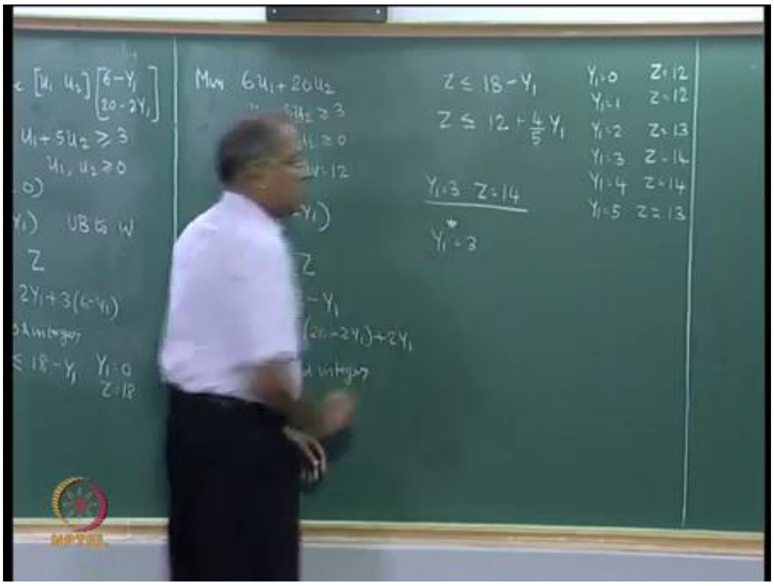

Now, we go back and substitute here into this dual, the known value of Y 1 which is Y 1 equal to 0. We go back, and substitute to get minimize 6 u 1 plus 20 u 2 subject to u1 plus 5 u 2 greater than or equal to 3; u 1, u 2 greater than or equal to 0. Now, you get your first known value of Y 1, in the earlier case we did not get known values of Y 1; therefore, we picked on upper bound here, now we have a way that we know Y 1.

So, substitute that Y 1 back to get 6 u 1 plus 20 u 2, because Y 1 is 0; so, when you substitute Y 1 equal to 0 you get this therefore, you have. Now, you have to solve this linear programming problem, which can be solved easily; write now, it is a single constraint. So, when you put u 1 equal to 3 you get 18, when you put u 2 equal 3 by 5, you get 20 into 3 by 5 which is 12.

So, the optimum solution is u 2 is equal to 3 by 5 with W equal to 12. Now, go back and check what happens you have a solution here with W equal to 12, you have Y equal to 0;

so the solution from the dual side is 12 plus 0 which is 12, the solution from this side is 18.

So, the solution here is bigger than the solution there. Which simply means that this actually has to come down; the solution that we get here which is 2 Y 1 plus 3 X 1 from this side is 18 write now. The solution from the other side is actually 12 from the dual side plus 2 Y 1 which is 0, so the solution there is 12; automatically, if you have an optimum solution to a set of LPs, after all we have converted an LP here, and we have written the dual there.

So, when we have that we at some point at the optimum the objective function values of the primal, and the dual will have to be equal. So, right now this 18 is higher than 12 plus 2 Y 1; therefore, we proceed further to see what we can. So, now what happens to this, now when u 2 is equal to 3 by 5; the objective function value here that you have for 3 by 5 is 3 by 5 into… So, this is 0 into 6 minus Y 1 plus 3 by 5 into 20 minus 2 Y 1, that is your objective function value that you get. Now, ideally this should be an upper bound to this; therefore, we put a constraint making this as an upper bound.

So, that this value comes down. So, now you write maximize Z, we already have the first upper bound for Z, that is comes from Z less than or equal to 2 Y 1 plus 18 minus 3 Y 1 which is 18 minus Y 1, which is the first upper bound. And your second upper bound that comes, because one second this is a feasible solution to the dual. So, this has to be an upper bound; therefore, this will give you Z less than or equal to 3 by 5 into 20 minus 2 Y 1 plus 2 Y 1; the plus 2 Y 1 comes from here, because all these the dual is written for the linear programming problem that involves only 3 X 1. So, this is an upper estimate of the 3 X 1; whereas, the actual Z is 3 X 1 plus 2 Y 1, therefore we write 3 by 5 into 20 minus 2 Y 1 plus 2 Y 1. And then, you write Y 1 greater than or equal to 0 and integer.

And then again we go back to the same conflict, that actually Z need not be integer; Z can be continuous, but then you get a MILP within an MILP which we do not want. So, we constraint Z also to be an ILP, and then we write this. So, now we have to solve this, before we solve this simplify this first.

So, we write Z less than or equal to 18 minus Y 1; Z less than or equal to… Now, this becomes 12, 3 into 20, 60 by 5 is 12; this is plus 2 Y 1 minus 6 by 5 Y 1 plus 2 Y 1 minus 6 by 5 Y 1. So, that is 4 by 5 Y. So, 12 plus 4 by 5 Y. So, now we want to find out the optimum value of Y 1, actually we can solve this ILP by either using a cutting plane algorithm or any ILP algorithm, but since we are having a very small problem, we try a use substitution to get the best value of Y 1.

So, you start with Y 1 equal to 0. so Y 1 equal to 0 will give you Z less than equal to 18, Z less than equal to 12; so Z will be equal to 12. Y 1 equal to 1 will give you 17 here, this is 12 plus 4 by 5 into 1, 12 one 4 by 5, Z is an integer. So, Y 1 equal to 1 will also give Z is equal to 12. Y 1 equal to 2 will give 16 here, this is 8 by 5; so Y 1 equal to 2 will give Z equal to 13, because this is 18 minus 2, 16; this is 12 plus 8 by 5 lower integer value. Therefore, 8 by 5 is lower integer value is 1, so you get Z equal to 13. Now, Y 1 equal to 3, Z is equal to 18 minus 3 is 15; this is 12 by 5 which is 2, so Z will be equal to 14. Now, Y 1 equal to 4; this is 14, this is 16 by 5. So, 3 plus something 4 into 4 16 by 5 is 3.

So, Y 1 equal to 4 also gives Z equal to 14, so Y 1 equal to 5 will give you, this will give you 13, this will give you 25 by 5 which is 17. So, Z equal to 13. So, we get solution Y 1 equal to 3 or Y 1 equal to 4 which are optimal to this. So, let us take Y 1 equal to 3 first, and Z is equal to from here, Z is equal to 14. So, we take Y 1 equal to 3, and Z is equal to 14 from this one. Now, we go back and check now what happens, here the value of Z is 14. Here the objective function value is actually 3 by 5 into 20 minus 2 Y 1; so 20 minus 2 Y 1 is 42 by 5; Y 1 equal to 3.

So, this is 14 into 3 by 5, 42 by 5 plus 2 Y 1, 42 by 5 plus 6 is 72 by 5 which is greater than 14; therefore, you have now reach the optimum with 14, because this the value as sufficiently come down. The dual is giving you a solution of 72 by 5, the primal is giving you a solution of 14, your duality theorems are satisfied; the only reason why there are not equal is the fact that you have restricted this to be an integer.

Now, this value as come down, and has become less than that of the corresponding dual computation, so you will go back and say that Y 1 star equal to 3 is optimum to the IP or to the IP portion of the problem. And once you substitute known value, then you come to this, now you are maximizing 3 X 1 subject to X 1 less than or equal to 3, and 5 X 1 less than or equal to 14. So, out of this X 1 will become 14 by 5, because X 1 less than equal to 3 X 1 less than equal to 14 by 5.

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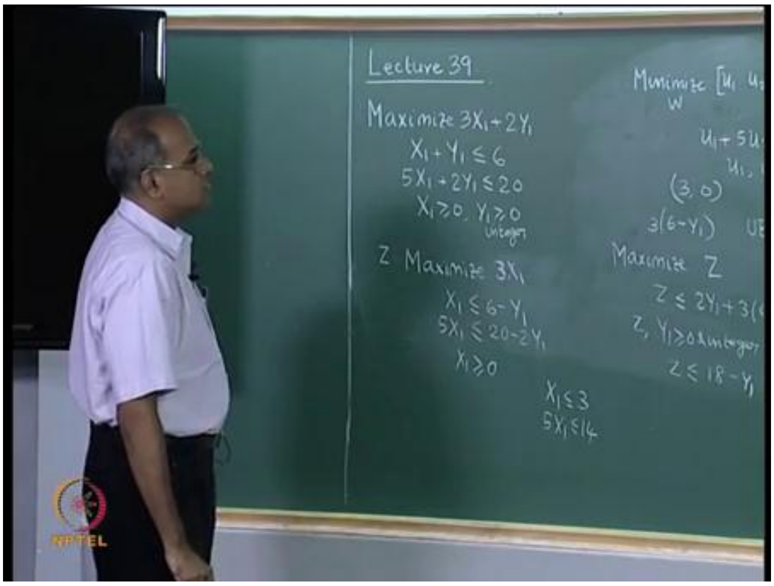

So, 14 by 5 into 3 is 42 by 5. So, Y 1 star equal to 3 X 1 star is equal to 14 by 5, and Z is equal to 72 by 5 is one answer. (No audio from 31:35 to 32:01) Now, you go back now you one can always go back, and say that we got another solution with Y 1 equal to 4 which also gave as the value of 14 there; so when you substitute Y 1 equal to 4 there, you get a solution 20 minus 8 which is 12 into 3 by 5 is 36 by 5, because we when we substituted Y 1 equal to 4, we got 18 minus Y 1 as 14, 4 into 4 16 by 5 gave as 3. So, 12 plus 3, 15; so less than equal to 14, less than equal to 15; so, we got another solution with 14. So, let us go back, and see what happens to that solution.

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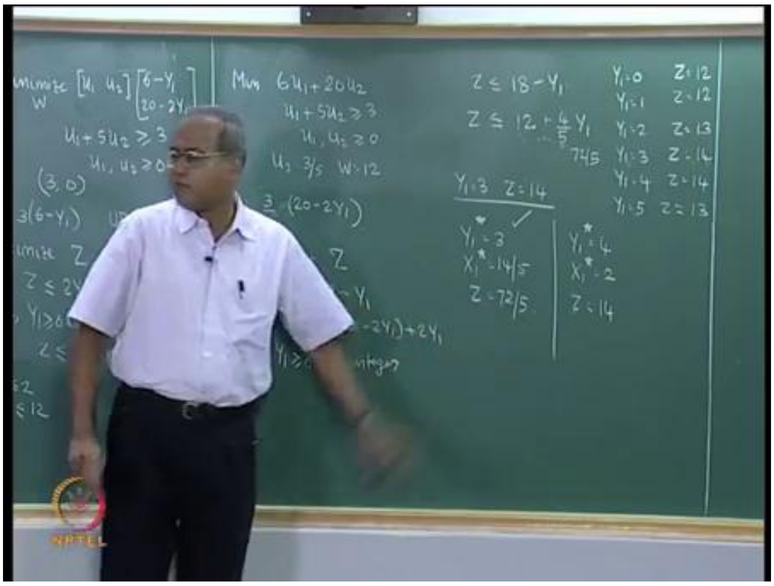

So, when you put Y 1 star equal to 4; now the dual value will become 20 minus 8 which is 12, into 3 by 5 36 by 5, 36 by 5 plus 8 which is 76 by 5. So, dual value was 76 by 5 here, primal value was 14. So, one second it was less; so, now you go back and substitute for Y 1 equal to 4, you would get X 1 less than or equal to 2. And it would give you 5 X 1 less than or equal to 12 from which X 1 will take value two, the smaller of that; so this will give you X 1 star is equal to 2, and Z is equal to 3 X 1 plus 2 Y 1 which is 14.

So, you would go back, and choose this solution, I had of the other one, because this solution has a higher value; so maximization. The other one happened, because in some sense the error that you have between this two, this is 14 and 2 by 5, and this is 14; both of them have lower integer component of 14, and because we actually solve the ILP instead of MILP, we got this. If we go back, and actually calculated the Z value here for a corresponding MILP, then this would have become 14 and 2 by 5, whereas this would become only 14.

We can even see that see, when Y 1 equal to 3, you got 15 here, but then we got here 12 plus 12 by 5, 4 into 3 by 5 12 by 5. So, we got 72 by 5; whose lower integer value was 14, but we could have got 72 by 5. Here we could have got 72 by 5 instead of 14 when we actually substituted and not treated Z as integer. Whereas, when you did Y 1 equal to 4, we would get 14 exactly from here, we would get 15 plus something from here. 15 and 1 by 5 which is 76 by 5, but then you realize the 14 dominates, and you would get only a 14; which is the 14 that you got here.

So, if you had carefully solved the MILP component write here, instead of an ILP for Z or instead of saying Z is integer; if we at Z is continuous. Now, 3 would have given as 72 by 5, and 4 would have given as 14. So, we would not have looked at this, we would have looked only at this; and this is the optimal solution, because we restricted Z to integers, we have to look at both, and then at the end go back. And say this is the optimum solution, because this has a higher value of the objective function to the maximization problem.

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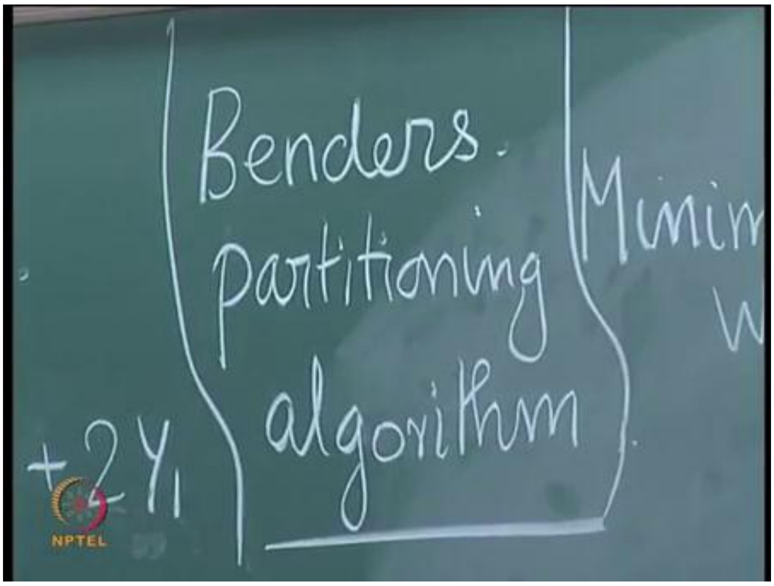

So, this is how we use a partitioning algorithm to solve this, and this partitioning algorithm is called the Benders partitioning algorithm.

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Called the Benders partitioning algorithm which can be used to solve MILP. Now, the benders partitioning algorithm has to be suitably interpreted and solved; if you are doing a minimization problem. What we have seen is a version for a maximization problem, because if it is a minimization problem, we would start with a minimization primal leave out the integer component, then the dual will be a maximization. And when we apply the week duality theorem, we have to carefully apply it that the week duality theorem is very clear that, every feasible solution to a minimization problem is greater than or equal to that of every feasible solution to a maximization problem. So, instead of solving maximize Z; Z less than equal to something, it will become a lower bound; so the problem will become minimize Z; Z greater than or equal to something.

So, this kind of a primal will become minimize Z; Z greater than or equal to something, and because of that here we first had larger values, and it was coming down to smaller values. There when you solve minimize Z; Z greater than equal to something, you will first have a smaller value, then tending to larger value. So, when we compare the primal and dual, we have to actually understand which is the maximization problem, which is the minimization problem. And at the end we have reached optimality, when either both of them are equal or the maximization problem shows as slight increase over the minimization. In this case, it the maximization showed a the minimization showed a slight increase over the maximization, because of the rounding .

Because of the fact that Z is taken as a ILP instead of an MILP. So, we need to carefully look at that the basic idea is essentially the same, but if the given problem is a minimization the dual will become a maximization, this problem will become a maximization. And then we will go back, and solve minimization problems here; and then we have carefully compared the solution of this minimization verses the solution Z we get from this maximization. And then understand at what point the optimality has been reached.

So, constraints will be added to this; just like we have two constraints here; there could be another Z greater than equal to constraint in the other case, and so on. The other difference is as we iterate we realize that this particular problem is getting bigger and bigger. Initially, we started with a single constraint and then it became one with 2 constraints, and then if there is a third iteration it would become one with a third constraint, and so on. So, solving this problem this is a we solve this ILP, because our example is a very small example with one integer variable, and one continuous variable. One could follow this approach to try and get to the optimal solution of this ILP, because irrespective of the number of constraints its essentially a single variable optimization, because a maximizing Z; Z less than. If you had multiple Y's, then this approach will not work, and one has to solve the ILP using on ILP algorithm or using a solver. So, this portion is actually a difficult portion here.

In a similar manner all these single constrained LP so I one could look at the constraint and then say this is the optimal solution. Whereas if, there were multiple constraint, now when will we have two variables for u here, because we have two constraints here for X 1. Now, if there been another X 2 variable here, then that would have put on one more constraint.

So, the more number of X variables here, the more constraints we will have in the dual. And so that l p has to be solved as a separate sub problem to try and get the optimal solution. So, the method invariably is not very amenable to a hand computation, because this sub problems themselves are l p problems and ILP problems, which have to be solved separately. In a very small example like this it is possible to show, because the corresponding lp is an ILP are now solved by inspection.

Simply, because we have one variable and one constraint to do, so when you solve a larger MILP, then this one is not amenable for a hand calculation, but this is quite efficient when it comes to solving it using computer programs. Reason being if the problem is can be solved within very few additions of constraints here, we will get to the optimum solution. In this case there, where two; if the problem is nice, and if you are little lucky; then within 2 or 3 constraints here, we will able to get the optimal solution.

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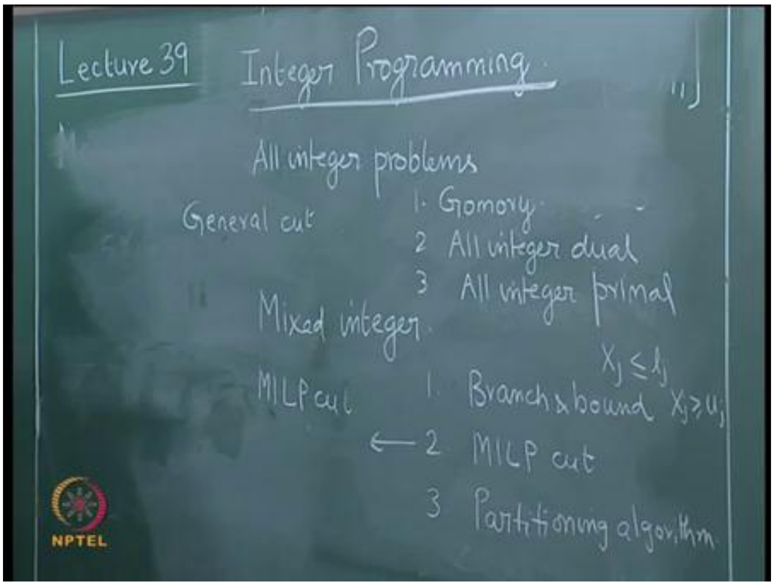

So, one second let us have a recap of what we have, seen in the last 4 lectures. The last 4 lectures essentially concentrated on integer programming - the integer programming we have seen in these 4 lectures, we have seen all integer problems, and mixed integer problems. In the last 4 lectures, we have not addressed the 0 1 problems, but 0 1 problems, have been addressed in earlier lectures in this lecture series. So, in the last 4 lectures we have looked at a few algorithms, we have looked at essentially 3 algorithms here: The Gomory cutting plane algorithm, the all integer dual algorithm, and the all integer primal algorithm.

Now, under MILP we have seen 3 algorithms, we have seen a branch and bound algorithm for MILP, we have seen an MILP cut or a cutting plane algorithm for MILP, and we have seen the partitioning algorithm for MILP. The branch and bound for all integer problems has been explained earlier in this lecture series. In addition, we saw we derive the equation of the general cut, here for all integer problems; and we also derived the expression for the MILP cut for the mixed integer problems.

We derive the expression for the general cut, and we also showed that when h is equal to 1 to get the Gomory cutting plane, when h is a positive fraction between 0 and 1, we use that for the all integer dual, and all integer primal. And both of them, since they are all integer algorithms, the h value was carefully chosen considering certain other aspects of the problem; if it is an all integer dual algorithm, h is chosen such that the pivot is minus 1, pivot equal to minus 1 maintains the integer property of the table. Pivot equal to minus 1 also ensures that at least one primal infeasible variable now become feasible, and h is also carefully chosen such that the dual feasibility is maintained. Which we saw we also try to get as large a value of h as possible; so that the increase or decrease in the objective function is as large as possible.

So, that made as look at the absolute value of the dual variable and divide them, and then see the fraction to which it is going to affect the value of h, and so on. We also saw the all integer primal algorithm, where h again was suitably chosen; such that the pivot is plus 1. So, pivot equal to plus 1 would one second maintain the integer characteristics of the table would also ensure primal feasibility. Now, this aspect was addressed in the earlier lecture, but all integer primal cuts invariably result in large number of degenerate solutions.

And therefore, would involve a large number of iterations before it actually terminates. So, while the all integer algorithms are very nice for hand computation, because the integer nature of the table is maintained therefore, it becomes lot easier to do the computations by hand. They are useful to solve problems, nevertheless the Gomory cutting plane algorithm is more popular, and Gomory cutting plane algorithm is used extensively more than the all integer algorithms.

Similarly, when in the MILP we derived the expression for the MILP cut, we also said at the branch and bound for MILP is very amenable and easily usable, because an MILP does not restrict all the variables to integers. Only some of them are going to be integers, so which our variables that are restricted to integers; it is enough to branch on them, and keep solving l p's. And even then very nature of the branch and bound, it is only going to introduce constraints of the type X j less than or equal to l j or X j greater than or equal to u j which are essentially bounds and not constraints.

Therefore, when we introduce an X j less than equal to l j or X j greater than equal to u j, we are not actually introducing a constraint which would make the problem bigger. They can always be treated as bound, and some calculations can be used to keep the number of constraints the same as the original problem. And treat them as bounds or one could use a sensitivity analysis and then proceed; so, branch and bound we said is a very effective, and a very efficient way to solve MILP's.

MILP cuts are good MILP cuts are equivalent of cutting plane algorithm, but again MILP cuts can be a little weak, and more MILP cuts can be required to solve the same problem. Nevertheless MILP cut is another way by which it can be solved, where could be instances where as in our example a single cut would give the optimum solution. Same is the case with the partitioning algorithm is completely unique way of looking at the problem, where the variables are partitioned into two sets.

Now, some interesting results from linear programming, now by fixing value is to the integer variables; the problem becomes a LP, and once it becomes a LP we borrow interesting relationships from between the primal, and dual of the LPs . And then end of creating problems, where the number of constraints increases with every iteration, an optimum solution of such a problem would give. So, there would be examples, where a partitioning algorithm will work. One of the advantages of the partitioning algorithm is that, the number of variables that we actually solve is less.

When we solve the LP portion we solve only for the use, which are the dual variables - dual variables corresponding to the continuous variables in the original problem. So, more constraints in the original problem more dual variables there, but then we solve only for those set of dual variables when we solve the LPs. Similarly, when we solve for the ILP's here, we solve only for the number of Y variable.

So, we do not solve for example, if there are 10 continuous variables, and 10 integers variables in the MILP. In the branch and bound or in the cutting plane, we would be solving a 20 variable problem; whereas in the partitioning algorithm, if there are 10 continuous variables, and say 5 constraints. Then will be solving for 5 dual variables at a time, will be solving for 10 integer variables plus 1 Z at a time. So, the size of the problem in terms of variables is actually very less in the partitioning algorithm. So, to that extend there is a gain when we use the partitioning algorithm. So, this we close the treatment of integer programming in this lecture series. .