Advanced Operations Research

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Lecture- 8
Primal Dual Algorithm

We continue the discussion on the primal dual algorithm.

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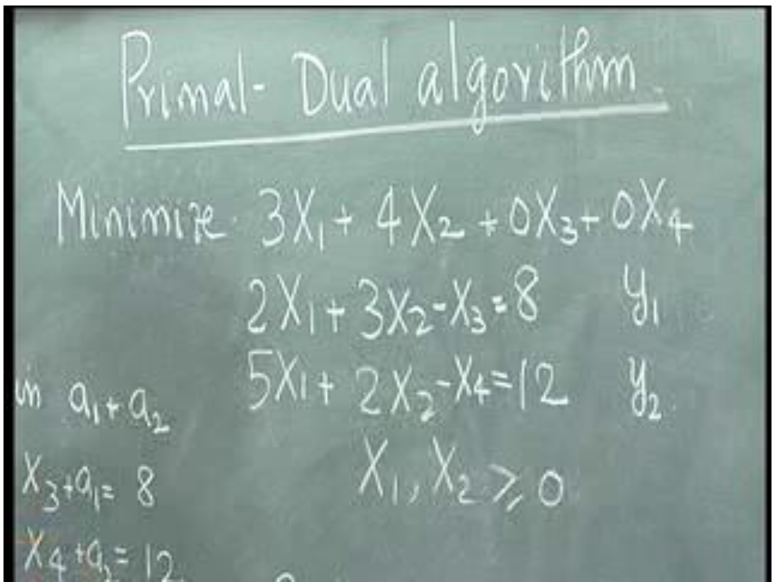

In the last lecture we were solving this problem, minimize $3X_1$ plus $4X_2$, subject to $2X_1$ plus $3X_2$ greater than or equal to 8, $5X_1$ plus $2X_2$ greater than or equal to 12. The standard problem is a minimization problem with all greater than or equal to constraints. We first converted the inequalities to equations by adding the surplus variables or negative slack variables X_3 and X_4, to get $2X_1$ plus $3X_2$ minus X_3 equal to 8. $5X_1$ plus $2X_2$ minus X_4 equal to 12. This gives a contribution of 0. Now we write the dual of this problem because we have already seen that for a minimization problem with all greater than or equal to constraints here. Strictly positive values 0 0 is not basic feasible to the primal, so this involved use of artificial variables or the two phase method to solve this. We are now going to write the dual of this problem.

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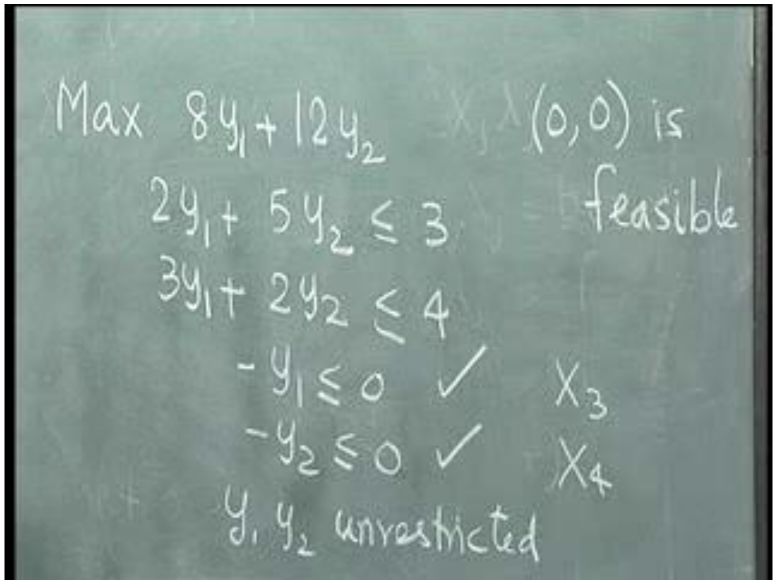

When we write the dual of this problem including the surplus variables, we get a dual like this. Maximize $8y_1$ plus $12y_2$ which comes from $8y_1$ plus $12y_2$, two dual variables subject to $2y_1$ plus $5y_2$ is less than or equal to 3; $3y_1$ plus $2y_2$ less than or equal to 4; minus y_1 less than or equal to 0; minus y_2 less than or equal to 0 and y_1 and y_2 unrestricted. When we look at a dual like this we quickly realize that 0 0 which is y_1 is equal to 0 and y_2 equal to 0 is feasible to the dual. The way the dual is written, if the given problem is a minimization problem with all greater than or equal to constraints and strict possible coefficients here then we will have a dual where, 0 0 is feasible. We have identified a feasible solution to the dual. The next thing we do is to apply complementary slackness based on this feasible solution to the dual. If we look at 0 0, this is satisfied as an inequality, this is satisfied as an inequality and these two are satisfied as equations. So with these two being satisfied as equations, we write the corresponding primal after applying complimentary slackness conditions. This corresponds to variable X_3, this corresponds to variable X_4, and so in the primal corresponding to this dual, after we apply complimentary slackness, it is equivalent of solving minus X_3 equal to 8 minus X_4 equal to 12 because X_3 and X_4 are basic that comes out of this. So, minus X_3 equal to 8 minus X_4 equal to 12, we need to find the solution.

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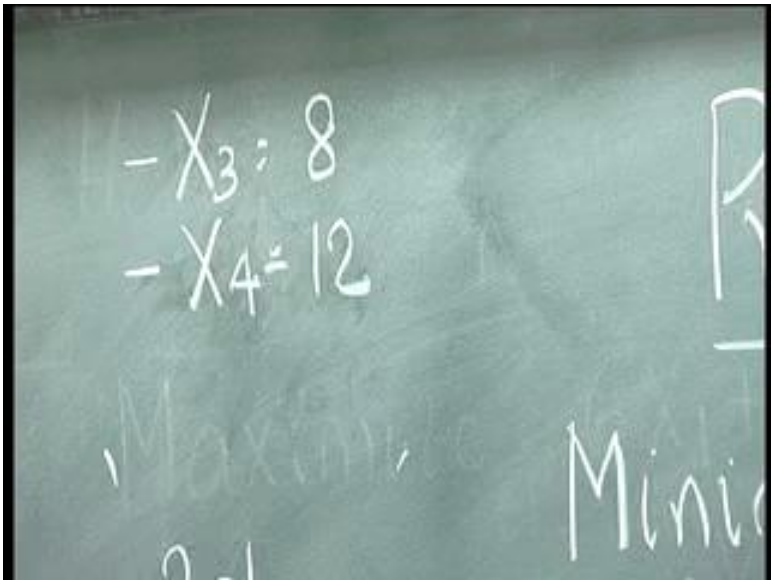

Right now, the solution for minus X_3 equal to 8 and minus X_4 equal to 12 is easy, but we also know that solving equations can also be done through linear programming. What we do is we rewrite this as follows:

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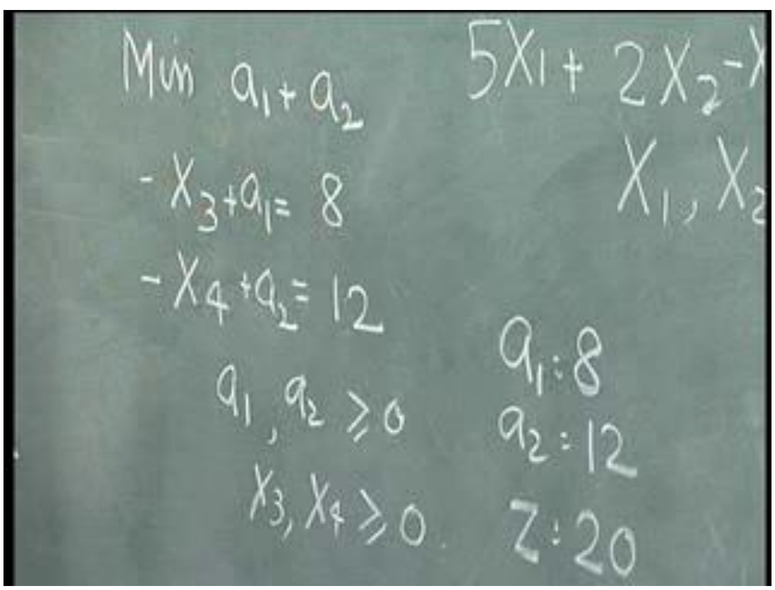

Minus X_3 plus a_1 equal to 8 minus X_4 plus a_2 equal to 12, a_1 a_2 greater than or equal to 0, X_3 X_4 greater than or equal to 0 and we minimize a_1 plus a_2. If this system has a solution, also satisfying X_3 X_4 greater than or equal to 0, then it will automatically force a_1, a_2 to 0 and give us a solution with Z equal to 0. If this plus X_3 X_4 greater than or equal to 0 does not have solution then, one of the artificial variables a_1 will be in the basis. This will give us a Z value as 1 0, which is non zero, which is 1 or 2 depending on the number of a's that are in the solution. We solve this problem using the simplex algorithm here.

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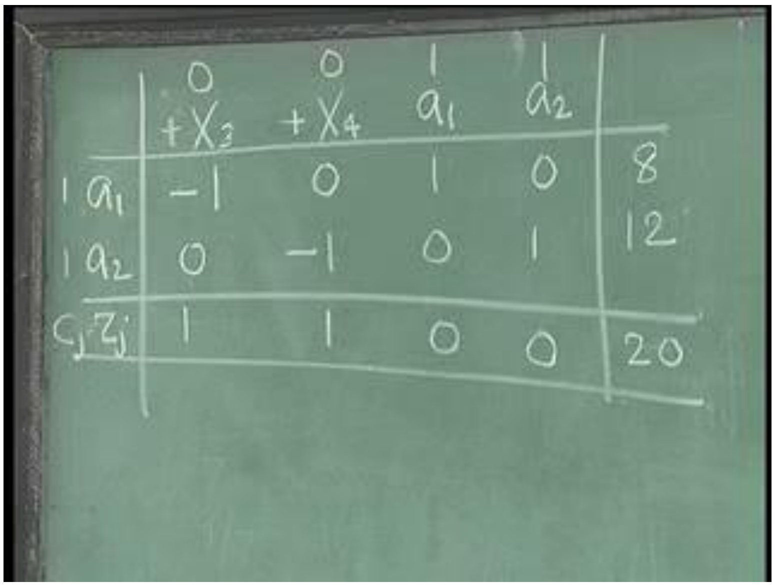

We found out that the solution a_1 equal 8, a_2 equal to 12 with Z equal to 20 is optimal. This gives us a solution a_1 equal to 8, a_2 equal to 12, Z equal to 20.

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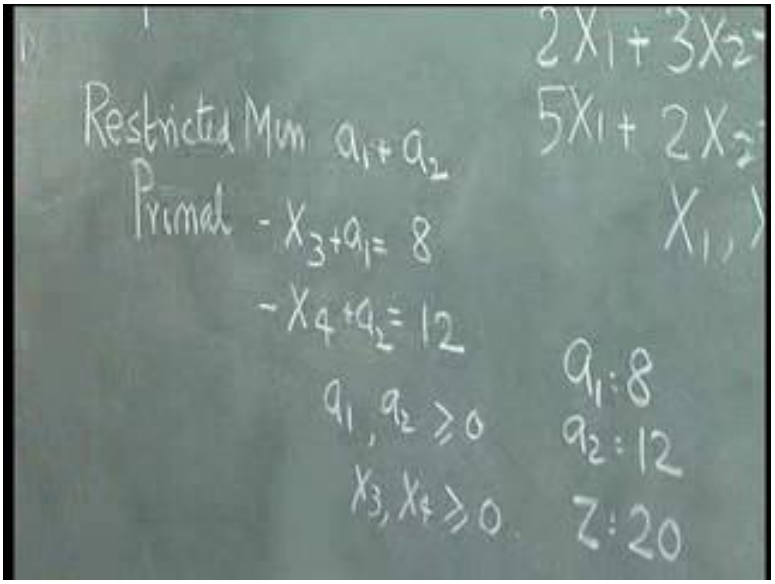

This problem is called the restricted primal; restricted because it is restricted by the feasible solution to the dual. We apply complimentary slackness and then we get the restricted primal. The basic idea is this. If I have a feasible solution to the dual, I apply complementary slackness and then if I solve the restricted primal, then the optimum solution to the restricted primal will be feasible to the original primal. Therefore, it will be optimal to the primal and dual based on duality relationships. Because, if there is a feasible solution to the dual, there is a feasible solution to the primal and they satisfy complementary slackness; then they are optimal to the primal and dual respectively. So this method will start with the feasible solution to the dual, apply complementary slackness, create a restricted primal and if the restricted primal is feasible then, it is optimal. Right now the restricted primal is not feasible to this because, this gives a solution a_1 equal to 8, a_2 equal to 12, which is not feasible to this one.

We need to get one more dual feasible solution. Now in order to get one more dual feasible solution, what we do is, we go back and try to find out the dual of the restricted primal and see what happens. When we look at the dual of the restricted primal, the dual of the restricted primal will now have two variables, in this case v_1 and v_2 and that will be to maximize 8 v_1 plus 12 v_2.

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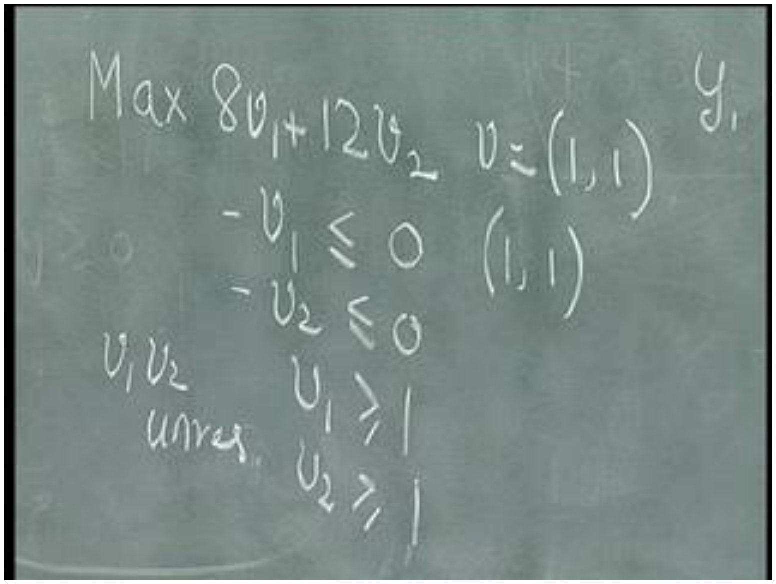

The primal has two constraints, so the dual will have two variables, which will be v_1 and v_2. So this will be maximize 8 v_1 plus v_2. Now as far as this is concerned, minus v_1, X_3 appears only here, so minus v_1, the coefficient is 0. All values are greater than or equal to 0. So, the maximization problem will have less than or equal to 0; minus v_1 less than or equal to 0, minus v_2 less than or equal to 0 from this. This appears only in the second constraint. From this, v_1 greater than or equal to 1 and v_2 greater than or equal to 1, because this variable appears only in the first constraint. v_1 objective function coefficient is 1, so v_1 greater than or equal to 1 and v_2 greater than or equal to 1 and v_1 v_2 unrestricted because the two primal constraints are equations.

A solution to this is given by 1, 1. v_1 is equal to 1 and v_2 is equal to 1 with Z is equal to 20, or objective function value equal to 20. Please note that, this has an objective function value of 20, which is the same as the objective function value of 20 here. So what we do is, this we call as some v which is the solution to the dual of the restricted primal. Now we have a solution which we call here as y, which was the starting dual solution.

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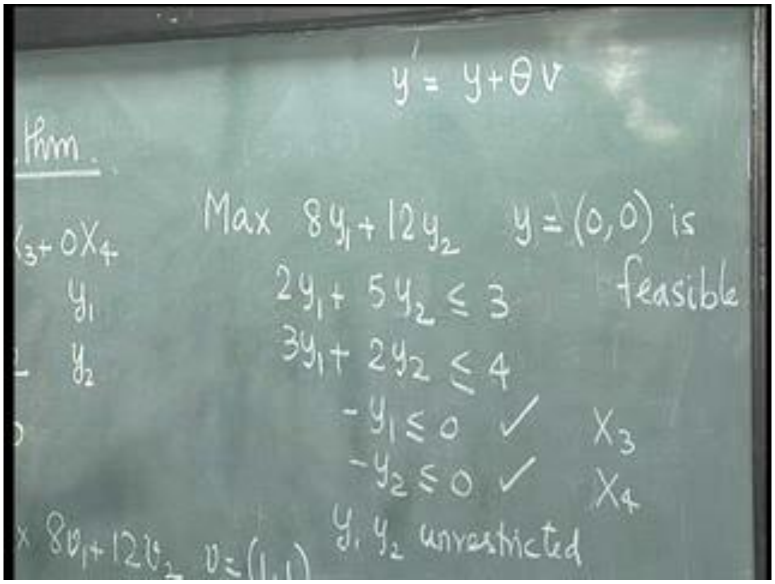

So what we will do now is, we want to create one more dual feasible solution and then, apply complementary slackness to solve the restricted primal. Let us now define the new dual feasible solution as y-dash, which we are going to have and let us call this y-dash as y plus theta v, where y is the original 0 0, the existing dual solution, v is the solution of the dual of the restricted primal. So v is 1 1 and we need to find out theta. If we need to find out theta, then, we get another feasible solution to the dual, which is based on the original y, as well as v. What do we want this y dash to have? When we started with y equal to 0 0 and wrote its restricted primal and solved it, it turned out to be infeasible. Now we want to define a y-dash, which is feasible to the dual, the restricted primal of which we will now solve. So we want the restricted primal to be different from this because, this is not giving us the solution.

When will the restricted primal be different? It will be different when the new y-dash satisfies at least one more new constraint as an equation, so that the corresponding variable will now enter into the basis here. Theta should be such that, at least one of the existing constraints which is satisfied as an inequality by y, should now be satisfied as an equation by y-dash, so that a new constraint satisfied as an equation here which means, a new variable will appear. It may replace an existing thing but, the only nice thing is that there will be one candidate, which can come in as a basic variable here, which means theta should be made out of all those constraints which are currently satisfied as inequalities, so that, one of those which are currently satisfied as inequalities will also become an equation. Second thing is that, in general principles of linear programming, this by itself is infeasible which is given by this, but this is a non-optimal solution to this.

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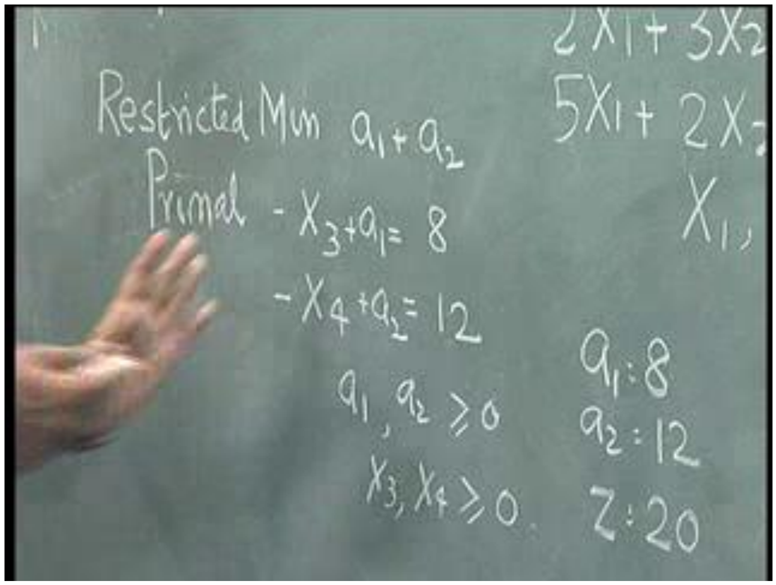

So non-optimal solution in principle should have a new variable that is entering and we relate the entering variable on the primal to the feasibility of the dual, then dual infeasible is primal non-optimal. Therefore, right now for all v that we have here, this will satisfy va less than or equal to c. So we need a variable X_j into this primal, such that v a_j is greater than zero. This will satisfy the condition v a less than or equal to 0, so we will look at those variables which have v a greater than 0.

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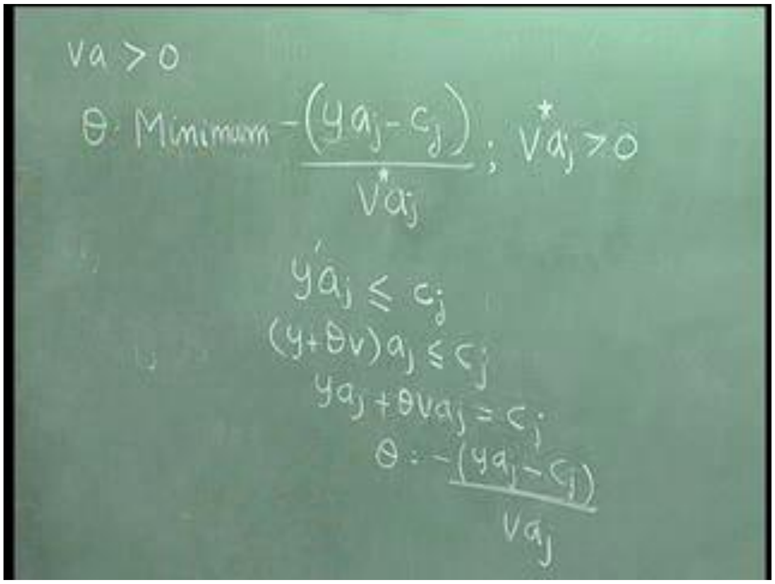

We would like to look at these constraints which are right, now satisfied as inequality and then try and get the best value of theta. The best value of theta that we will have, will be given by theta is equal to minimum of minus y a_j minus c_j divided by v star a_j, where v star a_j is greater than or equal to 0. There is also a little more theory involved in trying to get this. But without getting in too much of theory, I have tried to explain this; this comes out of two things. One is we will consider only those j's which are satisfied as an inequality. They are not satisfied as an equation because, we want eventually one of them to be satisfied as an equation. So we will not look at those that are right now satisfied as an equation. The second thing that I mentioned, is we want v-star a_j greater than 0 because this will satisfy the condition v-star a_j less than or equal to 0. So we want a new variable that enters, which will have a v-star a_j greater than 0, which is in principle an infeasible dual of the moment, which will be non-optimal primal and such a variable will enter.

If we go back and check this, this comes because, if you look at this dual, this dual is of the form ya_j less than or equal to c_j. For example, a_j's are the corresponding columns, c_j's are the objective functions and y is the dual variable. So a dual constraint is typically of the form ya_j less than or equal to c_j. Now what we have written is, we want y-star is equal to y plus theta v. So if there is a dual feasible solution y-star or y-dash then, y-dash a_j

should be less than or equal to c_j. Now y-dash is written as y plus theta v, so y plus theta v into a_j is less than or equal to c_j. Now ya_j plus theta va_j is less than or equal to c_j. We want it finally to be an equation, so that one of these, as we said, will become an equation so for something. For one of them to become an equation, let us quickly change this inequality to an equation. Therefore, theta will be of the form minus ya_j minus c_j by v-star a_j which is the same as c_j minus ya_j by v star a_j. Basically, this comes from this derivation that, that value which is the minimum will force one of these two to be an equation and the other would still be feasible. Keeping that in mind, we write this expression for theta. With the present solution, let us try and find out what we do. We first have to find out v-star a_j for these two.

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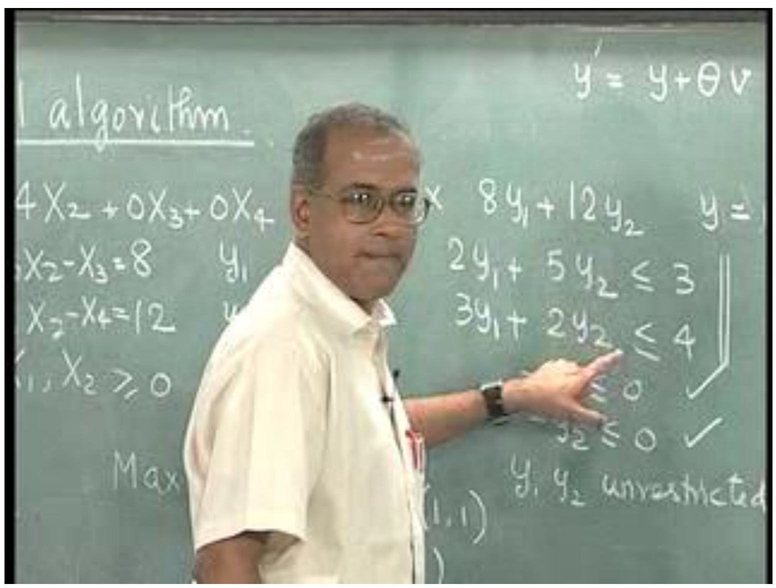

v-star a_j from here, this is a_j which is the same as this. Now v-star is 1,1. So the value here is 2 plus 5 equal to7, 3 plus 2 equal to 5; so for both of them v-star a_j is positive.

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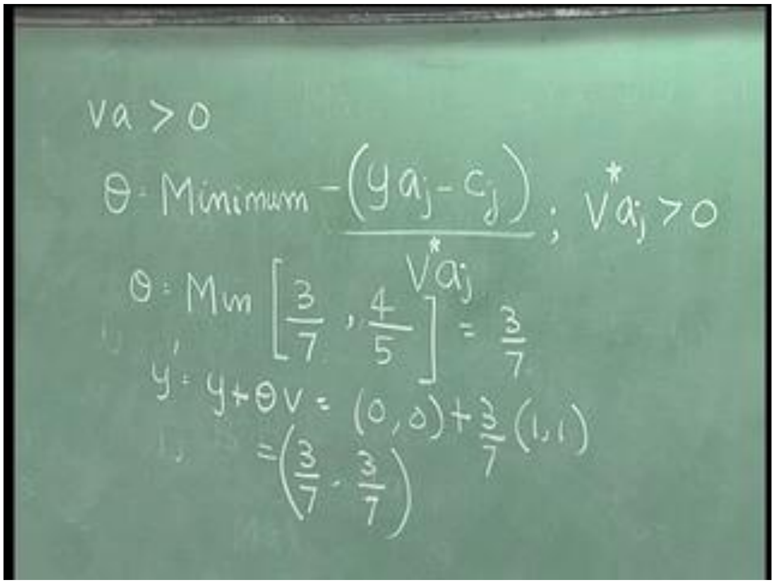

So theta will be equal to minimum of minus ya_j minus c_j, so y is 0 0, that we have here. So all y a_js are 0, so minus of minus c_j is plus c_j. So c_1 is 3 by v-star a_j, which we just now found out, v-star is 1,1, a_j is 2 5, so v-star a_j is 7. So, minimum over 3 by 7, for the second one here, this corresponds to the second coefficient. So c_j is 4 by v-star a_j corresponding to this, v-star is 1,1, a_j is 3,2, so v-star a_j is 5. So theta will be minimum over 3 by 7, 4 by 5; v-star a_j greater than 0. So theta is equal to 3 by 7. When theta is equal to 3 by 7, y-dash is equal to y plus theta v will be 0 0 plus 3 by 7 into 1,1 which is 3 by 7, 3 by 7. Now we have a new solution y_1, so we are going to say y is equal to 0 0, is what we started.

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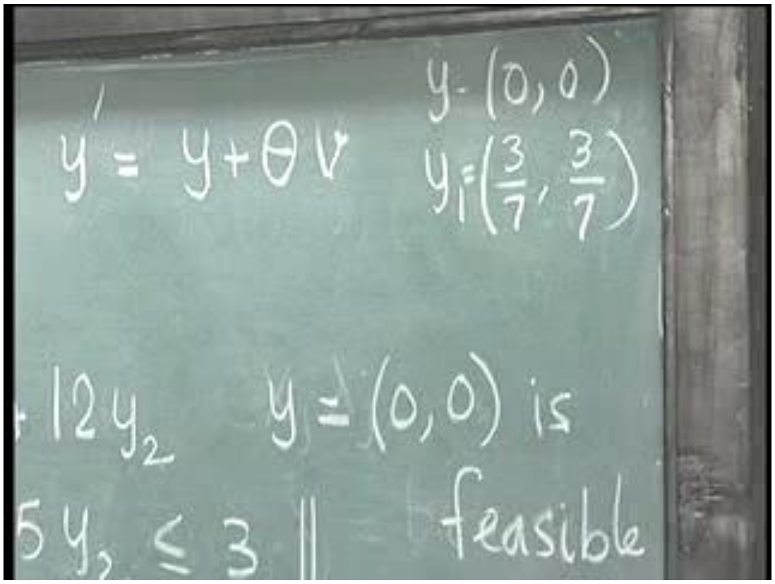

y_1 is y plus theta v, which is 3 by 7, 3 by 7. Now if we look at 3 by 7, 3 by 7 and start substituting here, this is 2 into 3 by 7 plus 5 into 3 by 7 which is 7 into 3 by 7, which is 3.

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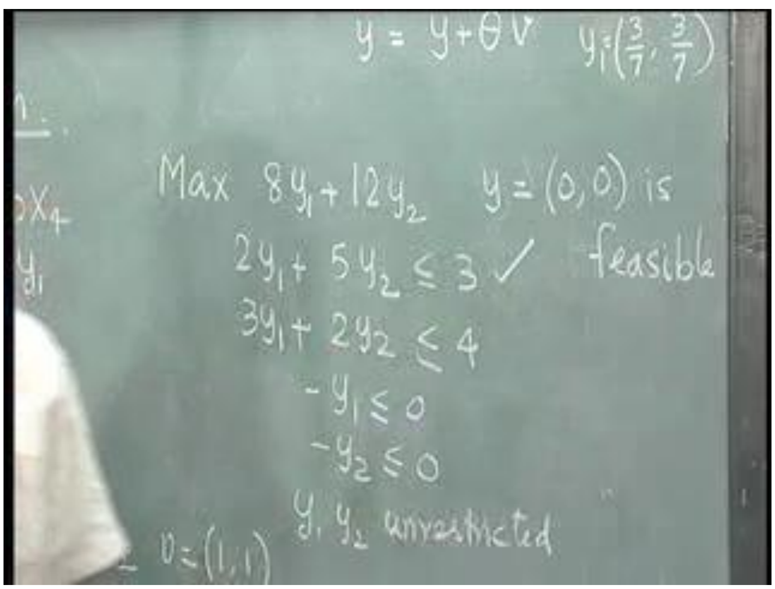

So this is satisfied as an equation. Now this one is satisfied as an equation; 3 into 3 by 7 is 9 by 7, plus 6 by 7 is 15 by 7 which is satisfied as an inequality. y_1 greater than or equal to 0, satisfied as an inequality, y_2 greater than or equal to 0 satisfied as an inequality. This is the only one that is satisfied as an equation. So this will force variable X_1 into the basis. The basis is different so we have to write the restricted primal. X_1 alone is in the basis.

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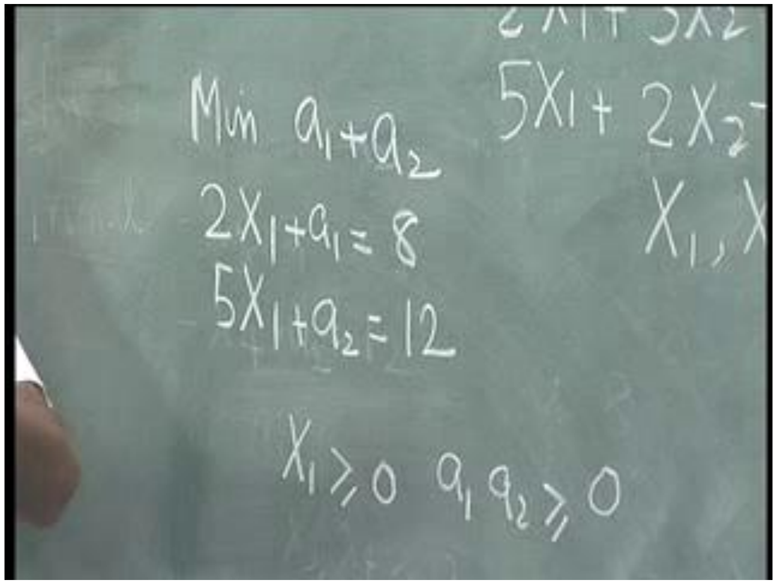

So we are trying to solve here $2X_1$ is equal to 8; $5X_1$ is equal to 12 is what we are trying to solve now. Once again this is made into a linear programming problem with X_1 greater than or equal to 0. This will become $2X_1$ plus a_1 is equal to 8; $5X_1$ plus a_2 is equal to 12. a_1 a_2 greater than or equal to 0 and minimize a_1 plus a_2. Now we actually solve this linear programming problem; it is like the 2 phase method. We are essentially solving only this set of equations $2X_1$ equal to 8, $5X_1$ equal to 12, but we are actually solving it as a LP by adding two variables, which may be like artificial variables, with the plus 1 coefficients and it is like a two phase method to solve this LP. We go back and solve this; we have X_1 a_1 a_2.

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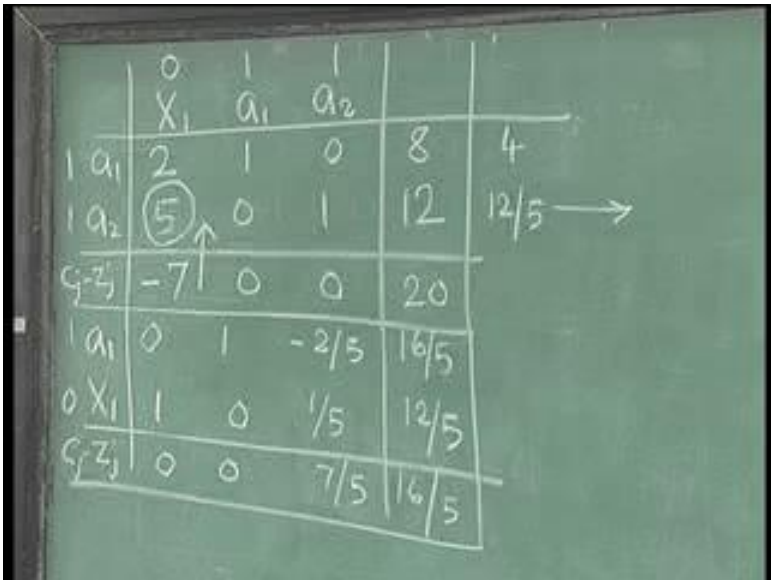

a_1 a_2 will be the basic variables, so $2X_1$ plus a_1 0 8; $5X_1$ 0 1 12. These have objective coefficients of 1; it is a minimization problem. These also have objective function coefficient of 1, so c_j minus z_j; 0. So 1 into 2 plus 1 into 5 is 7, so 0 minus 7 is minus 7, you get 0 here, you get 0 here, so you get 20 here. It is a minimization problem with the negative c_j minus z_j, the variable will enter the basis. So, variable X_1 will enter the basis. Theta is computed as 8 by 2 is 4, 12 by 5; 12 by 5 is smaller than 4. So the variable a_2 leaves the basis. This is the pivot element.

We do one more iteration with a_1 and X_1 coming here. c_j minus z_j, this is 0, this is 1 divided by the pivot element; 1 0, 1 by 5, 12 by 5; this minus 2 times this, 0 1 minus 2 by 5, 8 minus 24 by 5 is 16 by 5. So objective function value is 16 by 5 0 0 minus 2 by 5, 0; 1 minus of minus 2 by 5 is 7 by 5. This is c_j minus z_j, so no entering variable; algorithm terminates with the optimum solution X_1 is equal to 12 by 5.

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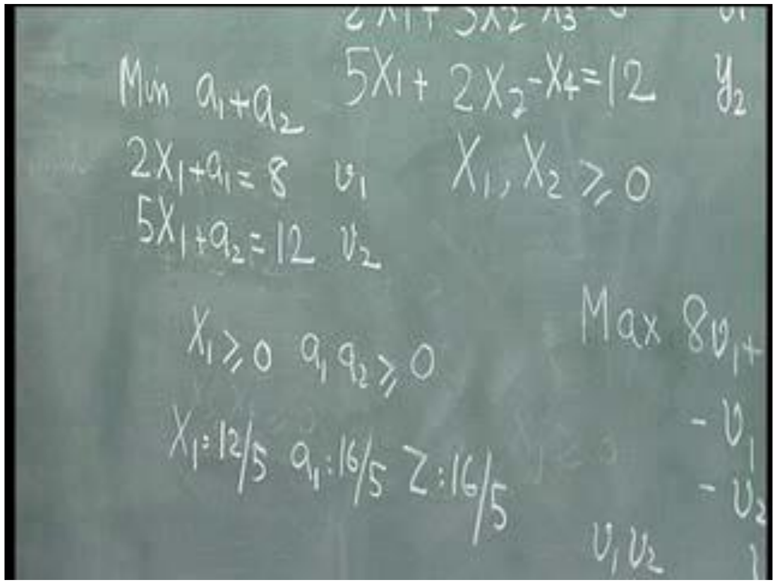

The solution here is, X_1 is equal to 12 by 5, a_1 is equal to 16 by 5 and z is equal to 16 by 5. Now the restricted primal has an optimum solution with an artificial variable still in the basis. Therefore, the solution to the restricted primal is not feasible to this. So we need to do one more iteration here because it is not feasible to this. We will now go back and try to find out the dual of the restricted primal as we did. This restricted primal will have two dual variables v_1 and v_2. The dual of the restricted primal will now have maximize $8v_1$ plus $12v_2$, which is the same here.

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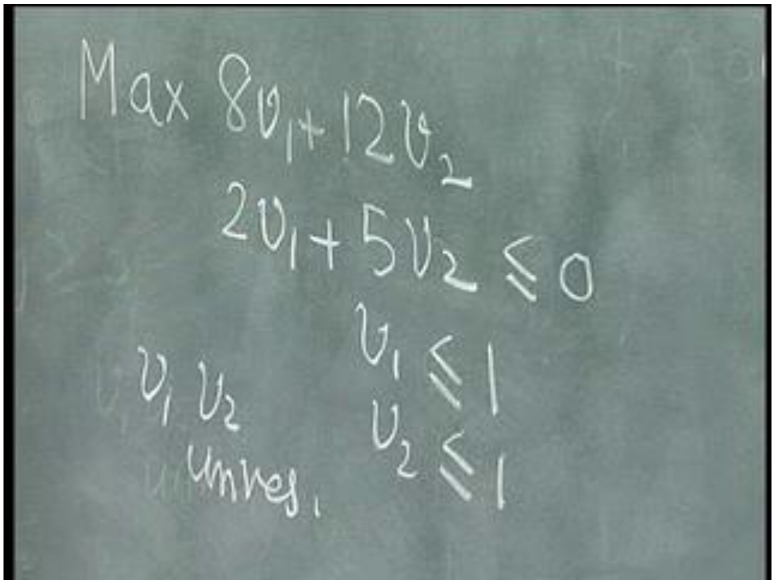

$8v_1$ plus $12v_2$, this one is $2v_1$ plus $5v_2$; this constraint is $2v_1$ plus $5v_2$ is less than or equal to 0. So $2v_1$ plus $5v_2$ corresponding to this is less than or equal to 0, corresponding to this is v_1 is less than or equal to 1 and corresponding to this, v_2 is less than or equal to 1; v_1 v_2 unrestricted in sign. So $2v_1$ plus $5v_2$ less than or equal to 0, v_1 less than or equal to 1, v_2 less than or equal to 1, so we have to solve this to try and get the optimum solution to this. We can either solve this by the simplex or we can solve this by the graphical method because we have only two variables that are there, so we will quickly try and solve it using the graphical method.

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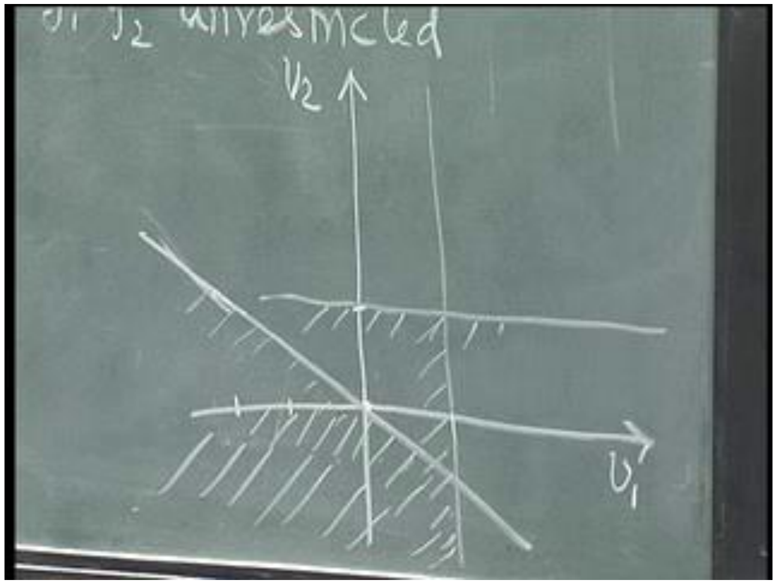

Note that, v_1 v_2 is unrestricted in sign. So just solving it by the graphical method will give us v_1 less than or equal to 1. This is same; this is $v_{1,}$ this is v_2. v_1 less than or equal to 1 is here. v_2 less than or equal to 1 is here, $2v_1$ plus $5v_2$ less than or equal to 0; this is one point; 0,0 is one point; the other point could be if v_2 is equal to 1, then v_1 is equal to minus 5 by 2.

v_2 is equal to 1, which is here, v_1 is equal to minus 5 by 2, this is 2, 2 and 1 by 2. So this is another point; so the third line will look like this. This is $2v_1$ plus $5v_2$ less than or equal to 0. So this is the feasible region, this is the region corresponding to this one, this is the region corresponding to this, this is the region corresponding to this. This will be the entire region that we will have.

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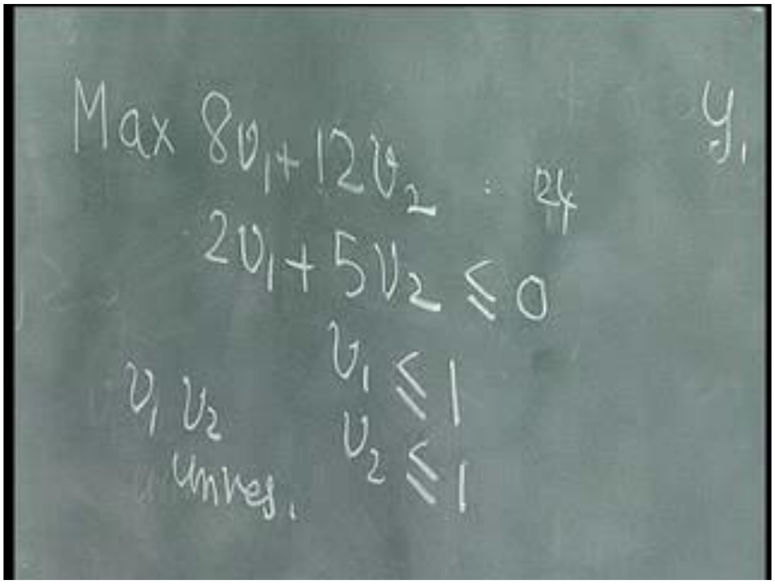

So the objective function maximize 8 v_1 plus 12 v_2 suppose we take 8v_1 plus 12v_2 equal to 24, then we have 3,0 and 0,2 that come here.

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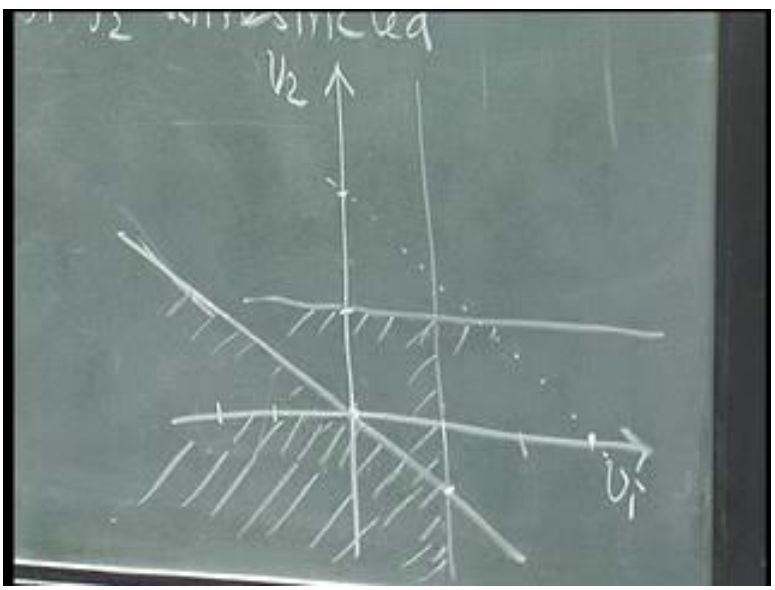

Now this is 1 2 3,0 0,2; so the objective function line is like this. As it moves here, the last point that it will actually touch is v_1 is equal to 1 and v_2 is equal to minus 2 by 5, so this will be the point. This is 0 0, this is v_1 equal to 1; so as it moves, this is the last point which will come here, so v_1 is equal to 1, v_2 is equal to minus 2 by 5.

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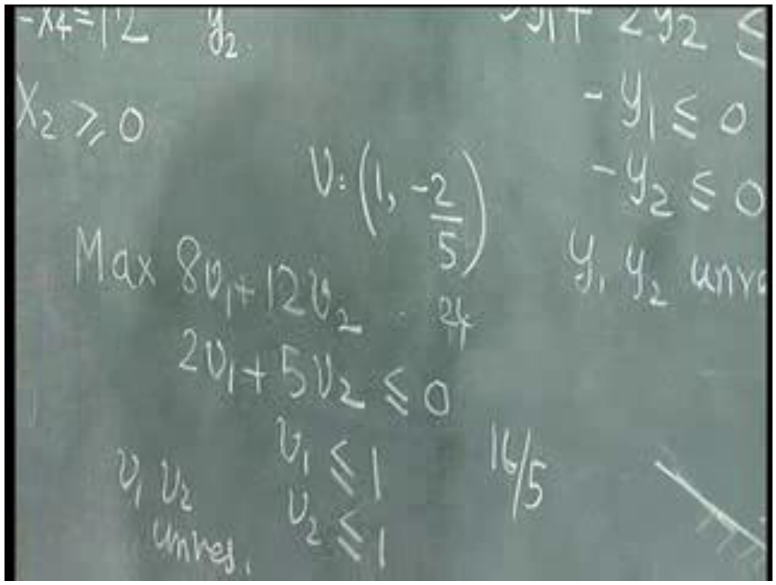

The optimum solution is v_1 is equal to 1, v_2 is minus 2 by 5. Now note that the objective function value here, this is minus 24 by 5 plus 8, which is plus 16 by 5 which is exactly what we got here as plus 16 by 5. We now have a new v which is given by 1 and minus 2 by 5. Now we need to find out again a new value. First we need to find out theta and then we also need to find out new y-dash is equal to y plus theta v.

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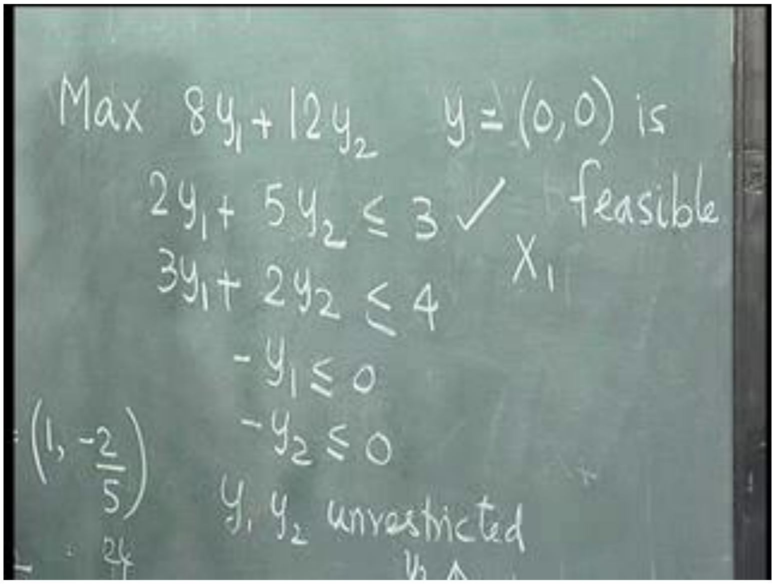

From this we know that this is the one that is satisfied as an equation. So all these three now become potential candidates to evaluate theta. We also need to find out for which of them v-star a is greater than 0. We need to find out v-star, so first let us find out the second one; 3,2.

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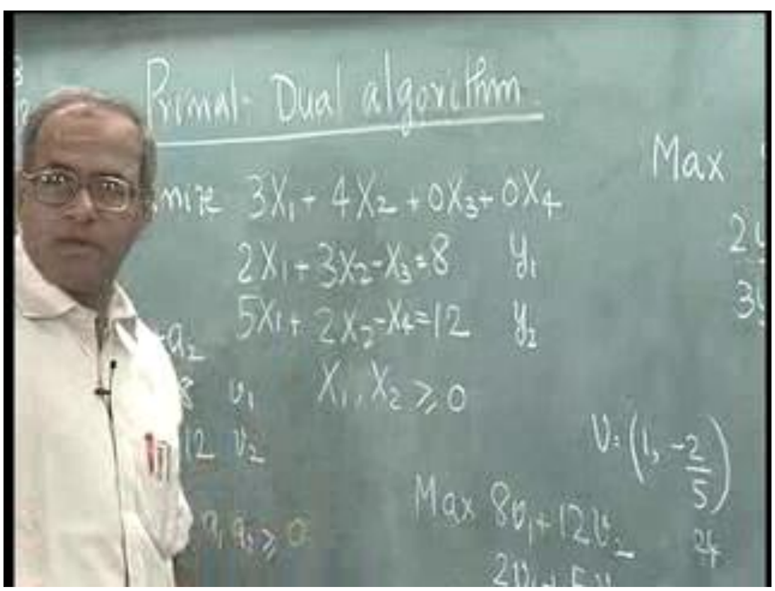

v-star a is 3 into 1 minus 4 by 5; 3 minus 4 by 5 is 11 by 5, so v-star a is greater than 0.

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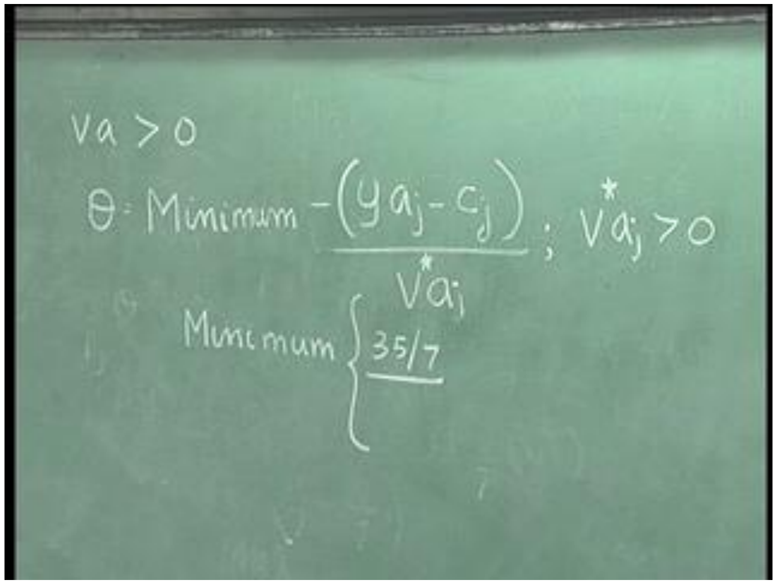

So we will write minimum of, for the variable X_2 ya_j minus c_j; y is 3 by 7, 3 by 7. So ya_j is 9 by 7 plus 6 by 7 which is 15 by 7. ya_j is 15 by 7 minus c_j minus 4, negative of that, so 4 minus 15 by 7 is 35 by 7, so 35 by 7 divided by ya_j minus c_j.

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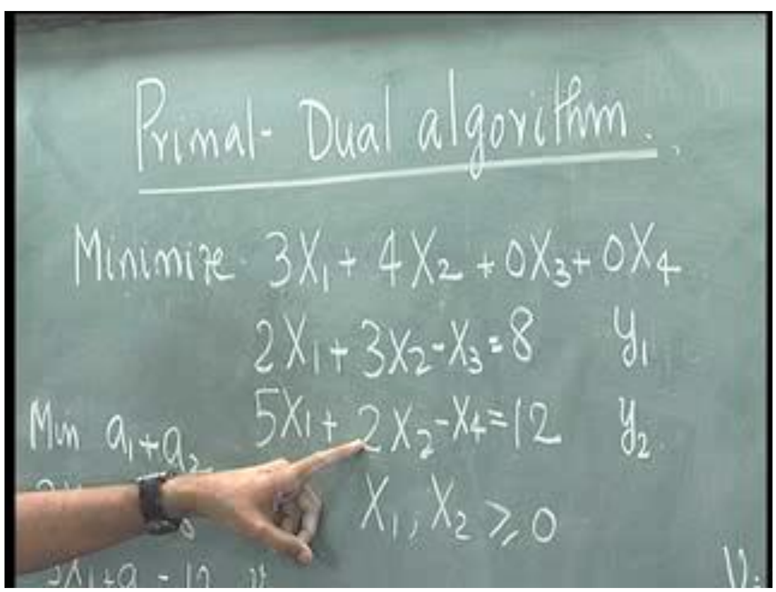

y_1 is 3 by 7, 3 by 7; so, 3 by 7 into 2 is 9 by 7 plus 6 by 7 is 15 by 7, so 4 minus 15 by 7 is 13 by 7.

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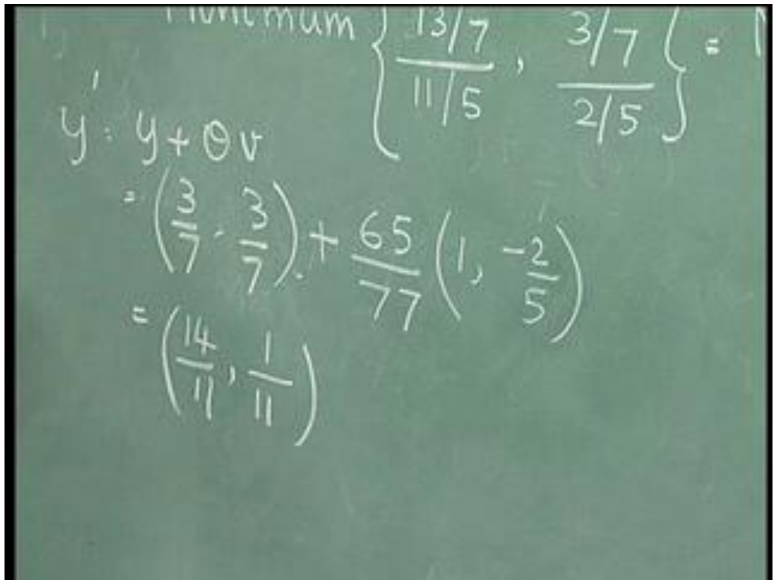

This is 13 by 7 divided by v-star a_j, which we just now calculated as 3 into 1, 3 minus 4 by 5, which is 11 by 5, so 13 by 7 divided by 11 by 5. Now for the third one here, which is variable X_3, so this is variable X_3 minus 1 0 into 1 minus 2 by 5, so v-star a_j is negative, minus 1 0 into 1 minus 2 by 5 is negative.

Now for the fourth one, 0 minus 1 into 1 minus 2 by 5 is positive, so you get 2 by 5 here. This is minus 3 by 7, so we get c minus ya_j; 0 minus, minus 3 by 7 is plus 3 by 7 divided by 0 minus 1, 1 minus 2 by 5 so plus 2 by 5; so 3 by 7 divided by 2 by 5. So this is minimum of 65 by 77, 13 by 7 into 5 by 11. So 65 by 77 and 3 by 7 into 5 by 2, which is 15 by 14. You get 65 by 77 as the value. So now y-dash is equal to y plus theta v, so y is 3 by 7, 3 by 7; so 3 by 7, 3 by 7 plus 65 by 77 into v, which is 1 and minus 2 by 5. This is 3 by 7 plus 65 by 77. This is 33; 65 plus 33 is 88, 88 by 77 is 8 by 7, this is 3 by 7, 65 into minus 2 by 77 into 5. This will go 13 times minus 26 by 77. This is 33 by 77, so 7 by 77, which is 1 by 11. This is 65 into minus 2 by 5 minus 26 by 77 plus 33 by 77, 7 by 77, which is 1 by 11. Let us look at this, this is 33 plus 65, this is 11 times, so 33 plus 65 is 98 by 77. So 98 by 77 is 7 into 14 is 98. 14 by 11, 1 by 11; so the value is 14 by 11 and 1 by 11. So that is your y-dash.

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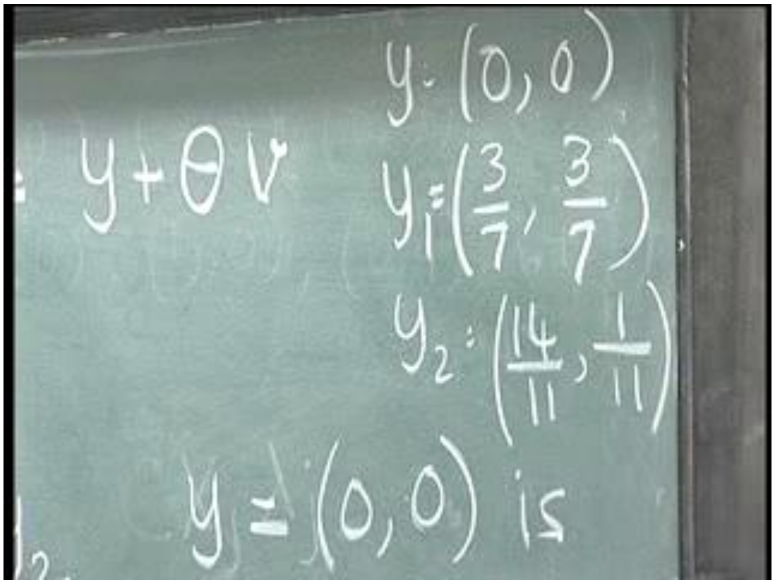

So we call that as y_2 equal to 14 by 11, 1 by 11. Now, we go back and check from this.

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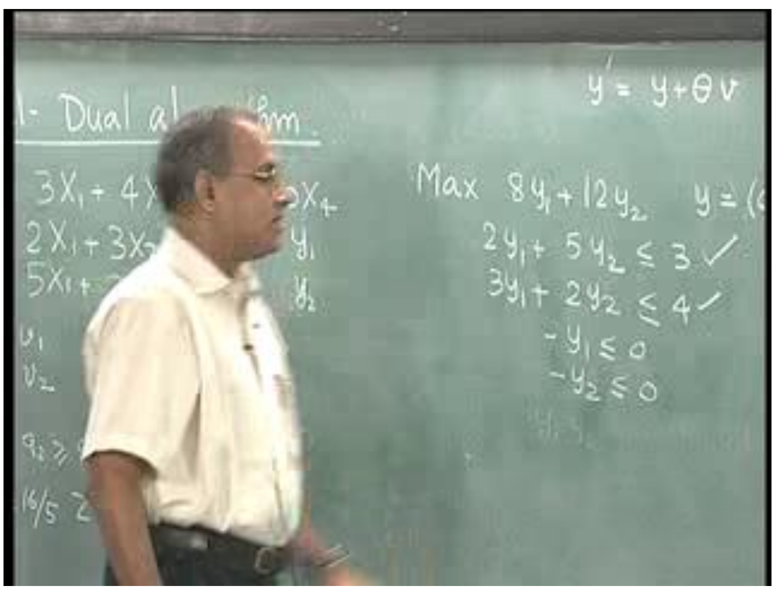

14 by 11; this is 28 by 11 plus 5 by 11, is 33 by 11, which is equal to 3; so satisfied as an equation. 42 by 11 plus 2 by 11 is 44 by 11; so satisfied as an equation, these are satisfied as inequalities. Now we go back here and write the restricted primal. Now corresponding to these two X_1 and X_2 will be the basic variables.

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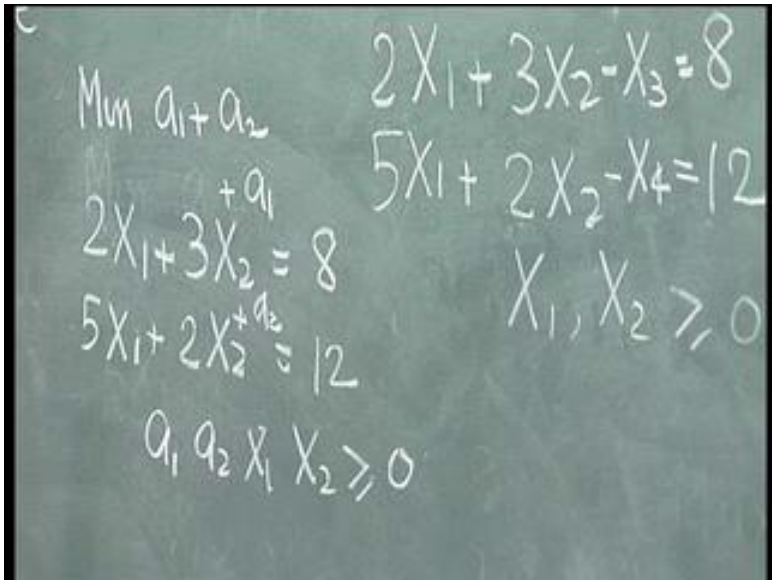

We should solve for $2X_1$ plus $3X_2$ is equal to 8 and $5X_1$ plus $2X_2$ is equal to 12. So in the same way, we add two more variables plus a_1 equal to 8, plus a_2 equal to 8 and then we minimize a_1 plus a_2 and say that a_1 a_2 X_1 X_2 greater than or equal to 0. One way is to solve it directly, the other is to go to the linear programming and do it. Since we have consistently been using the linear programming approach, we will do it using the same approach.

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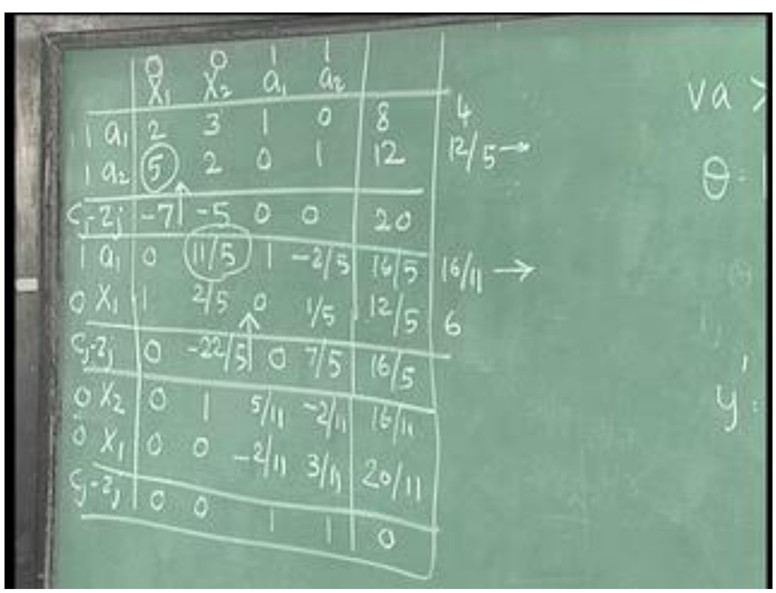

We start with a_1 and a_2 as the basic variables. So $2X_1$ plus $3X_2$ plus a_1 is equal to 8, $5X_1$ plus $2X_2$ plus a_2 is equal to 12. We have 1 and 1 here, so we have c_j minus z_j. This is 2 plus 5 7, so minus 7 minus 5 0 0 and 20. So minimization problem, so negative c_j minus z_j will enter, so 8 divided by 2 is 4, 12 by 5. So 12 by 5 leaves the basis, this is your pivot; your a_1 X_1 coming in. Now this is 1 and this is 0 1 2 by 5, 0 1 by 5, 12 by 5. This minus 2 times this is 0; 3 minus 4 by 5 is 11 by 5, 1 minus 2 by 5, 8 minus 24 by 5 is 16 by 5. Now this is again c_j minus z_j, this is 0; this is minus 22 by 5. This goes, so you get minus 22 by 5. This is 0, this is 1 plus 2 by 5 is 7 by 5 and 16 by 5.

Once again minimization problem, so negative c_j minus z_j will enter. So this enters, so 16 by 5 divided by 11 by 5 is 16 by 11; 12 by 5 divided by 2 by 5 is 6, 16 by 11 leaves the basis. This is your new pivot. So you will have X_2 and X_1 with 0 0 here, c_j minus z_j divided by a pivot element 0 1 5 by 11 minus 2 by 11, 16 by 11. This minus 2 by 5 times 1 is 0 so 0 0 0, minus 2 by 5 into 5 by 11 is minus 2 by 11. So this plus 2 by 5 into 2 by 11, this plus 4 by 55, 15 by 55 which is 3 by 11. This minus 2 by 5 into this, so this is 12 by 5 minus 32 by 55, this is 12 into 11 is 132 minus 32 is 100 by 55, which is 20 by 11. So this will be 0 0, these all 0, so you get 1, you get 1, you get 0. So the restricted primal now has a solution which is X_1 equal to 20 by 11.

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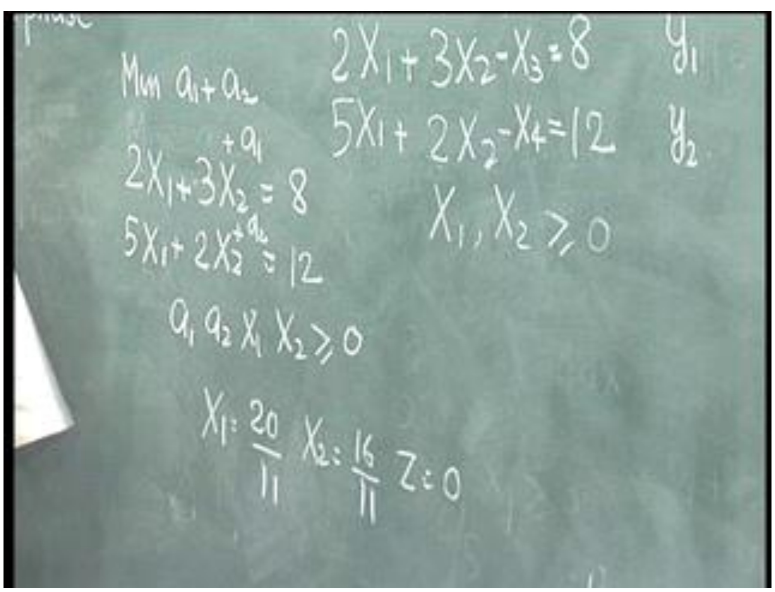

X_2 equal to 16 by 11, with Z is equal to 0. So we can just quickly check 40 plus 48 is 88 by 11, which is 8, 100 plus 32, 132 by 11, which is 12. Now the restricted primal has an optimal solution with equal to 0. so the moment it has an optimal solution with z equal to 0, it means the artificial variables are not there in the solution, which means, you have a solution based only on X_1 and X_2. Therefore, this solution is feasible to this. Right? Which is, 2 into 20 by 11 plus 3 into 16 by 11 is equal to 8 and so on. We get a feasible solution to this, so right now we have a feasible solution to the dual complementary slackness is satisfied.

The corresponding restricted primal gives the solution that is feasible to the primal. Therefore, this solution is optimal to primal and dual respectively. The solution 20 by 11, 16 by 11, with z is equal to $3X_1$ plus $4X_2$; 3 into 20 by 11 plus 4 into 16 by 11. This is 20 plus 60, 60 plus 64 is 124 by 11. That is $3X_1$ plus $4X_2$, so from here we get 14 into 8 is 112 plus 12, 124 by 11. So the solution 20 by 11, 16 by 11, z equal to 124 by 11 which is what we got here 16 by 11, 20 by 11 plus from the dual side 14 by 11 into 8 plus 12 into 1 by 11 gives the solution. This is how the primal dual algorithm works by giving us this optimal solution. So the underlying principle is that, for a certain type of problem, which is like this minimize, write the dual, work with the feasible solution to the dual, try to get a feasible solution to the primal and if it is feasible then it is optimal. Otherwise, try and get a new feasible solution to the dual, such that there is at least one new candidate in the restricted primal. That is the fundamental principle that there is at least one new candidate in the restricted primal.

Once we start getting a new candidate in the restricted primal, the solution to the restricted primal becomes better. This theta also will ensure that the dual feasibility is maintained. At the same time, a new inequality which is right now satisfied as an inequality will be satisfied as an equation. So in this process I am alternately updating this dual variables and then writing the restricted primal and solving it till the restricted primal has z equal to 0, which means, it does not have artificial variables is the essence of the primal dual algorithm. So this is how the primal dual algorithm works and many a time we will be tempted to say that this algorithm is somewhat similar to the two phase method.

Now instead of doing this, we could have straight away started with this problem, apply the two phase simplex algorithm by simply adding two artificial variables a_1 a_2, giving them objective function values of 1 and in the first phase eliminating them and then proceeding to do that. Computationally, it is the same as the two phase method but it brings a very important idea, that by making the dual different and by bringing a new candidate in the restricted primal, we can get to the optimal. In fact, the primal dual algorithm is used extensively. The Hungarian method that we saw for the assignment is a direct application of the primal dual algorithm.

Now let us take a very small example of an assignment problem and try to show how the Hungarian algorithm is actually a primal dual algorithm; we will do that with a very small example here.

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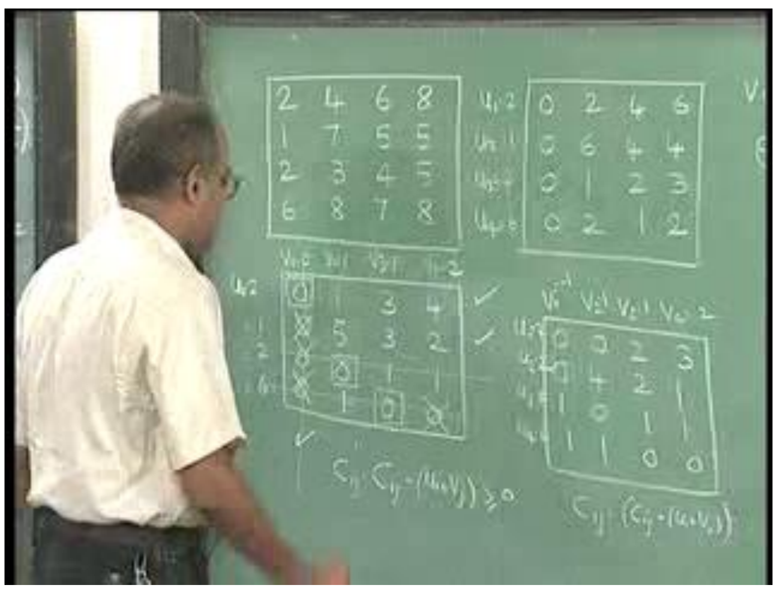

Let us look at a small 4 by 4 assignment problem. Now let us do the row subtractions and simply write u_1 equal to 2, with 0 2 4 and 6, u_2 equal to 1 0 6 4 4, u_3 equal to 2 0 1 2 3 and u_4 equal to 6 0 2 1 2. Let us do the column subtractions to get 0 0 0 0, 1 5 0 1, 3 3 1 0, 4 2 1 0. Now we have u_1 equal to 2, u_2 equal 1, u_3 equal to 2, u_4 equal to 6, v_1 equal to 0, v_2 equal to 1, v_3 equal to 1, v_4 equal to 2. Let us quickly make the assignment in this. Now we realize that we can make an assignment here. This goes, this goes, this goes;

now we can make an assignment here and we could make an assignment in any one of them and say this goes. We have not been able to make four assignments, so what we do is we do the line drawing procedure and tick an unassigned row, if there is a 0 tick that column, there is an assignment tick that row, draw lines through unticked rows and ticked columns and then change this, such that, if there are two lines, find the minimum theta, which is 1. If there are two lines add theta, if there is one line retain it and if there is no lines subtract theta. So we get 0 0 2 3, 0 4 2 1, 1 0 1 1, 1 1 0 0. We know that whether we use this matrix or whether we use this matrix, the solution is the same; the reason the solution is the same comes from this.

When we created this, we have written c-dash$_{ij}$ equal to c_{ij} minus u_i plus v_j . By doing the row column subtraction, we have done this and we have ensured that c-dash$_{ij}$ is greater than or equal to 0. Therefore this is dual feasible to the assignment problem because assignment problem constraint is of the form u_i plus v_j less than or equal to c_{ij}. If we ensure that c-dash$_{ij}$ c_{ij} minus u_i plus v_j is greater than or equal to 0, it means the dual is satisfying. Now we satisfy complementary slackness only by assigning in 0 positions, we do not make an assignment on any other position.

We satisfy complementary slackness and if we get a primal feasible solution, then it is optimal. Right now, we do not have a primal feasible solution because we do not have four allocations. Now what we do next? By doing this, we actually readjust the u_i's and the v_j's. They are now adjusted again; this has a different u_1, u_2, u_3, u_4. In fact, here there is a line that goes, so v becomes minus theta. So this is, minus 1 1 1 and 2. Here when there is no line, the theta actually reduces. So u_1 is equal to 1, u_2 is equal to 0. The theta gets adjusted here, so when there is a line theta remains the same.

So u_4 is equal to 6, this is equal to 2. When there is no line, theta becomes plus 3 and 2. For example, this 0 was 6 minus 6 plus 0 0, 6 minus 5 is 1, so we still again have c-dash$_{ij}$ equal to c_{ij} minus u_i plus v_j. Now what we have done is, we have a new set of dual variables. But in the process, what we have done is we have included at least one new assignable 0. If you see carefully, this is one new assignable 0, which means, this can bring one more variable into the primal basis. That is, exactly what we did here. We started with the dual solution and then after one iteration we computed the theta.

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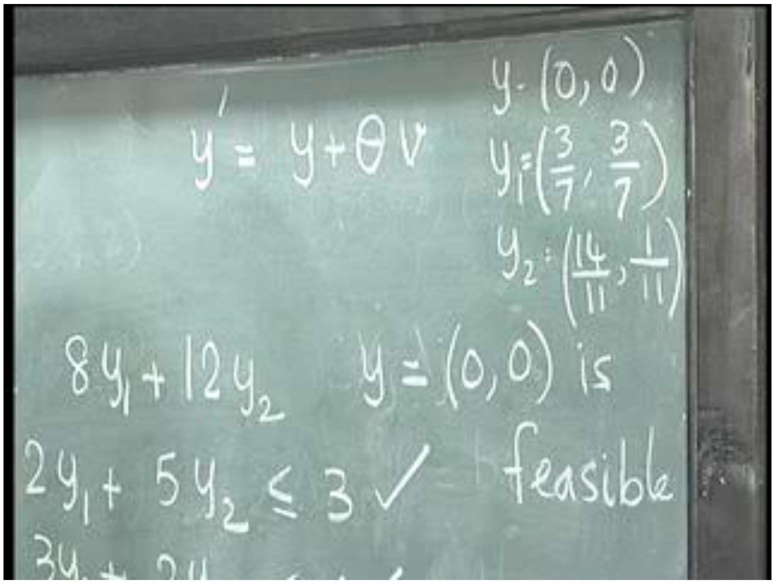

Then we make sure that by moving from y to y_1, at least one constraint which was originally satisfied as an inequality is now satisfied as an equation. So here satisfying as an equation is reflected by 0.

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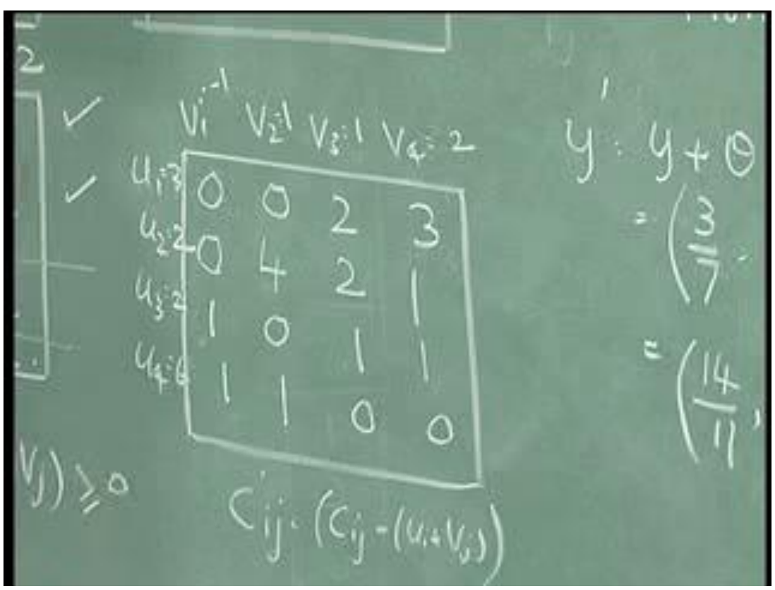

At least one new constraint is satisfied as an equation and the restricted primal works, with one more additional basic variable. So in that process, we get the optimum. In a very similar manner till we get the optimum we keep making these ticks and subtractions, till we finally get a feasible solution to the primal, which will give us optimal. Here also, the moment we get the feasible solution to the primal, we get to the optimal. So the Hungarian algorithm is a direct application of the primal dual algorithm.

Once again the primal dual algorithm starts with a minimization problem with greater than or equal to, writes a dual so that it is easy to find a feasible solution to the dual then computes the restricted primal. If the restricted primal is feasible then, it is optimal. Otherwise the y becomes y_1 by carefully ensuring that there is at least one new dual constraint, which is now satisfied as an equation.

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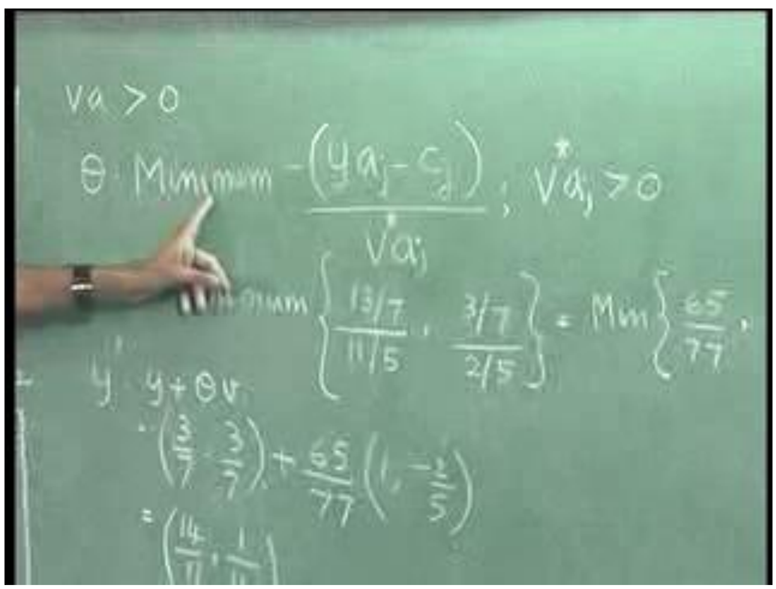

That is done by using this expression for theta such that dual feasibility is maintained and one new basic variable enters. With that once again the restricted primal is computed and this process is repeated back and forth till the restricted primal is feasible. The moment the restricted primal is feasible, it is optimal and we have also seen now that the Hungarian algorithm is the direct application of the primal dual method.

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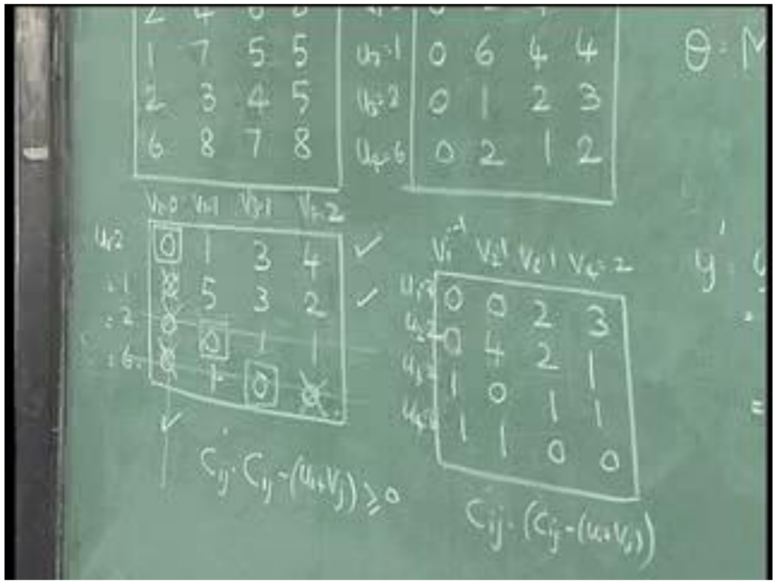

So far in the course, we have seen several aspects of linear programming. We started with the revised simplex algorithm which essentially tries to quicken the simplex and time taken per iteration. Then we saw how simplex is modified to handle bounded variables, by treating them separately and not by treating them as explicit constraints. Then we looked at the idea of column generation through the cutting-stock problem, where it is not necessary to store all the columns explicitly. Columns can be generated by solving sub problems and we saw an application to the one dimension cutting-stock problem.

Then, we moved to the Danzig Wolfe decomposition algorithm, where if the problem has a certain structure whereby removing certain constraints, we can decompose it into smaller sized LPs. Then we exploited the fact that, it is easier to solve a certain number of smaller LPs compared to solving one large LP, because the computational effort is cubic with respect to the number of constraints that we have. So the decomposition algorithm taught us a way by which we can split a bigger problem into smaller problems and by solving a series of smaller problems and also by using the column generation idea we generate entering columns into the basis. Then we have seen the primal dual algorithm, where by intelligently working with the dual and by modifying the dual solutions at each stage and by solving restricted primal which is a much smaller sized problem, one can go back and get the optimum solution to the linear programming.

So far, we have seen several aspects of simplex algorithm and algorithms related and associated with simplex. We have largely looked at it from the point of view of making it faster, by storing less and by generating entering columns as and when they have to be generated. So much we have seen about how to make the simplex better and handle enough variety. The next thing, we may have to look at is do linear programming or OR problems have only one objective, or can we have situations where these problems can have more than one objective or goal. From this, we move to a topic called goal programming, whose basics we will see in the next lecture.