Conduction and Radiation

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Module No. # 01
Lecture No. # 41
Two Dimensional Steady State Condition

In today's class, we will look at Analytical solution to problems, where more than one dimension is involved in steady state.

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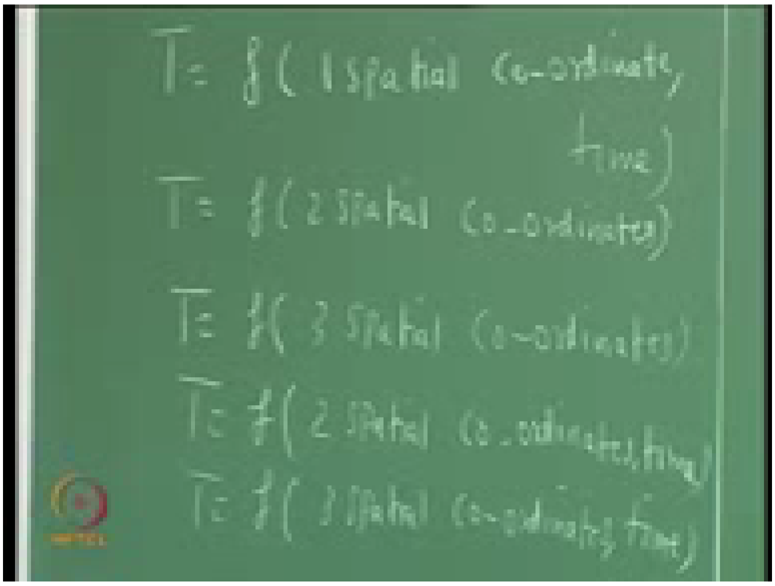

In conduction, you can have situations where temperature is... The first will be a transient conduction, where your lump capacitance cannot be used and your semi-infinite assumption also cannot be used. Temperature is a function of both the spatial and the temporal coordinates. I mean that it is a function of the spatial coordinates and as well as the time.

There is also a possibility, where the temperature is a function of two spatial coordinate. You can see that this represents an increase in complexity level. So, once you go to temperature, it is a function of three spatial coordinates and time. These spatial coordinates could be in the Cartesian coordinate, cylindrical coordinate or spherical coordinate. Beyond a certain level, it becomes extremely difficult to solve it analytically. That is why numerical methods have been developed, but still many of these problems are tackled using analytical method. Pc's came only in the 80's, before the advent of computers, lot of mathematicians, applied mathematicians and even in heat transfer, scientists like professor Ozisik have done and spent lot of time in applied maths. They have tried to get analytical solutions to these problems. How far they are relevant today, we do not know, but yet it is elegant because the temperature is found at every point in time and every point in space.

If it is a numerical method, you can make your grid; you can choose your grid such that you can get a temperature at any point, but you will not get a closed form solution, you will not get an analytical expression. If x, y, z and t are supplied, it will directly give the temperature, but there is still some elegance involved in this.

Now, needless to say, we will be considering one of the easiest problems. It is good for communication; it is good for the teacher and also it is easy for the student to follow. You can see that the asymptotic variants of the problem where, you don't really worry about the spatial dependents of time. It is basically called the lumped capacitance method of temperature; it is a lump capacitance method.

The other is a semi-infinite problem, where the thermal disturbance is only in the first few millimeters at the first few layers. For example, if this is 10 centimeter and in the early regime, if this body is at t i and suddenly the temperature is brought to t s. In the early regime, a large portion of the body will still be at t i. Therefore, some special techniques can be used. You can define a similarity variable, convert the partial differential equation into an ordinary differential equation and get the solution. That is much easier than solving the full partial differential equation.

The other thing is the late regime, where the whole body can be assumed to be at one temperature. You can work out the bio number, if it less than 0.1. You can use that assumption and the temperature is found to be dependent only on the spatial coordinate. In terms of dimensionless quantities, the non-dimensional temperature is just a function of bio number and fourier number. For all the other cases, non-dimensional temperature will be a temperature of bio number, fourier number and x by l or r by r c or r by r naught, where r is the radiant coordinate, r naught is radius of the cylinder or the sphere.

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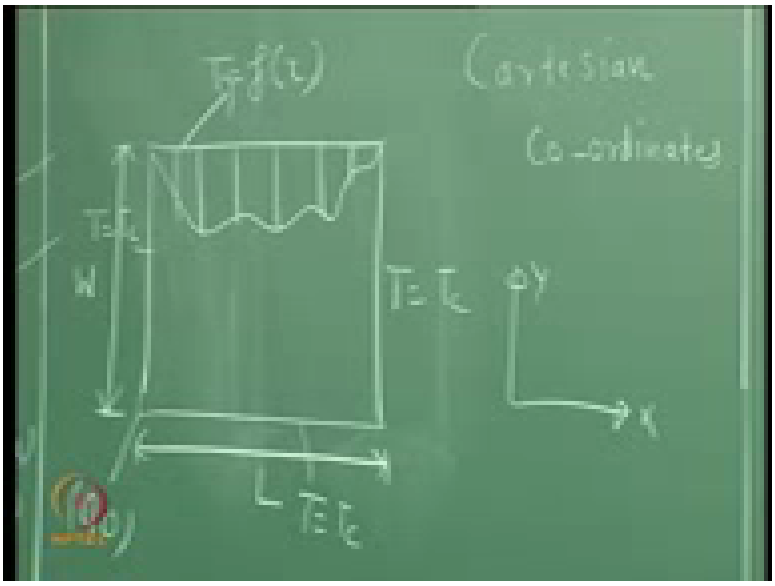

Let us look at temperature as a function of two spatial coordinates in cartesian coordinates. We will look at the laplace equation and typically the problem could be like this. We take a rectangular slab, which is infinite in the direction perpendicular to the plane of the paper. Typically, it has dimensions L, W. Now, I could have a prescription of temperature for simplicity. The three sides except the top are at temperature - t c; t c means cold temperature. At top, I don't need to have a constant temperature; I can have t as a function of x, which could be because of some electrical heating or something. It means that I can have something like t and it varies with x.

Now coordinate system is like this as (0, 0). When do we get this two dimensional problem? When can we consider this two dimensional conduction? If the slab is infinitely deep in the direction perpendicular to the plain of the paper, it is based on three sides by a very cold fluid, the high heat transfer coefficient. You can assume that the three sides as take on the temperature of the surrounding of the fluid steam, which is beading this. Otherwise, it is possible to have the general robin boundary condition or mills boundary condition of h and t infinity, which can also be tackled using the method of separation of variables, but that is going to be little more involved.

This is a precursor to many problems; the two-dimensional steady state conduction. For example, if there is a heat generation, you can have conduction as the heat generation in nuclear reactor and so on. If there is a chemical reaction, which is taking place, the first step is considering the laplace equation. If you add the heat generation term, it becomes the poisson's equation and so on.

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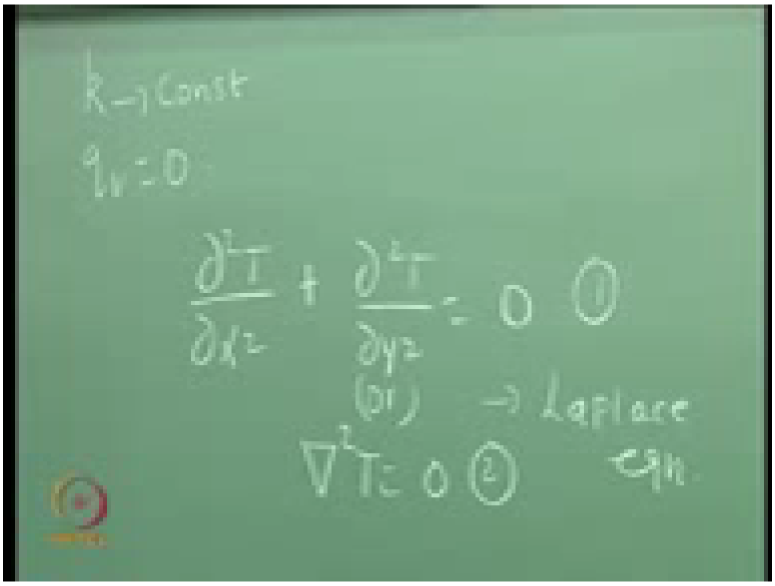

Let us consider two dimensional steady state constant properties, no volumetric heat generation. The governing equation for this problem, which we have derived in earlier class, where we looked at the general three dimensional heat conduction equation. You can remove unnecessary terms that are at steady state. Therefore, unsteady term goes, the heat generation term goes, the body is stationary. You can remove all those terms, so you will be left with. This is called the laplace equation and in compact form, you can write like this. Now, if you want to solve this analytically using a very powerful technique called the method of separation of variants.

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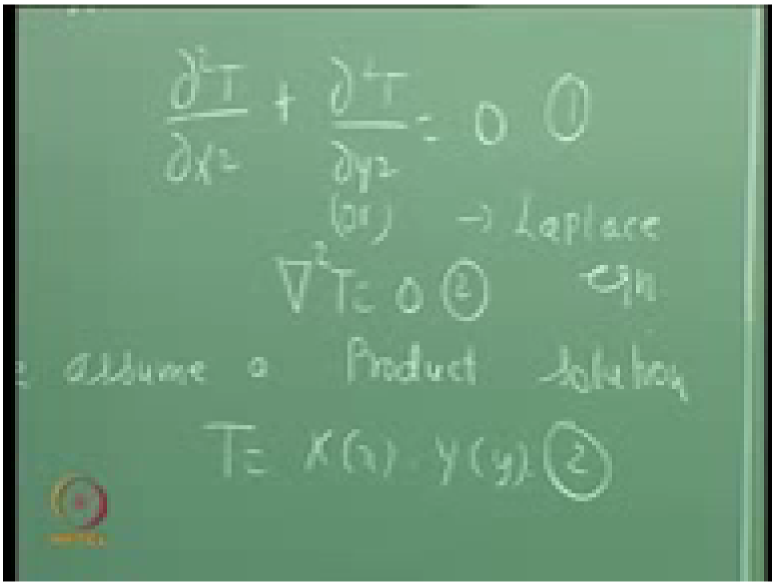

The method of separation of variables starts like this. We first assume a product solution that is t is a function of x and it is a function of y. It is possible if you have linear partial differential equations and all the boundary conditions are homogeneous. I will later explain to you that what this homogeneous boundary condition means, but this is a linear partial differential equation.

If you have terms like t dou t square by dou x square, then it is out. This method of separation of variables will not work. There is a difference, if you have x square into d square d by d x square that is not a non linear partial differential equation, but that is a partial differential equation with variable coefficients. If you have t into dou square t by dou x square is gone, then it is a non linear partial differential equation.

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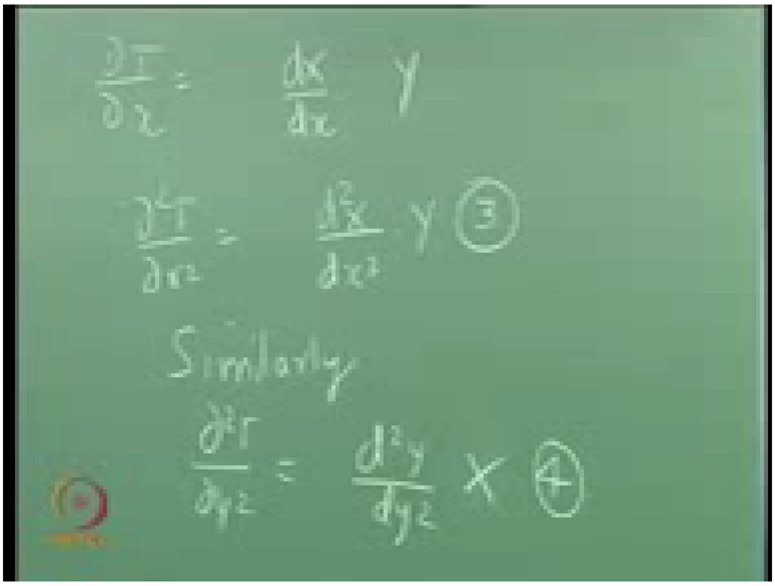

We first assume that a product solution exist and once you have a product solution, it is possible for us to get dt by dx. In order to be consistent, y is the solution and small y is the variable. Here, small y is the variable coordinate and y is the solution. Now, I can have d square t by d x square will be equal to … and similarly, this is small x.

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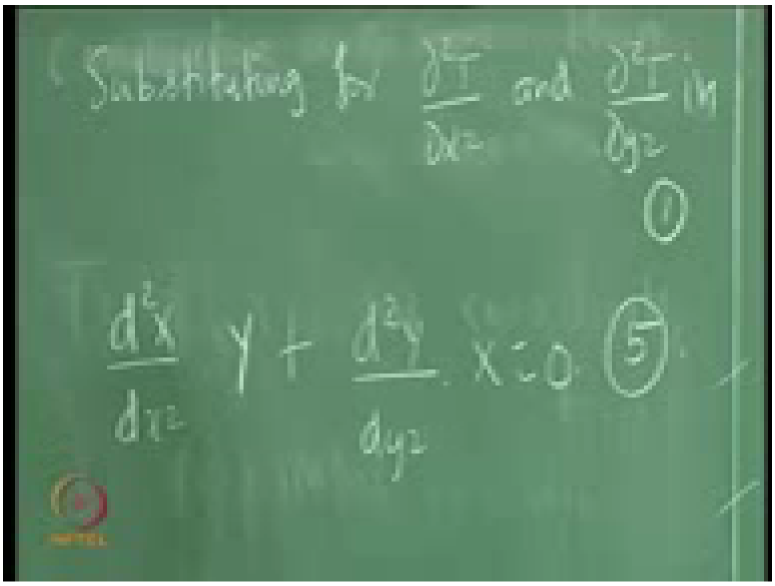

Now, we can substitute for d square t by d x square and d square t by d x square from equations 3 and 4 into equation, what we get? Is it clear up to this stage? Look at equation 5; equation 5 is very interesting because the left hand side is completely a function of x, the right hand side is completely a function of y. If a function, which is a function of x must be equal to the function, which is completely a function of y. Each of this must be equal to a constant; otherwise this will not hold good in general. Therefore, we say that this must be equal to lambda square.

Now, doubt arises, why should it be lambda square plus lambda square, can it be minus lambda square? There can be two doubts, why should it be plus and why not minus? Why you are taking it as lambda square instead of lambda? I am taking it as lambda square for convenience. Lambda is a parameter and lambda is a constant. I can choose lambda or lambda square lambda cube depending on my convenient. If I choose it as lambda square, I get it in a form, which is very convenient for me to handle. It is plus because of the boundary condition, which will become clear in a few minutes.

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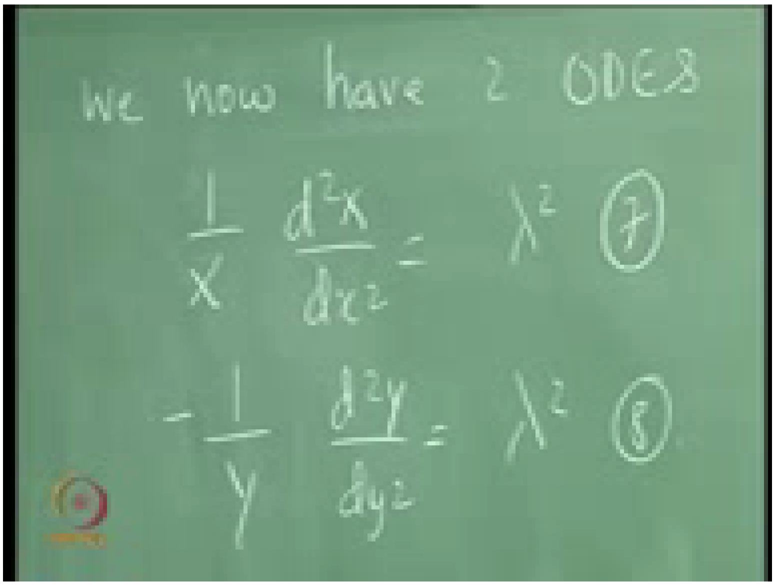

Now, can we solve? We now have two ordinary differential equations, dou square t dou x square dou square d dou x square became d squared x d… because we made a very major assumption that t can be written as a product of x into product of y. It is possible for linear partial differential equation with homogeneous boundary condition. What is homogeneous? I will come to it.

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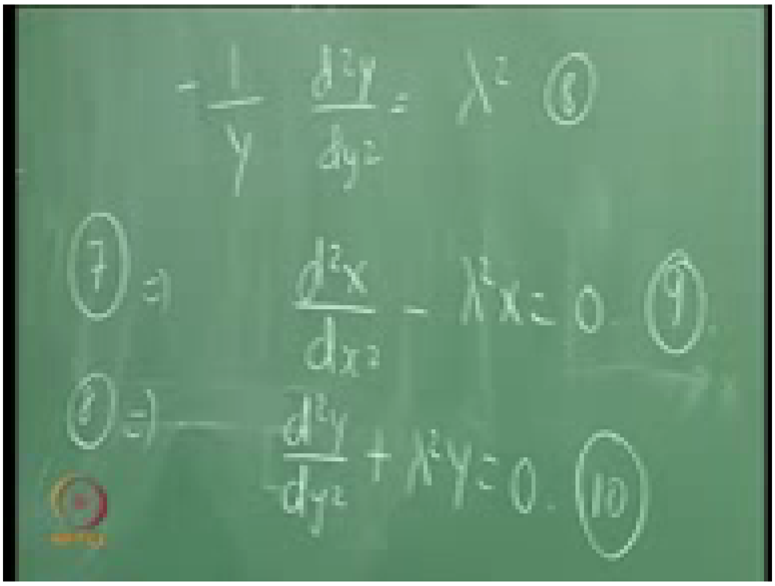

Now, we can solve this. Equation 7 can be rewritten and 8 can be rewritten as… If you put it as minus lambda square, what happens to this? What is the solution to that? e raised to… Now, let us look at the boundary conditions. Here, it is t c; here it is t c; here it is t c and here, it is f of x. Assume that it is uniform, now you expect that the temperature profile to be symmetric about x. Final temperature should be symmetrical because left side, right side boundary conditions are same; whereas the top of f of x is hot, bottom is cold. You expect the temperature distribution to exponentially decay in the y direction. If it has to decay in the y direction, you should get an exponential form of the solution in the y direction and you should get a periodic form of the solution in the x direction. Therefore, what we will do is nothing and I can change the minus here.

Now, let us look at this problem. Left side is t cold, right side is t cold, bottom is t cold. Let us forget about f of x, it is hot. If it is hot, let us say for the time being, it is t h -t hot. How do you think that the temperature in this direction at any section will be? How do you think the temperature will decrease exponentially? If you have something like this; this is like a finn equation, t square theta by d is minus m square theta. Finn equation gives an exponential variation. Therefore, if I choose or if I put this minus here, I am getting an exponential variation in x, which is unphysical. If this is 100, this is 0, 0 and I expect that this is the centre at any point. If it is 60 degrees here, it will also be 60 degrees here. I don't want an exponential variation in the x direction that is a wrong physics. I could have made an argument that I keep the minus here, but I want to use minus lambda square. Therefore, I make this plus and make this minus. Is it clear?

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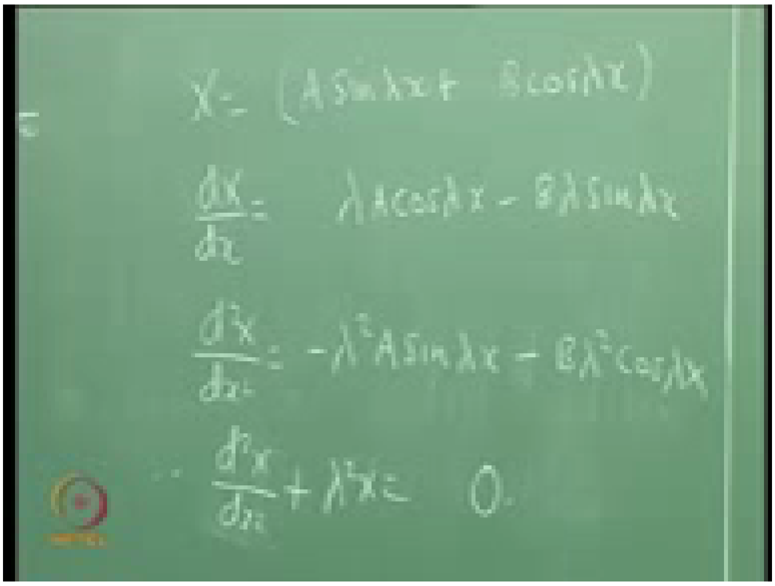

Now, what is the solution to 9? I don't like cos plus, I will have sin. So, the solution to 10 is. If you have forgotten your basic calculus, it is good to substitute it, substitute this back and we will do that. So, d x by… because the solution is a sin lambda x. If you multiply by lambda square, you get a lambda square sin lambda x; you get minus a lambda square sin x. Here, it is sin lambda x, you get b lambda square cos lambda x. So, here you get plus b lambda square minus b lambda square cos lambda x, here you get minus b lambda square. Therefore, that is correct and x equal to a sin lambda x minus b cos lambda x. It is indeed the solution to this ODE. So, we will keep that as 11.

By using an exact procedure, it can be proved that the solution to equation 10 is indeed what I have written on the board. This has to be small y, please note that cos h lambda y is e to the power of lambda y plus e to the power of minus lambda y divided by 2. Here, d by dy of cos h lambda will be sin h lambda and similarly d by dy of sin h lambda will be cos h. No interchange of minus plus sign will not come for the hyperbolic and in fact, when we change, please look when we change the minus from this to this. We were talking about an exponential solution, but I am not taking an exponential solution, I am taking a hyperbolic solution, is that correct?

Hyperbolic is already exponential, it has got its linear combination of the exponential term e to the power of m x and e to the power of minus m x. Why I am saying e to the power of minus m x? I am so obsessed with finn heat transfer, so that m is root of h b by k. So, theta by theta b is e to the power of minus m x and cos h m l minus x divided by cos h m l. Therefore, this is where d square y minus…. We have got the general solution to the problem, obviously it is an elliptic equation because for the method of characteristics, you can prove that b square minus 4 e is less than or equal to 0 or greater than or equal to 0. You can figure out whether a partial differential equation is elliptic parabolic or hyperbolic.

Once you have established that it is elliptic because it is second order in x and second order in y. It supports two boundary conditions in x and two boundary conditions in y. So, you require four boundary conditions for closing the problem. Mathematically, the four boundary conditions are indeed specified on the four side; whereas the left, right and the bottom are at t c, the top is at a temperature of f x. Therefore, four boundary conditions are specified and the problem is mathematically closed. It is possible for us to completely solve this problem.

We have four boundary conditions. If we apply these four boundary conditions, we have got four constants a, b, c and d. You can straight away evaluate it and get a, b, c, d and you can get the solution. So far it is very easy, but getting a, b, c and d. It is indeed the trick in the method of the separation of variable. So far it looks so simple, the general solution is so easy, but wants to attack and get the constant. You will have to invoke some higher mathematics; you have to know that we have to just get into orthogonal function of the property of orthogonality and we have to see all that.

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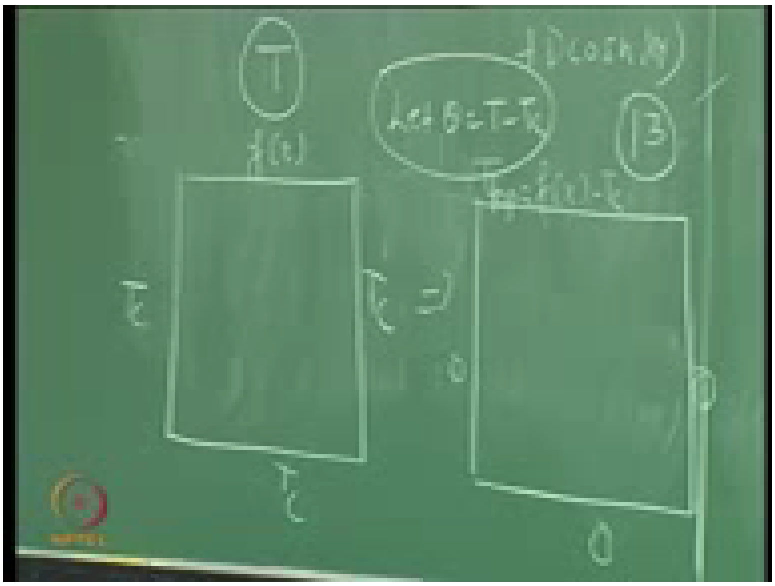

Therefore, what is the equation number 13? Now, t is the boundary conditions on the variable t and t is the temperature. Now, let me define (0, 0). If you want, you can say t top equal to t top of x. Therefore, it is equation number 13. If t minus t c equal to theta then this will also be a solution. Instead of t, if you define theta, it will be also be a solution except that a, b, c, d will be adjusted because after t c is a constant because dt by dx is the same as d theta by dx.

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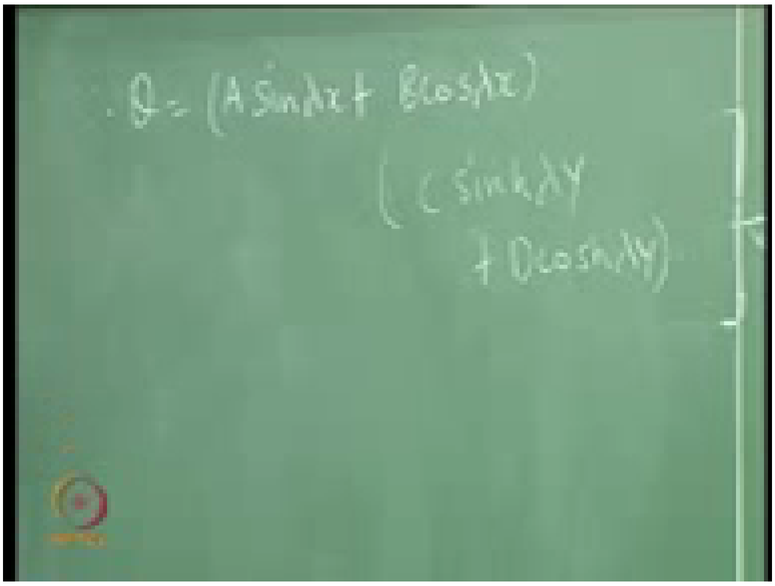

Therefore, theta is the general solution. Now, this x equal to 0; this x equal to l; this y is equal to 0. Here, what is y equal to? What is the height of the plate? Here, at x equal to 0, theta equal to 0. This is for all y, which means everywhere.

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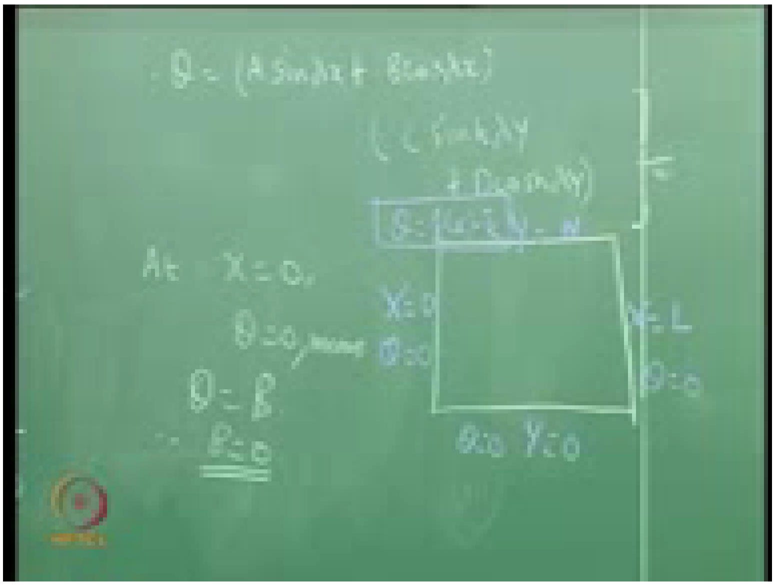

Here, x equal to 0; it is equal to 0.1 l, 0.3 l, 0.5 l, 0.6 l, 0.7 l and x is equal to 1 l. Now, x equal to 0 means this is vertical line and all y has to be valid. Here, x equal to 0 means, this c and d are untouched. Here, x equal to 0 means sin of 0 is 0, cos of 0 is unfortunately 1. Therefore at x equal to 0, theta is equal to 0. It means theta equal to b and therefore, b equal to 0. If b is not 0, we are having problem because cos of 0 is 1. If only that b is sin lambda 0, anyway sin of 0 is 0. First two terms are 0, second term whatever we have substitute value of y, c, d, theta will become 0. So, it satisfies the quadratic equation; this is the homogeneous boundary condition. I am saying that it is not possible to solve this equation for t c, t c, t c. This method of separation of variables can handle one in homogeneity that is why homogeneous means very crudely we make Dirichlet boundary condition to zero.

Therefore, I am defining new theta t minus t c such as it is (0, 0), (0, 0), (0, 0) and I am making on the three sides. Therefore, I can do some magic like this because I cannot invoke many of these things, if it is not 0. The method of separation of variables of numerical technique will work.

In method of separation of variables, for example, if you have a problem, where this is 100, 45, 35, 25, so I will subtract 25 and make it into an equivalent problem. If three of the boundaries are having a temperature not equal to 0, then I will split the problem into three or four.

In each of these problems, I allow only one wall to be at the temperature greater than 0 that is called super position; otherwise this technique will not work because you can appreciate that because I am out of a, b, c, d. I am keeping on cancelling something; it is not possible. If I do not have the zero boundary condition, it will become very messy for me.

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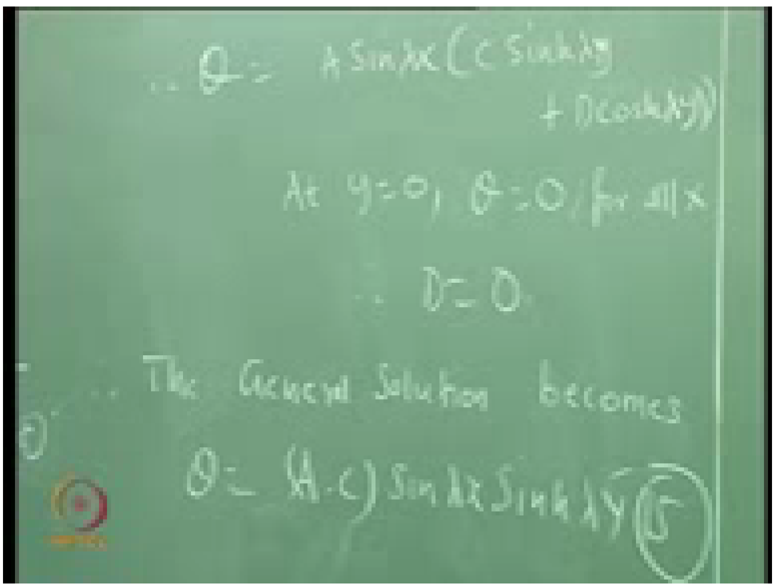

Now, at y equal to 0, theta equal to 0 for all x. What is that? That is the bottom wall. Before this, we will just write the solution. Please use small y, sometimes I am not consistent, so be careful. At y equal to 0, theta is again 0 for all x. You do not have to worry about a sin lambda x for any value of x. It should work and a cannot be 0. If a is also 0, then there is no problem, you have to go home.

Here, b is already 0, if a is already 0, theta is 0 everywhere, then where is the problem? So, a cannot be 0. Theta is equal to zero, it means a cannot be 0, it should be valid for any value of x. What is sin h of 0? It is 0. Therefore, c can take any value; c is a free bird, but d cos h of 0 is equal to 1. Cos h is equal to 1, but theta has to be 0 for any value of x, therefore d must be 0. Therefore, the challenge is to get the product of a c and call a into c as e. Is that all you have to get from the boundary condition? You have got two boundary conditions, is there only one constant to find? You have to find lambda also. So, two more boundary conditions are there, so with those two boundary conditions, we can find the product of a c as well as lambda. This becomes slowly more and more difficult.

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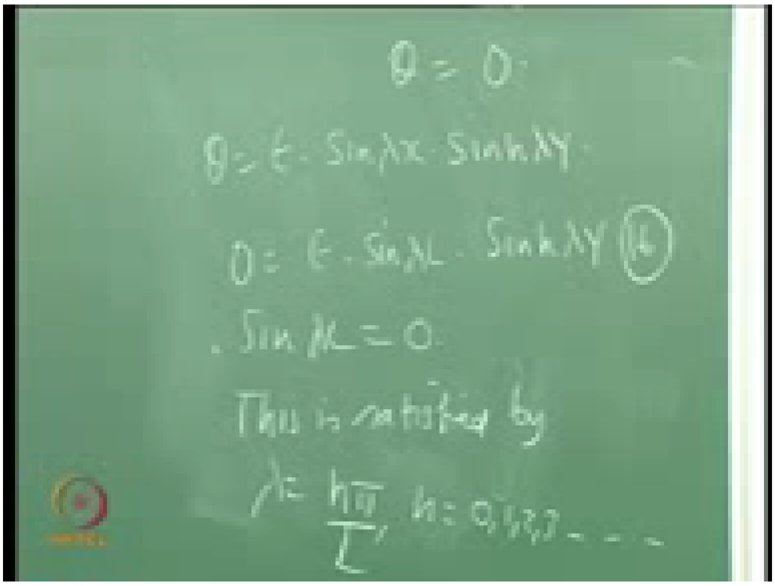

At x equal to l for all y, theta equal to, very good. I can call this as e. So, theta is equal to e into sin lambda x into sin h lambda y; this is a general solution. Zero is equal to e into sin lambda l into sin h for any y. Now, I am substituting x equal to l here and I am leaving this here. Again, theta equal to 0, if e is equal to 0 and again everything becomes 0. If e is 0, temperature is zero that is previous solution. It is not a solution to the problem e cannot be 0, it should be valid for any value of y. So, this cannot be made -0 because y is a variable in the problem. Therefore, sin lambda l must be 0 and this unfortunately is satisfied by a infinite f create some (( )).

Next step, there is a property theorem, which says that in such a situation when lambda has succession of values. Each of it satisfies that sin lambda equal to 0 and the general solution is given by summation of all these solutions.

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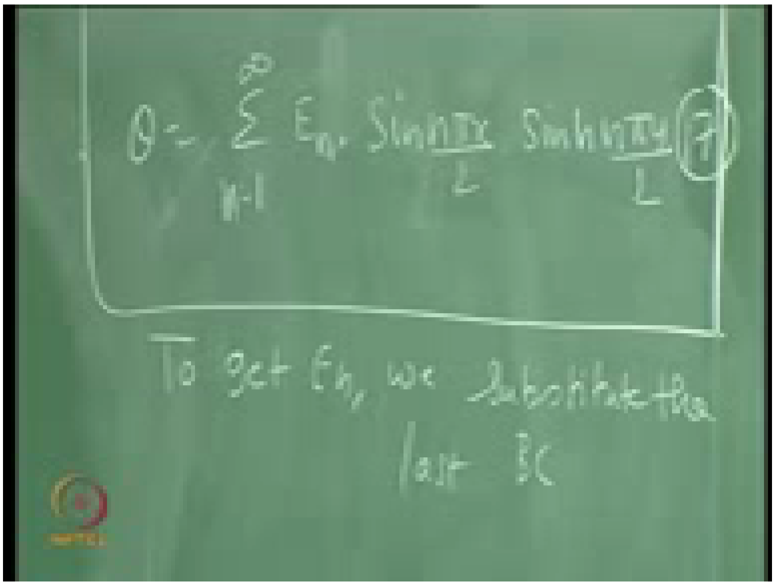

Therefore, sin h or I can replace this by… Please note, you will not get sin h n pi by w. So, the lambda is coming because of the l; it is a periodic boundary condition in l. In w, it is a exponentially decaying function. Where it can be satisfied? Why should e 1 be equal to e 2 equal to e 3?

Here, e is a constant and since it is satisfied by the succession of value, I cannot make a fatal assumption that everything is only one e. There is e 1, e 2, e 3 or e 4. Therefore, this is the solution to the problem; this is an infinite series. So, we started in very simple way and now it has come to this stage.

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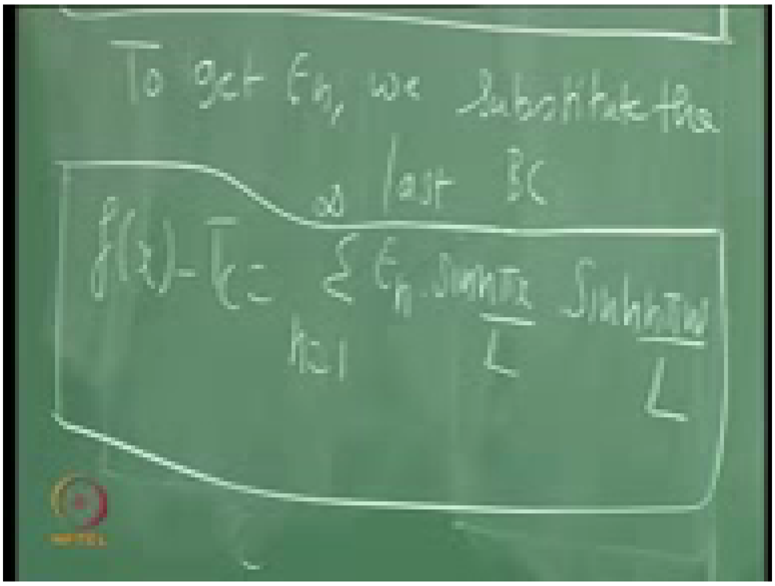

Now, to get this e n, you have to substitute the last boundary condition that is at the top and there is non-homogeneity.

It can be anything and not linear, but it can be general; it can be a periodic, it can be a sinusoidal function or a cosine function and all that. I am just giving for a general f of x, but I am going to reduce for a special case, where it is only one temperature. If it is non- uniform and exponentially decaying, then we are in trouble. It should be symmetric about the centre and that is understood as it is coming from physics. Now, we substitute the last boundary condition, so f x minus t c is. This is invariably specified to you in the beginning of the problem. You know what this value is and it can also be a constant value. It could be just t x minus t c, it could be 50 degrees or for a problem in which, you make this as t top, t left, t right, t bottom. I can make theta equal to t minus t c, so I can make this as… So, this could be just that 70 and left side is known.

Here, sin is n by w i will tell what the value of w is. You know what is l, it is square slab w by analysis and it is equal to 1. Here, n is nothing but it is 1, 2, 4 to infinity, but this boundary condition is valid for all values of x from 0 to l. If it is constructed in this way, is there a way to find out e n? What is the problem? Is there a way to find e n? If you don't know what is orthogonality, you are stuck here and you cannot proceed further. We have to invoke the property of orthogonality, which we will see in the next class. Before that I will give you a sneak peak of what is orthogonality.

You know that sin x and cos x are periodic functions. Lot of mathematicians have done research on expressing any function as a sum as sin series or cosine series. If any function f of x can be expressed as sin series or cosine series; this f of x minus t c can also be expressed as a sin series or cosine. If it is possible for us to represent it as a sin series, already there is a sin, which is coming here with that and there is hope for us to find e n. Don't worry about this sin n by x pi equal to w, it is just a constant. You can make e n into sin n h by w by l is e n dash e dash of n.

Now, the orthogonality property, where any function can be represented as sin series or cosine series and this is called the fourier series. If this is the fourier series and from the property of fourier series, we have to get the value of e n, which we will do in the next class.