Conduction and Radiation

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Module No. # 01
Lecture No. # 42
Analytical Solution for Laplace Equation

In the last class, we started with the method of separation of variables for the two-dimensional steady state conduction in Cartesian coordinates and we tried to solve the Laplace equation. We first proposed that a product solution exists that is t is function of x into function of y, which can be separated out into two solutions. That is possible only if it is partial differential linear equation and when you have homogeneous boundary conditions and so on. Now, we got the general solution, which had four constants. We started reducing, we started evaluating the constants one by one. Finally, we came to a stage where we got the solution as an infinite series and the constant e n needs to be evaluated.

I gave you a glimpse of what the e n is eventually likely to be because any function f of x can be represented as a sum of sin series or cosine series; this is what fourier series is all about. Therefore, there is some n pi x by l on the right side and all that. There is hope for us that we can get it in the form of a… We can use the series expansion and evaluate e n. Once you evaluate e n, then depending on the accuracy you want, you can stop your solution with e 1 or e 2 or e 3 or e 4 or e 5 up to e 10 or e 15 terms. Like grid independent study, you will check how the solution changes to n number of terms in the expansion. Usually, 5 to 7 terms are enough; first 5 to 7 terms are enough to get a solution.

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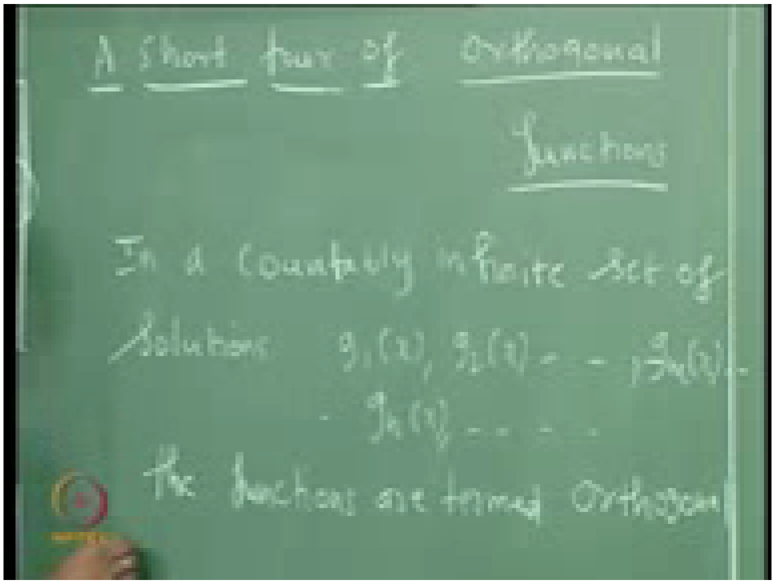

Now, because it involves some series expansion and it involves a property of orthogonality, we have to study about orthogonal function. I will call this as brief 15 minutes excursion and a short tour of orthogonal functions. In a countably infinite set of functions that is the way the mathematicians call it. It is solution g 1 of x, g m of x and g n of x. There is basically some mathematical jargon; in countably infinite set of solutions, there will be a lot of g 1, g 2, g 3, there is g m and there is also g n. It does not stop with g n, but it goes beyond g n. So, in a countably infinite set of solutions, the functions are termed orthogonal in the interval, a less than equal to x and less than equal to b.

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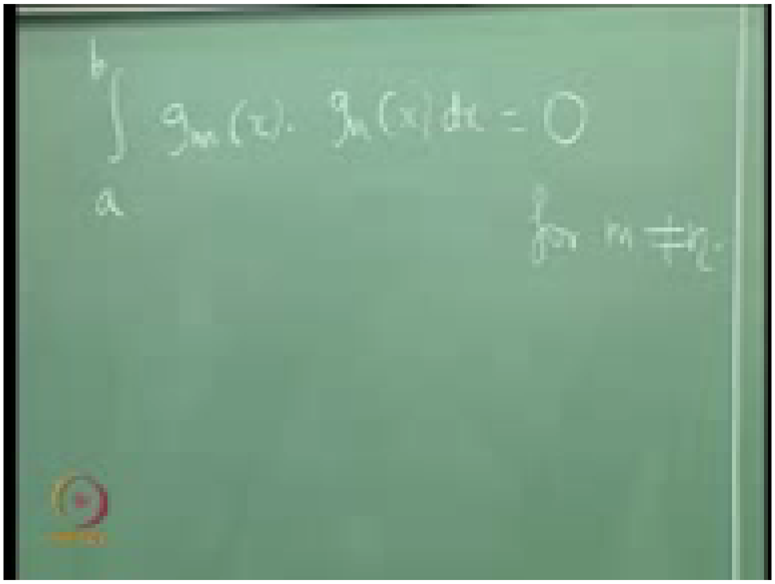

Integral a to b into g m of x into g n of x is equal to 0 for m not equal to n. We say it is orthogonal, we will say hard work and here are orthogonal to each other means that you will not work hard. If you take these two functions, their product becomes 0. It means they are orthogonal that means they do not like each other that is why this orthogonal is also used figuratively. Punctuality and somebody are orthogonal to each other means he is never punctual. So, g m and g n are orthogonal because whenever you multiply that integral, it becomes 0 only under the special case, when m equal to n has a finite value.

Now, what is the big deal about it? It is orthogonal, they found that some function g1, g2 are orthogonal, what is the big deal? If f of x is an arbitrary function of x, we are exploring the possibility of representing f of x as a linear combination of orthogonal function. The idea behind defining this is basically to see whether it can be used. So, where is it used in mathematics? In applied mathematics and engineering, take any function f of x that is general arbitrary function f of x. We want to see whether by using this orthogonal function g 1, g 2, g 3 and any function f of x can be represented as an infinite series. Here, f of x is different from g 1, g 2, g 3. They three are all certain functions, which exhibit orthogonality property. If you want some peace of mind, now itself I will say sin n by x by l is an orthogonal function. All functions are not orthogonal functions, there are certain functions, which are orthogonal, but our idea is to have a general function f of x. We want to represent f of x as a combination of orthogonal function that is I want to represent any f of x as a combination of sin pi x by l sin 2 pi x by l, sin 3e pi x by l and so on.

If I am able to do that somehow I will link it to this formula and get e of n, so that the problem get solved and till I find e n, my problem is not solved.

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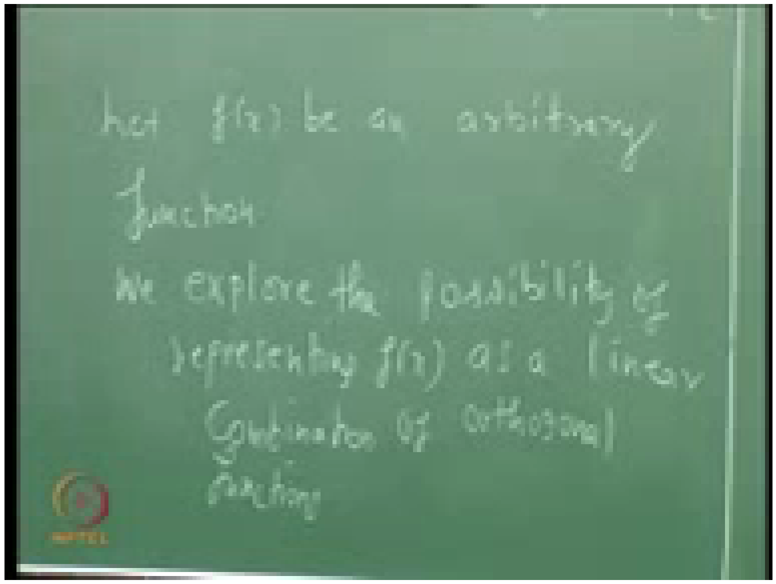

Let f of x be an arbitrary function, we want to explore the possibility of representing f of x as a linear combination of orthogonal function like this. What numbers you want to give? Shall we go ahead with the global numbers? Shall we call this 19? So, we will go ahead with the global number.

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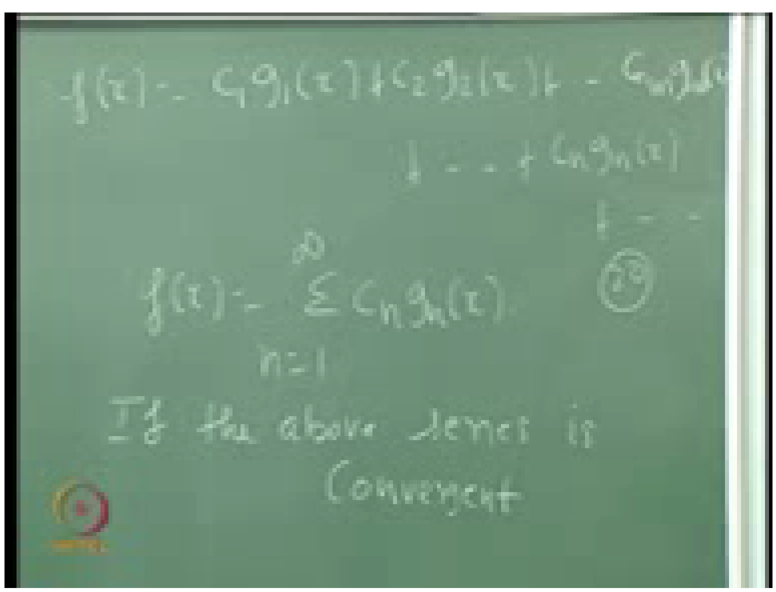

Now, let us say I want to make this f of x as g 1 of x plus c 2 into g two of x and so on upto c m into g m of x and so on plus g n into c n g n of x plus continuation of x. So, what I basically want to do now is it means f of x. Watch carefully, what is the logic we are trying to use any function f of x? I want to express it as a linear combination of the orthogonal variables. Please look at this equation 20, f of x is represented as an infinite series sigma n equal to 1 to infinity; it is c n g n of x. It is very similar to equation 18, except that we have the same. Instead of f x, we have f of x minus t c, but t c is nothing but a constant. So, left hand side is the same and in right hand side, I do not worry about e n into sin n by n.

Here, sin n by n w by l is a constant, I do not know whether sin n by x by l is orthogonal function or not that we have to check. Therefore, equation 20 is nothing great, but from equation 20 and using equation 19, if it is possible for me to extract c of n that means I have solved this problem. If I get c n there that means I will get e n here. Now, how do I get c n there? What do we do for that? We have to integrate. First of all, the above series must be a convergent series. If it is a divergent series, we have to go to home. We assume that it is a convergent series. Let us not get into the details of how mathematicians call it as, what are the mathematical properties for making it convergent series or whatever. According to me, numerically if it is convergent, if I take the first five terms, I get solution as 0.8. If I take eight terms, I get a solution as 0.82. If I get eight terms, I should get 0.85 like that. Suppose, if I evaluate something in integral and if I use five terms, I will get 100. If I do eight terms, I will get 3000. If I use ten terms, I will get 5000, then it is not a convergent.

With increasing number of terms, it should asymptotically reach some convergence. Therefore, there is some peace of mind for us. After eight or nine terms, if I add more terms, it will not materially alter the solution, then it is heading towards the convergent that is e 1, e 2, e 3 e 4 or c 1, c 2, c 3, c 4 will be very important compared to c 11, c 12, c 13, c 14 and later terms of the series will not contribute much to the summation.

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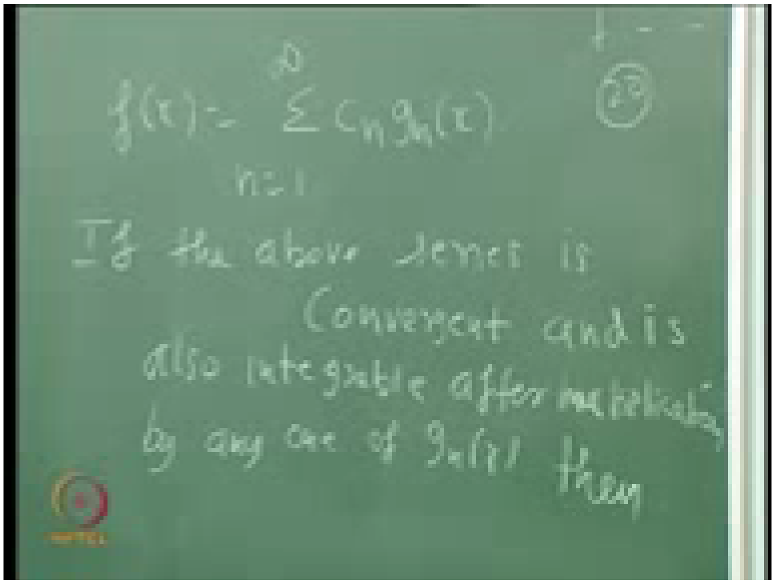

If it is possible that the above series is convergent and integrable and after the multiplication I will tell you what it is.

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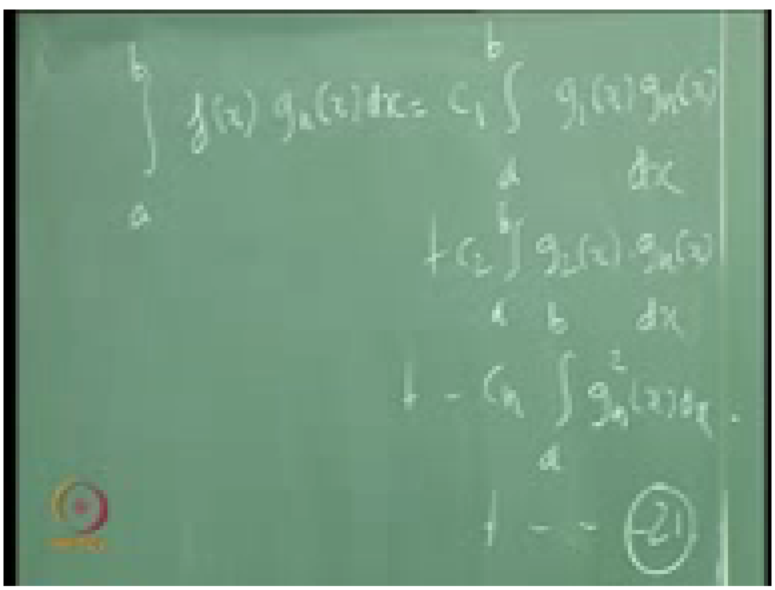

Integral f of x into g n of x into dx, I can replace f of x by c n g n of x. First term will be c 1 g 1, second term will be c 2 and third term is c 3 g 3. So, this will be c 1 and nothing great. I just did f of x g n of x for f of x. I will substitute it from equation 20, what is the big deal? The big deal here is g 1, g 2, g 3, g n, g m and they are all orthogonal functions. Therefore all of the integrals on the right side, except the g n squared of x will all be 0 because it satisfies the orthogonality property for any value of m not equal to n, the integral automatically becomes 0. That comes from the property of orthogonalities; it is a nature that some functions satisfy that because they are satisfying and mathematicians have used it to solve some equations. So, this is 0; this is 0 and everything is 0 except this one non-zero term that exists on the right hand side.

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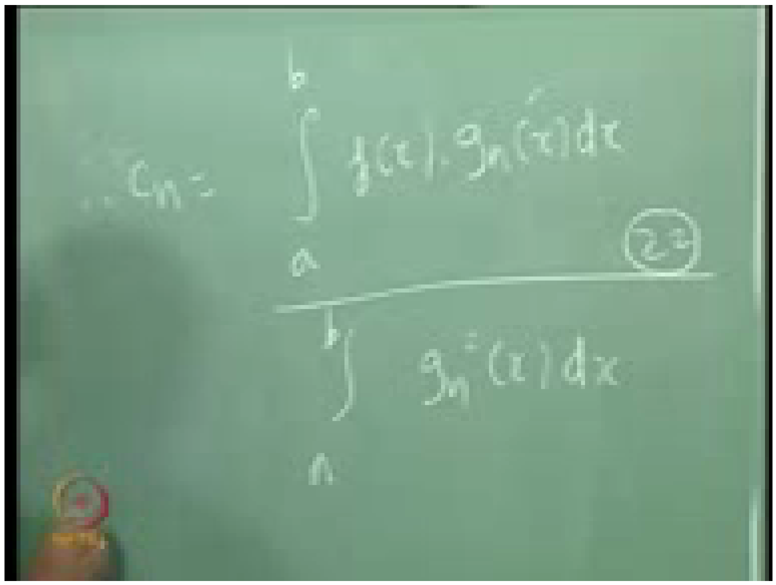

Therefore, c n is nothing but a to b that is you substitute n equal to 1, you get c 1; n equal to 2, c 2; n equal to 3, c 3 and you will get all the c n. If you fundamentally solve the problem of expressing any f of x as an infinite series and now by exploiting the properties of orthogonal function, you are able to get c n. It means that you have essentially solved the problem of steady state conduction in a two-dimensional slab. It is the same f of x g n of x and now one thing remains whether sin n pi x pi by l is orthogonal or not. If that is satisfied, then my g n is nothing but my sin n pi x. Do not worry about this; e is a constant. What are these?

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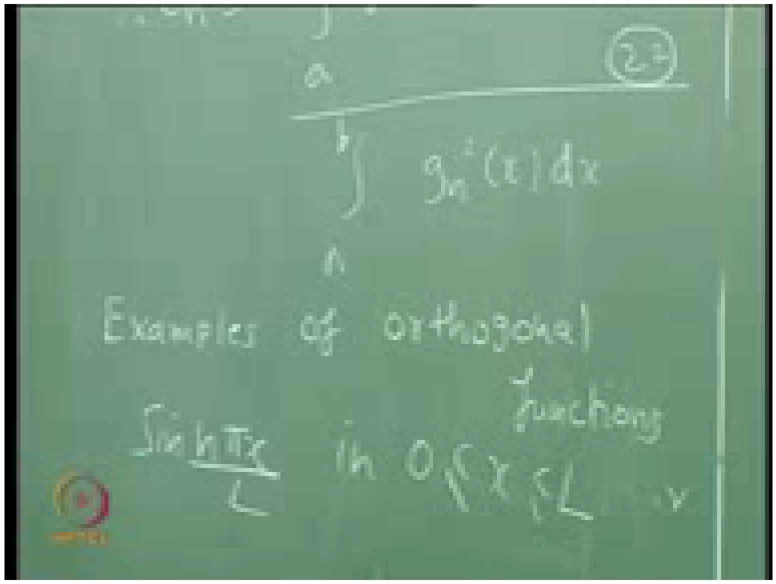

Can we give some examples of this orthogonal functions? We will see the short tour of orthogonal functions. Before it ends, let us take examples of orthogonal functions. Now, we want to see whether it is orthogonal or not.

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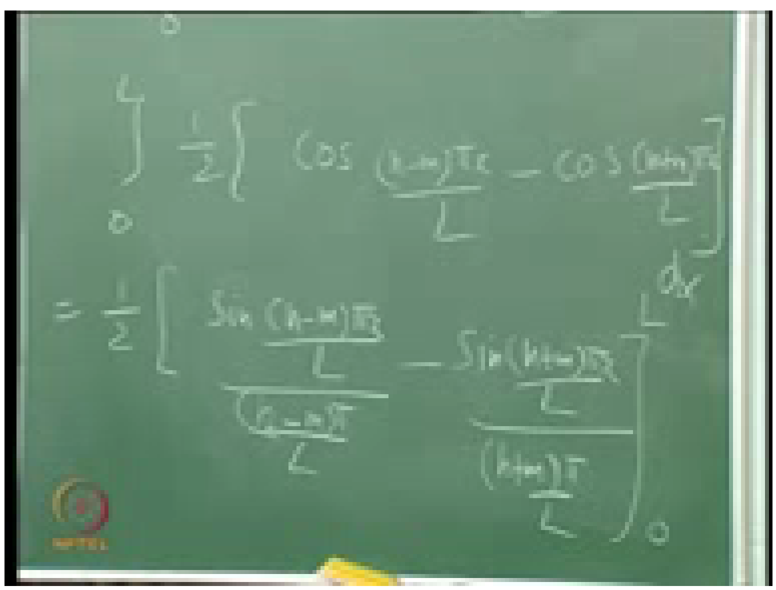

Therefore, I have to check in the interval 0 to l. It should satisfy two things - sin n pi x by l into sin m pi x by l integral is 0, until it becomes 0. When m is equal to n, it should be non- zero. If that is also zero, game is closed. Therefore, there are two things, which it has to satisfy. Now, can we do this? What is sin a sin b, cos of a minus b minus cos a plus b? What is the integral cos n pi? What is integral of cos? It is sin. Here, d of cos is minus sin, so integral of cos by something will come.

Now, it is n minus m pi x by l and all that. Let us not worry about the denominator; anyway it won't trouble. Both the lower limit are 0 and in upper limit, n is a number, m is a number. Here, n minus m must be 0, 1, 2, 3 and everything is 0. If everything is 0, sin n pi x by l is orthogonal. Let us do the other one, when m is equal to n. So, this is a big zero. There are indeed functions represented by general g 1 of x, g 2 of x upto g n and g m of x, where integral a to b g m of x into g n of x equal to 0 for m not equal to n.

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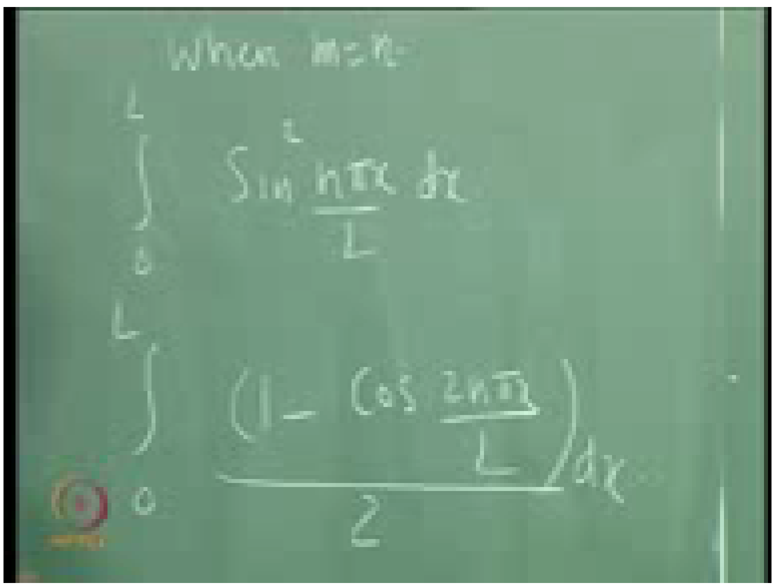

When m equal to n, let us get rid of m, we use only n. Therefore, 0 to l sin square. What is sin square theta? It is 1 minus cos theta by 2, very good. So, is it 1 minus the whole by 2? It is equal to... Here, sin of n pi sin of 0 and everything is 0. So, this is nothing but l by 2. What is l by 2? Half-width of the slab and that integral is nothing but half the width of the slab, if you translate it to our original problem. Therefore, here is 1 sin n pi x by l that is ideally suited for us. Sin n pi x into sin m pi x by l integral 0 to l is 0. When m is equal to n, it is non-zero.

Therefore, it is possible for us to exploit this property of orthogonality. Fortunately, for us, sin n pi x by l is already updating here and our f of x is also here. Therefore, it is possible to express f of x as an infinite series using this sin n pi x by l. Comparing the left hand side and right hand side that c n is nothing but the e n. This is the way to solve this laplace equation using a method of separation of variables. So, we close this general discussion on orthogonal function and now we rearrange that f x minus t c and all that. Please use whatever funda or logic I have given. Please get e n, substitute it into that. We will reduce it for the special case, when f x is t hot, so that f x is a constant, then we will get that general expression. We look at the general expression, we will look at the physical insight whether the temperature is decreasing in one direction and it is periodic in the other direction. What happens if more than one wall is at a temperature, which is non-zero? After discussing all this, we will close, but before we close, Somerjit has solved this problem using a fortran code for the simple case. We will see how the isotherms look like, so that is a program. Now, please use this l by 2 business; this concept of orthogonality and evaluate e n for the problem and question.

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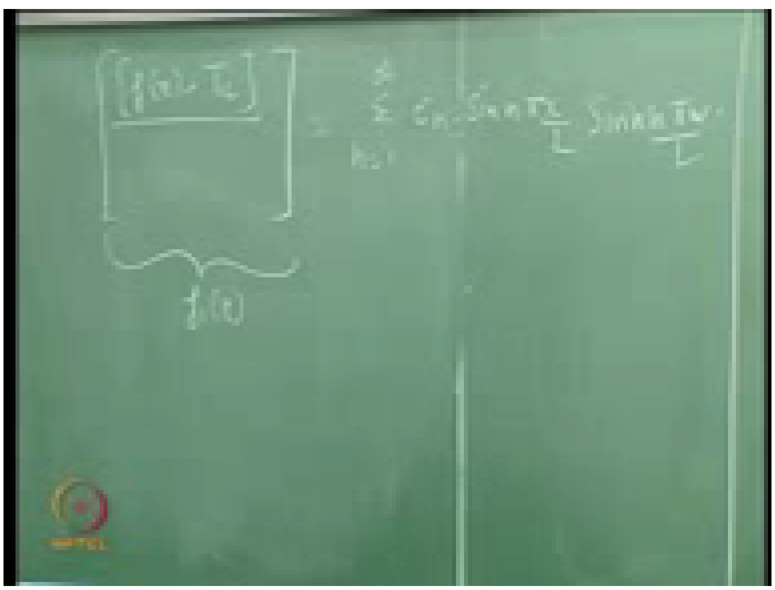

This is like your v code, we do not call it f dash; no f dash is a problem, so f 1. Here, e n must be f 1 of x into g of x. What is g of x? No, we cannot do this. Let us do this, so this comes here. We have to keep it here because this fellow will keep on growing and e is dependent on…

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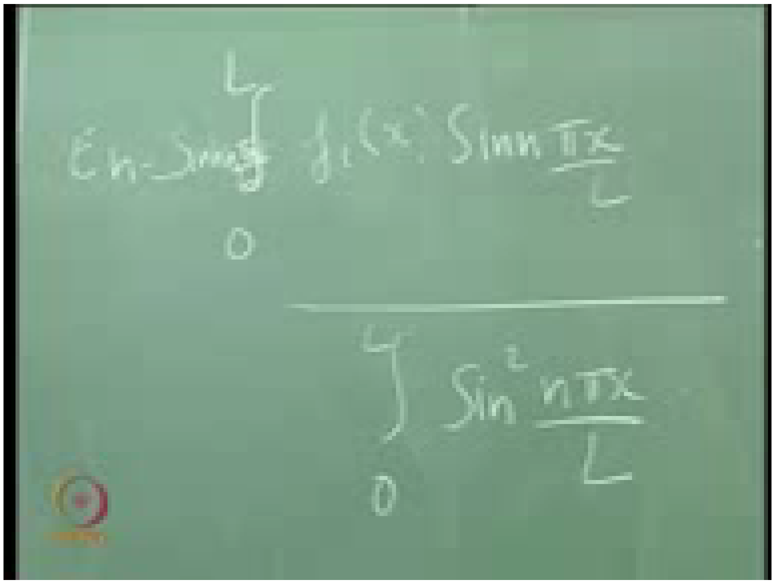

For this, e n will be equal to f of x into, take that sin h n pi w also. Please evaluate, so e n will be equal to f 1 of x into sin n pi x, correct that into that sin h n pi x that is coming. That can be taken out here, but this one should come because that won't come in integration from 0 to l divided by 0 to l sin n pi x by l. It is not coming in the denominator, is it? Are you clear about that sin is not coming in the denominator? Is sin n pi is in the numerator or the denominator?

Why? What is that sin square.

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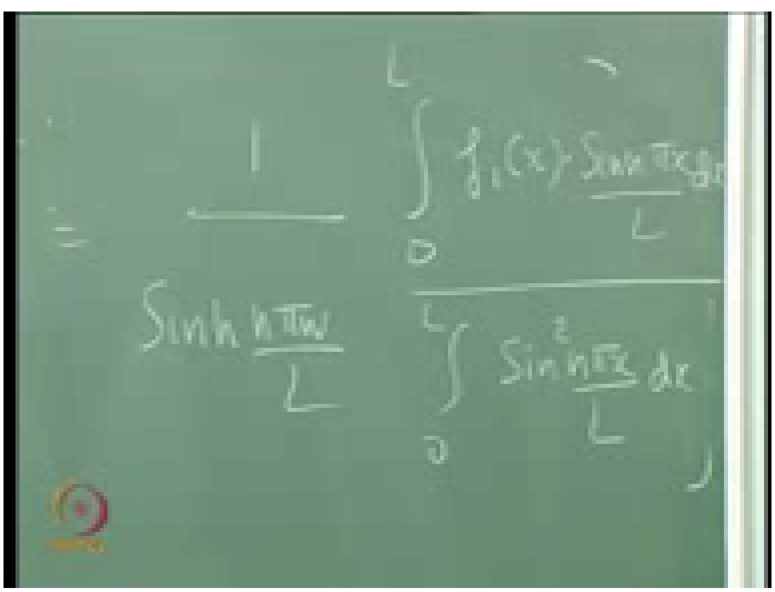

Is that clear, what he is saying? Our g n is not the sin n pi x by l into sin x because that fellow has got the he unfortunately it is having n. So, it is multiplying the numerator and also multiplying denominator by g n square, but fortunately for us, it can be taken out of the integral. So, the integration is with respect to x. correct We will cancel this, so e n will be equal to denominator is what? It is already evaluated as l by 2. Now, if you go on like this, it becomes very difficult. Now, I will say that f of x minus t c is constant. Somewhere you have to make that otherwise you have to tell what is this f of x? You have come to this stage, where you have to tell that.

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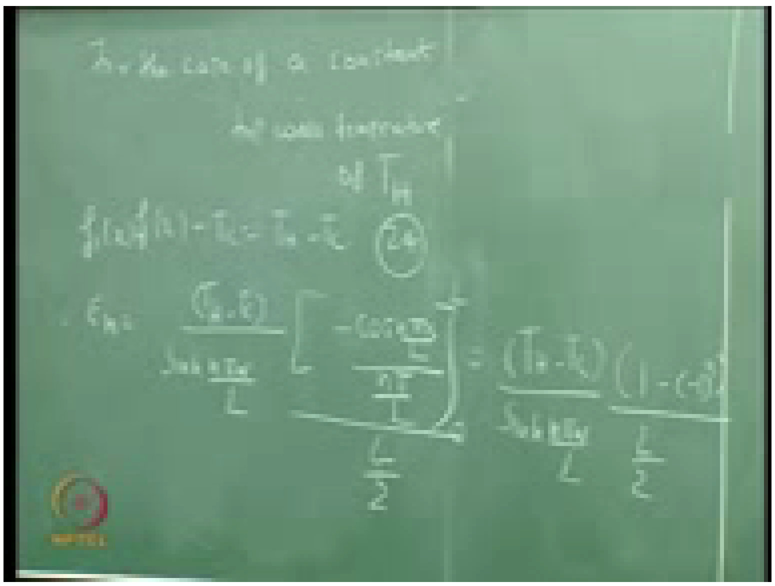

For a constant hot wall temperature of t H at the top, t c is the cold temperature at the three sides. Now, we have now stopped at 20. In orthogonal function, what was the equation number 20? So, this is 24. Therefore, I can pull that out and numerator, integral 0 to l sin n pi x by l. What is integral sin n x pi by dl? So, the lower limit is 1 for the numerator. Cos of 0 is 1. Upper limit will be keep on changing, why? Odd and even will change that is all silver tea cups. 180, 360, 720 and for that they will alternate. Therefore, we can call this as and you know that all silver tea cups. Let it be something like this. Is that okay that l should get cancelled, is that correct? I put minus 1 to the power of n that is it will alternate and it becomes 0. So, 2 will come and l is it getting cancelled. What is happening to l? Here, you are getting l by 2, is it getting cancelled or becoming l square? Tell me, I am not getting and I am not able to cancel it. Where is it in the numerator divided by n pi by l? So, n pi by l get cancelled.

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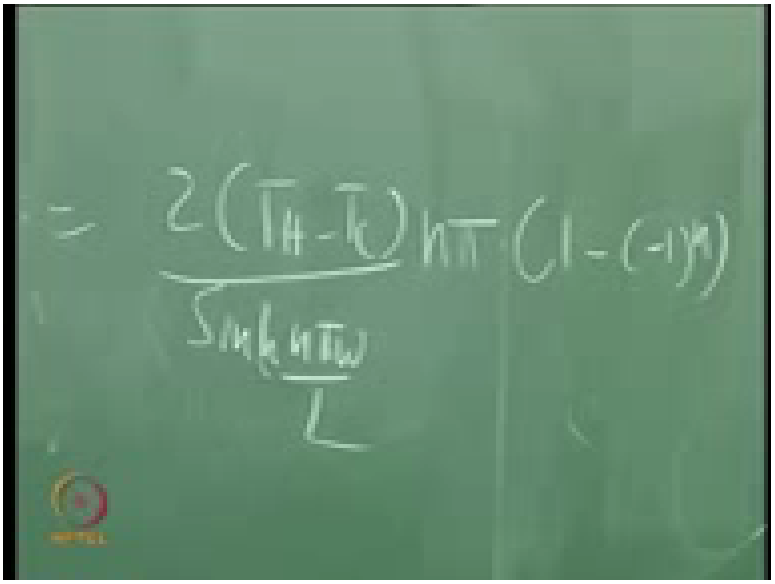

Here, L is cancelled. So, what you get? 2 t x minus t c n pi is there. Are you sure you want to learn the method of separation variable? So, these are all the stories to get e of n. Now, substitute your e of n into the original solution.

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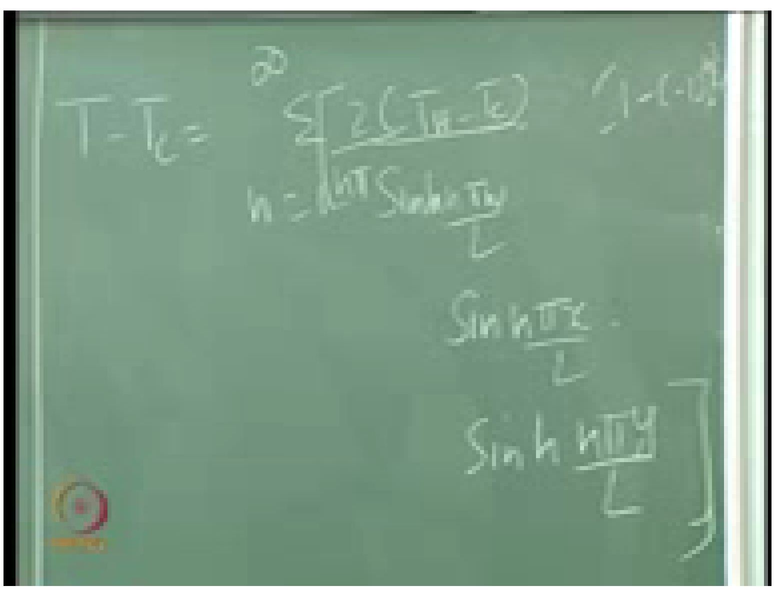

Therefore, t minus t c is equal to... I am substituting it to the solution 18. This 25 into I am not sure about that n pi is coming. Is it okay? Where is it going? Yeah, 2 by n pi because I was not happy with this sin h n pi y by l and do not put sin h into w by l that is there here and let us solve this.

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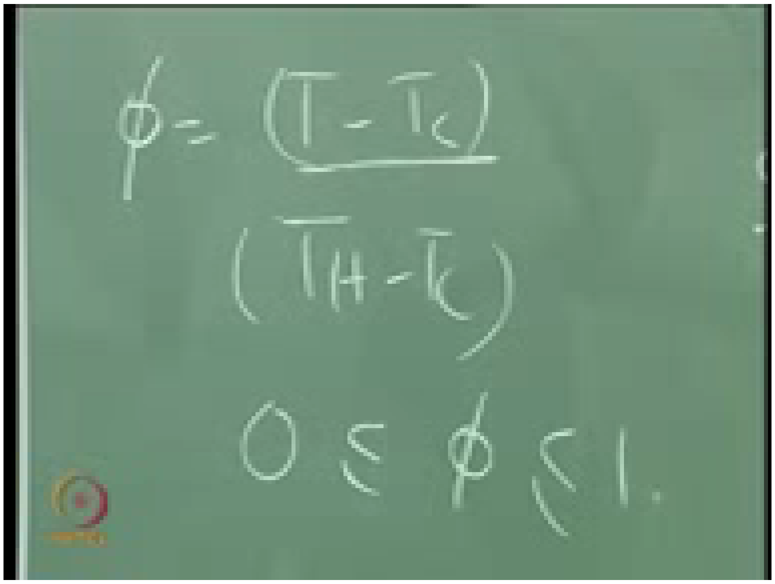

Now, let the dimensionless temperature be t minus t c by t h minus t c. So, this dimensionless temperature p varies between 0 and 1 in the problem.

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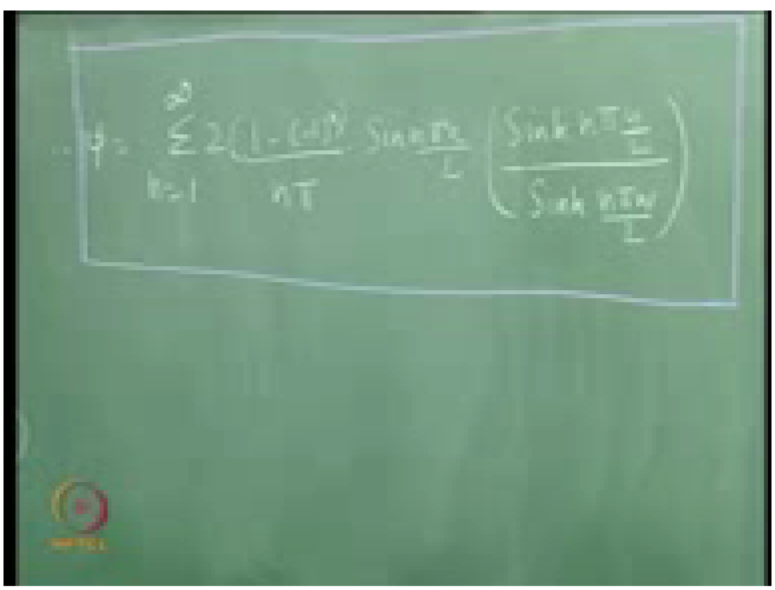

We can put that two there. No, it is the constant; t h minus t c is a constant. Everywhere in all the terms, I can pull it something like 1, 2, 3 and it is taken out as a constant. Now, it is quiet a formidable expression. 20 years back, it is a great skill to do all these derivations. If you do it for a cylindrical coordinate, you will get bessal function and if you do it for a spherical coordinate, you will get legendary polynomial and so on. This is before the advent of computers this was the only way you can attack that is why mathematicians contributed largely to growth of this subject. After the computer has taken over, you have got matlab and also got fluent and other things, where these things are numerically solved. In matlab, you can write your code script and in fluent, it is numerically solved and you can also use your finite volume method to solve it.

The most important thing is the temperature, it is periodic about x, which we believe that 0 less than or equal to x or less than or equal to l. So, the temperature is periodic about x. This is a constant and sin h n phi y by l is exponential exponentially decaying. Therefore, if the hot wall is at the top, then the temperature is exponentially decaying with y. Watch carefully, the temperature is exponentially decaying with y. The temperature is periodically fluctuating with x that means it is symmetric about x.

All the physics, which was used in the formulation of the problem is intact in the solution. The physics is not at all altered and therefore, the final solution should abide by the physics. So, regardless of your procedure, you may use orthogonal function, bessal function or whatever. If you get something, which fundamentally violates your common sense and understanding the solution, then that solution is not correct. The other most important thing is since n is in the denominator and it is increasing, the contribution of the various things decrease. It will be a convergent series, where n equal to 1 will have the maximum place, tenth term will be 1 by 10 and all these things also will have that part because in sin x, you can see what will happen to sin h e to the power of m h minus e to the power of minus m h and all that. You can see all this, but when since n is in the denominator, it is very clear that this will be a convergent series. The solution will be periodic about x and it will be exponential with respect to the y coordinate.

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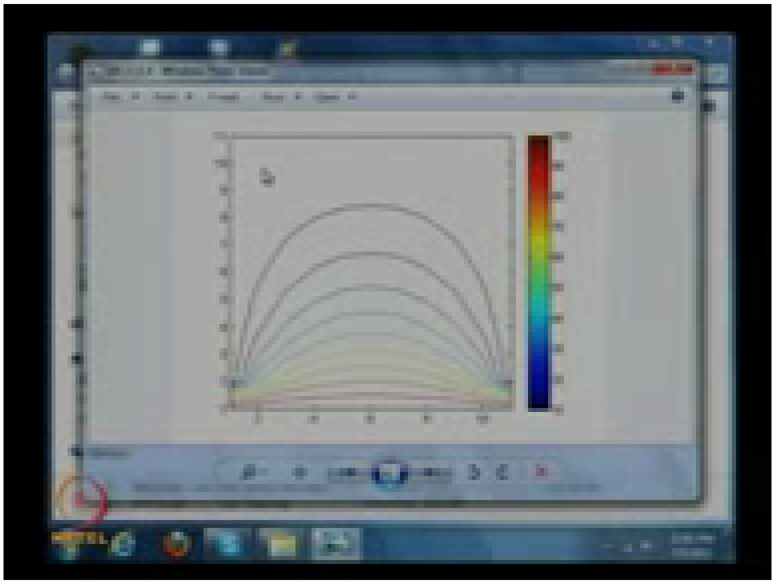

Shall we see this? Yeah, when n tends to infinity that seems you have already added so many terms and you do not have to add that is all. This gives you a… Please look at this graphic, this gives a solution to the problem; space problem for a square slab, which is 10 centimeter by 10 centimeter as you can see. Though this is a problem on steady state conduction, k is not at all used in this problem. Thermal conductivity will enter the picture, if you want to determine the heat flux of the surfaces, where it will be k dt by dx and use the fourier's law. Temperature distribution is independent of the thermal conductivity, whereas the heat flux is… We are not worried about the heat flux here that is point number one. Here, this is 10 centimeter by 10 centimeter.

Now, the temperature; this is 100, this is 0, this is 0, this is 0. These are basically the isotherms, so you can see the colour code. This is hundred, this is going to be 100, this is 100 that is why the red colour is here, so this is 100. As you can see, the temperature decreases, so this portion of the slab almost reads the temperature, which is between 0 and 10. You have to look at what is the value here. Therefore, the temperature is exponentially decaying in this direction that means if you take this section and if you plot the temperature distribution along y, it will follow the exponential decay, which is picked up by this time h n pi y by l. If you look at this section and you look at temperature on the left side and right side, they are at equal distance and they will be the same here.

You can see at x equal to 8 and x equal to 4, the temperature is the same that is picked up by sin n pi x by l. So, this is symmetric boundary condition. This is the test we give to all our M.Tech students as the first exercise to solve the laplace equation and you should get isotherm, which are consistent with common sense.

There is a further check. If this is 100, 0, 0, 0, you know that the temperature here from finite difference must be east plus north plus west plus south by 4. The temperature at the centre should be equal to 25 degrees centigrade. As you increase the number of grids, the temperature may approach 25 either from 24.5, 24.6 or it may approach from 26.2, 26.1, 25.8 like that it will approach, so that is another test of consistency.

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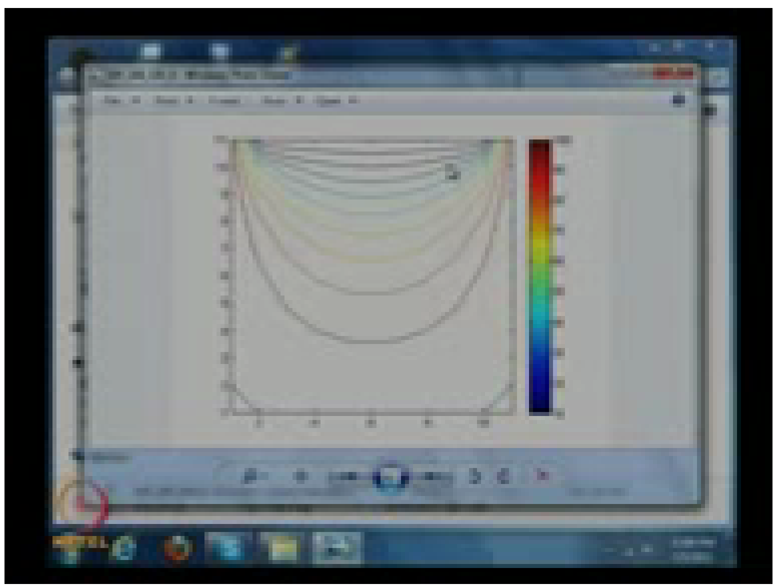

Now for other conditions also, we have got the…. This is basically three walls at 100 and one wall is at 0 degree centigrade. Which wall we will have to find? Which wall is at 0 degree centigrade, can you tell that? It is top wall because the red colour here is the top wall and it is at 0 degree centigrade.

Suppose, more than one wall is at 100 degree centigrade that means this fellow cannot handle because it can handle only one homogenous equation because using the non- homogeneous boundary condition and see all dirchlet boundary conditions were 0. We made t minus t c and theta equal to 0, 0, 0 on three sides. We made it homogeneous that is called homogeneous that is if dirchlet condition and first type of boundary condition is 0. Second type of boundary condition is making dt by dx equal to 0 and it is called homogeneous boundary condition. Insulated boundaries are very easy to handle in finite elements. We call it as natural boundary condition. Now, if we have more than one inhomogeneity, this funda of using orthogonality and all will not work because you can get only one constant.

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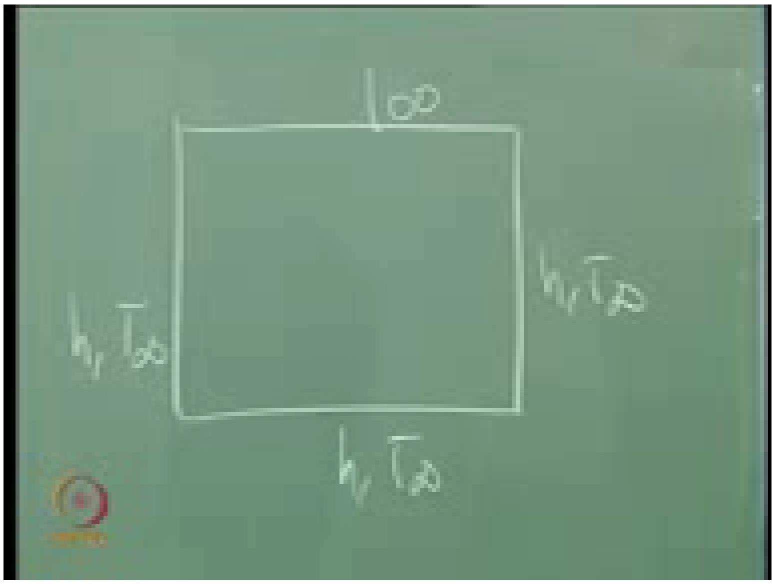

How do you solve that problem? You split it using super position. You use what is called one by one; you can handle inhomogenity in all sides. More involved problem like this or you have to put minus k dt by dx equal to h into h delta d on the three walls. You will get some solution like lambda n tan lambda n l is equal to 0. So, you will have transcendental equations to get the lambdas. You have to solve a transcendental equation to get the value of lambda. Now, you got lambda as n pi by l because of periodic boundary condition and even these kinds of boundary conditions can be handled.

Now, what about this? Is it correct? There is no minus, plus 2 n pi and everything is correct. So, the challenge is to program it with variation of varying number of n. Now, I have solved it compared with what you have written in your own good. That is a good test and that is a good way strengthening your learning. Take a problem as 10 centimeter by 10 centimeter, generate pi at various values of x and various values of y. Compare it with what matlab or fluent or a finite difference and give a good check.

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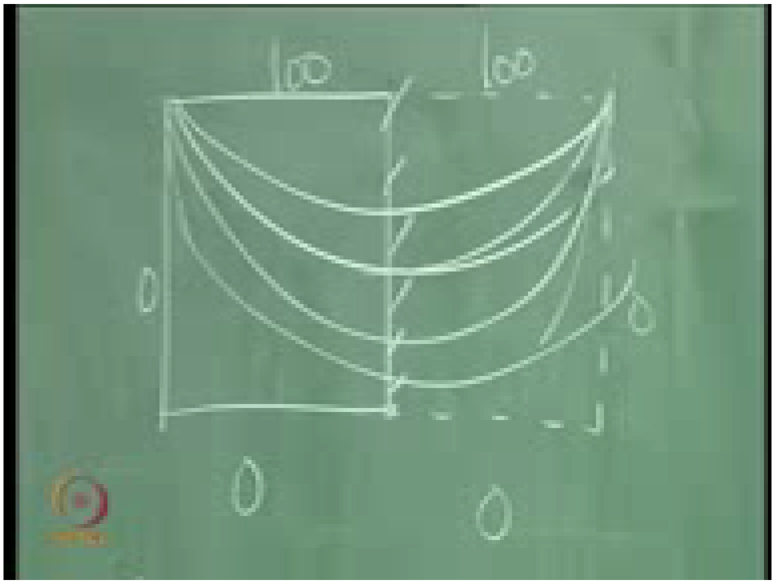

Now, if you have problem like this. You can use the super position and make it like this. Now, it will just become a 100,0,0,0 problem. So, you may get isotherms like this, then you cut the isotherm upto here. You just increase the geometry; take the mirror image.

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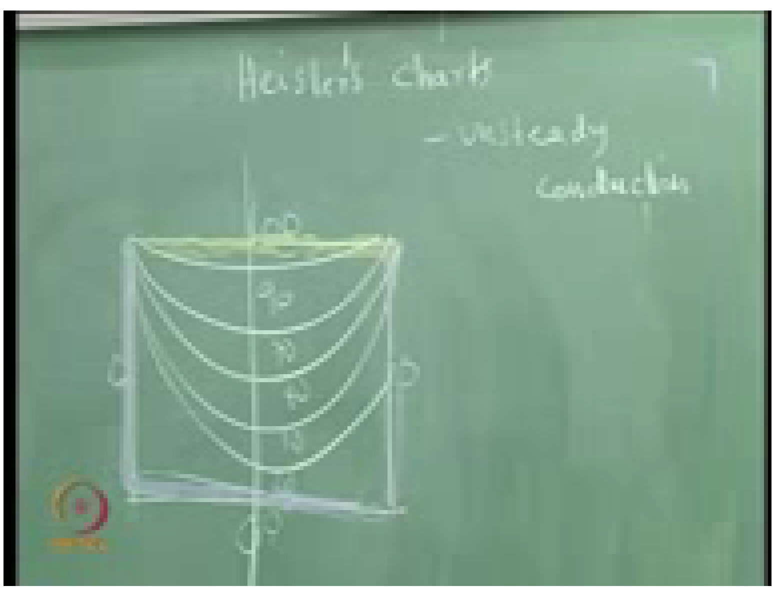

At the end of the day, for this 100, 0, 0, 0 problems, this is the 100-degree isotherm and it will be here only. The 0-degree isotherm is on the three walls, the 100-degree isotherm represents the maximum temperature. In the problem, the 0-degree isotherm represents the minimum temperature in this problem. We also know that temperature is periodic about this and all the isotherms have to lie in between. Finally, this is 0, 20, 30, 40, 70, 90. So, when you read an isotherm, after you solve a problem and plot that it must be intuitively apparent that there is no violation of physics in the problem. This is the way to interpret and it should be symmetric about x. The temperature must decrease exponentially from the top to the bottom. In fact, our solution and the numerical solution confirms this.

This method of separation of variables can also be used for solving the transient one dimensional conduction problem. You will use a minus lambda square, but what is equation one? Alpha 1 by alpha dou t by dou tau is dou square d by dou x square. If you use the method of separation of variable, you will get a periodic solution in x, but you will get an exponentially decaying solution in t. If you put minus, you will get a into e to the power of minus lambda square. Finally, you get the dimension; you will get biot number, fourier number as well as x by l for that problem. In the solution, instead of having sin x and cos x for the transient conductance problem, you will still have sin n pi x by l and cos n pi, no problem, but steady conductance in cylinder, you will get vessels function.

To recapitulate, we use the method of separation of variables. We had a product solution, we split the solution and then we wrote the general solution. By homogenizing the boundary conditions, we knocked off several constants invoking the periodicity. We developed and we found out that there is a lambda, which is n pi by l. So, the solution first becomes an infinite series, then to get the constant of the infinite series, we got stuck and we exploited the property of orthogonality.

Fortunately, sin n pi x by l is orthogonal and finally, we got a solution and we got a physical and intuitive field of the solution. We also saw a numerical solution on the board, whatever isotherms you are getting is consistent with our common sense and it is also a conceptual understanding of the problem. This can be extended to transient conductance in a 1 d slab. Where you can again use the method of separation of variable exponential in time and periodic in x? You will get same and you will get sin n pi x cos n pi x. This has been nicely worked in the form of what is called Heisler's charts and it is available in all books and handbook.

We either take an analytical solution and work out as an infinite series or you can use Heisler's charts or you can write your own code in matlab or you can numerically solve it or you can use a finite differential or finite volume solver like fluent to solve it. This method of separation of variables can be applied to select class of problems. Even for the cylinder and sphere, it will lead to vessel function and legendary polynomial. It becomes more and more involved and these are called classic conductance heat transfer. The other is conduction in heat transfer. You can completely by pass all these and start with finite difference, finite volume, finite element boundary element that is applied that is a modern approach to conduction, but this is classic. At least one or two of these elegant solutions must be known to you because you are able to get a closed form solution, you can get a physical field of the solution. When you are numerically solving, you do not know what it is giving. We will stop here.