Conduction and Radiation

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Lecture No. # 44
Numerical Methods in Conduction Contd…

So, in the last class we were looking at the finite difference method and then we found out a way, to express the first derivative using the forward differences backward differences as well as central differences. The power of the central differences lies in the fact that a central difference results in a second order accuracy scheme. Therefore, for a given grid size the order the order of the error in so far a the truncation error is concerned the truncation error arises, because you are replacing a derivative by only a finite number of terms in an infinite series. So, delta X squared by 2 factorial is much better than delta X therefore, the central difference is much more powerful. So, we apply the central differences to the problem of Laplace equation and then we found out that for an equispaced grid the temperature at any node is just the algebraic sum is the algebraic sum of all the four is the weighted average of all the 4 neighboring nodes. So, the weights are 25 percent if delta X is equals to delta Y the weights may be different from x 25 percent if they are not equispaced if they are not equispaced.

If you remember in one of the earlier lectures, I looked at the Lagrange interpolation scheme so Lagrange interpolating polynomial. So, you can use a second order Lagrange polynomial for x and second order Lagrange polynomial for y or third order, and then get d squared T by dx squared and d square d by Dy squared. You will again get T i j as a function of all the neighboring nodes but, here you would not get 25 percent 25 percent 25 percent 25 percent depending on your grids. Suppose you first algebraically generate the grid using a cosine function or a tan function or sin theta cos theta or using a geometry progression, then for a particular node we can take nodes ahead and behind or two nodes ahead and two nodes behind, use a Lagrange interpolation and develop a higher order schemes. So, with higher order schemes the the advantage is with less number of grids you can get a solution, which otherwise would involve a larger number of grids for a lower order scheme.

So, for example we considered only 4 points as a neighbor you can consider one more point in the north, one more point in the east, one more point in the south, one more point in the west that means, you look at 8 neighbors' for each of the points.

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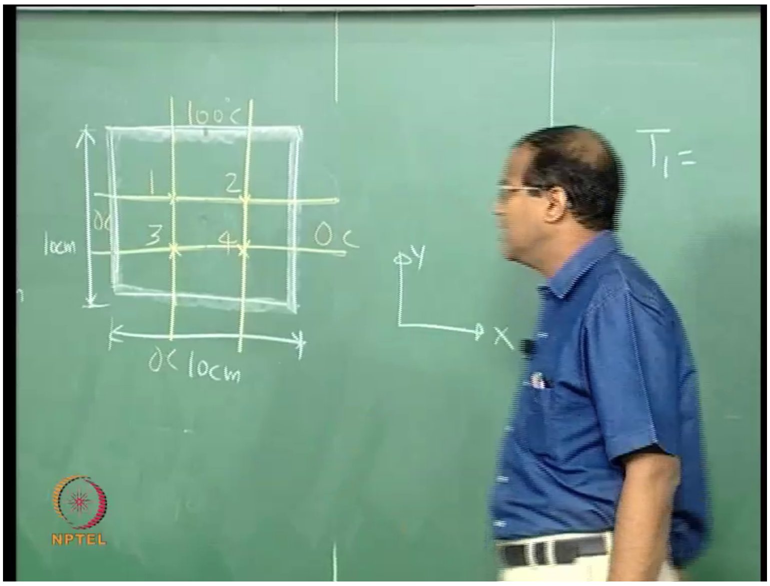

So, that is called a 9 code stencil or the last code could be like this correct for this point for for this point you considered 1, 2, 3, 4. Let us take no it is not very clear for this point what you can do is you consider 1 2, 1 2, 1 2, 1 2 or you can do one you can consider 1, 2, 3, 4 so for this point all the corners can be considered. So, that is that is what is called as a 9 point stencil.

Now let us take a simple problem, 2 D steady state heat conduction. So, the slab is 10 centimeter by 10 centimeter it is equispaced for simplicity we are having only 4 points 1, 2, 3, 4. So, we want this use the central difference formula to numerically solve this problem, instead of using the series expansion the series solution which we obtained in the last class.

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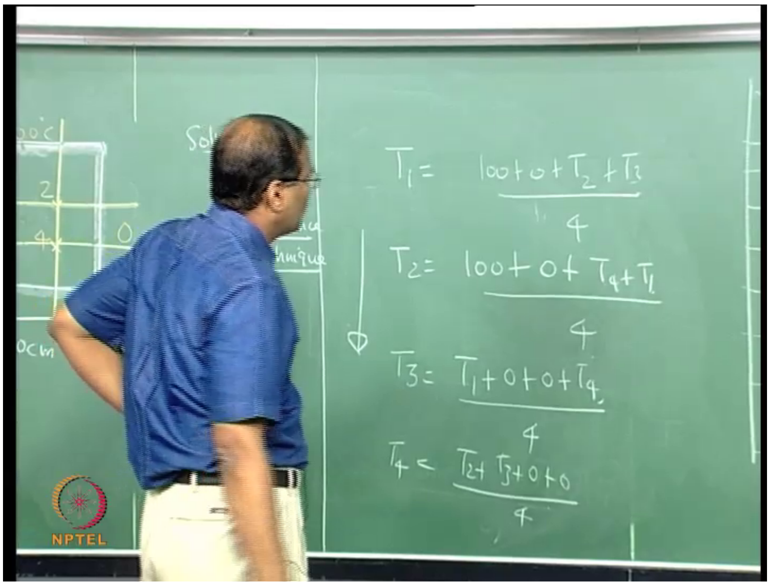

So, T 1 please note you can write the equation for T 1 is 100 plus 0 plus T 2 T 2 will be 100 plus 0. So, the question is solve using finite differences. So, T 3 will be equal to T 3 will be equal to neighbors' naught T 1 plus 0 plus 0 plus T 4. T 4 is T 2 plus T 3 plus 0 plus 0.

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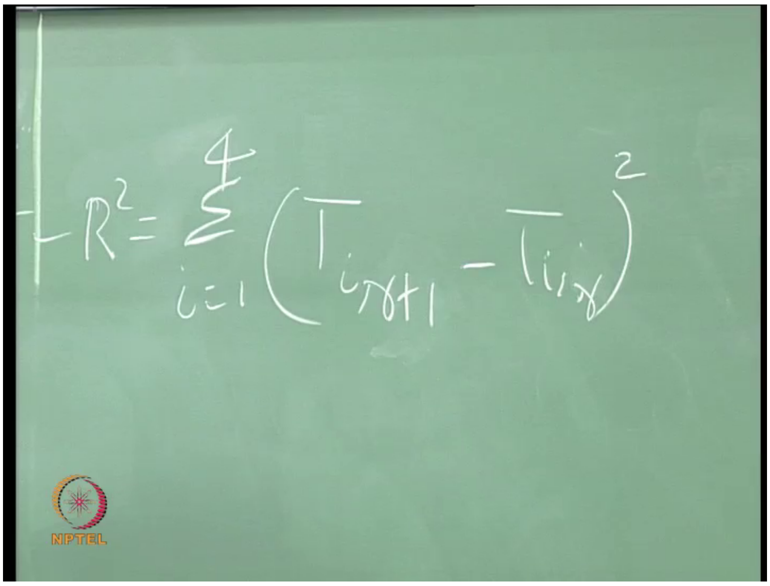

So, we put up a tabular column serial number. I will call the error as it is the iteration number or you do not like i you put i plus 1 whatever. So, j r so this will be T i r plus 1 minus this is r plus 1 th iteration minus rth iteration. I will start with 20 20 20 20 for all the 4 I have not started iterations.

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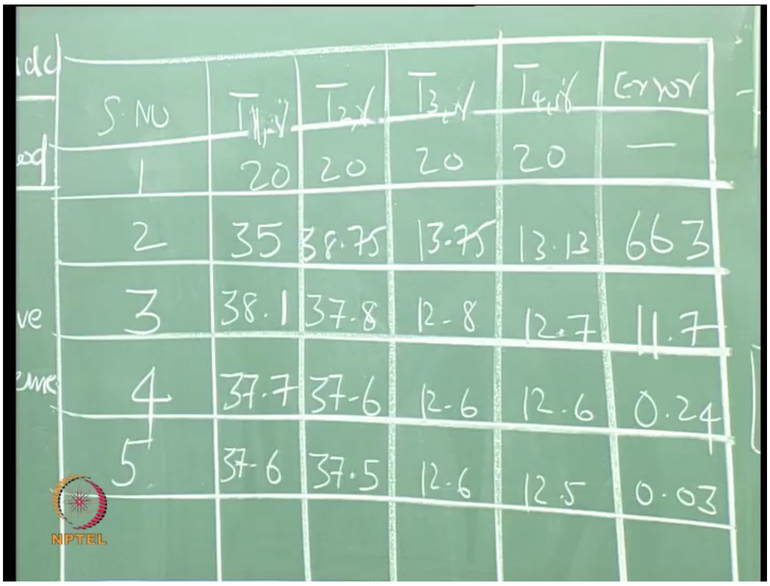

So, I cannot calculate error I have put dash, because I have not started. Now I want to use gauss seidel method, what is a gauss seidel method now it will go in this sequence. First I will use T 2 and T 3 with that I will get T 1 wherever T 1 comes I will use the latest value of T 1 T 4 is not done yet, I will use the old value of T 4. Now after T 2 is done no no after T 2 is in this equation I can only use T 1, but when you come to this equation I can use T 2 and T 3. I use the latest values or the updated values of the temperatures as the iterations proceed, I do not do all the things one at a time for example, all T 1, T 2, 3 4 I do not take 20, once I calculate T 1 latest value of T 1 I will use. So, what is that T T 1 will become 35 then T 2 155 by 4 then error you have to just take 35 minus 20 whole square plus 38.75 minus 20 whole square plus plus plus. Good 663 everybody through with this, I have not given the problem number, because I have lost track of the problem number.

Now T 1 please use the latest value, you say 38.1 T 2 is it 12 point is it correct just check.

Student: 37 point.

37.05.

Now 12.67 error has drastically come down see. So, we are using the gauss seidel method which I have already discussed which I have already discussed in this course. So, 11 point see how much it has drastically third iteration it will convert 37 point I will make it 7 then.

Student: 37.978.

37.6 12.6.

Student: 12.55

12.6 error it is 3 and 4 are symmetric.

Student: 0.25 4.2.

So, we almost reached so may be is R squared you can make it less than point naught one or something, you want to do one more iteration shall we do just one more, you can see that it is rapidly converging.

Student: 37.6.

37.6.

Student: 37.5.

12.6.

Student: 12.5.

Point naught 3 so we have we have come to the we are very close to the true solution. So, please remember that the central difference gives you only a numerical scheme, then you get a the set of the resultant algebraic equation. We have used one more technique which is the gauss seidel method gauss seidel method is we are taking an initial case for all the values T 1 T 2 T 3 T 4 right side is completely known left side is unknown the latest we substituted and then iteratively we keep on solving till the error reduces. We can actually invert the matrix and solve for T 1, T 2, T 3, T 4 also, but beyond 100 by 100 it is very difficult to solve a it leads to lot of errors and all memory problems and all that gauss seidel will work very well. The same gauss seidel is also used in commercial software for solving the equations also its.

Now, you can see that 37.6 37.5 12.6 12.5 5 now common sense tells you, that the this centre temperature must be 100 plus 0 plus 0 plus 0 by 4 the centre temperature must be close to 25. Now you have got T 1, T 2, T 3, T 4 please take the average of T 1, T 2, T 3, T 4 and see whether it is close to 25 25 point no 25.0 very good so the finite difference works.

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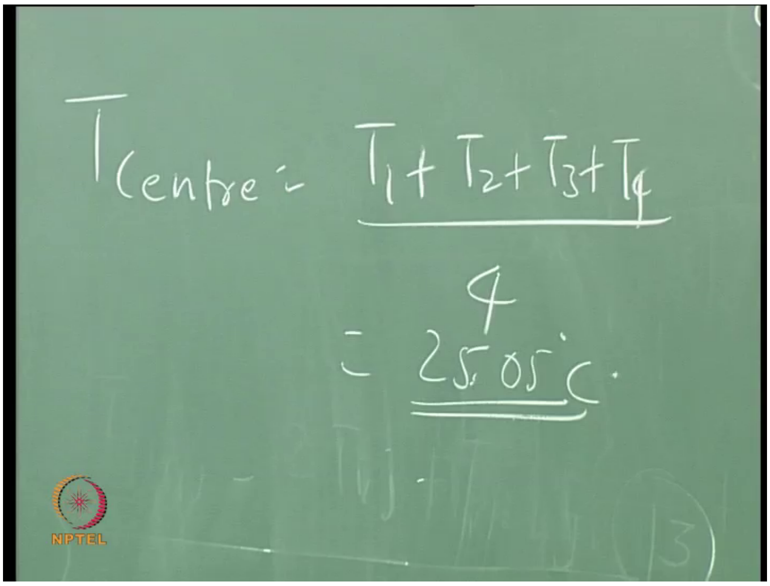

So, you can make if you see for 4 by 4 you can do with hand calculation, you are making it ten by 10 20 by 20 30 by 30 then you have to write a program. Gauss seidel is very easy to program do i equal to 1 to j j equal to 1 to j therefore, do i equal to 1 to 20 j equal to 1 to 30 for all i j T of i j is some T initial then keep updating and then evaluate the criterion. If R squared is less than this thing come out of the loop if R squared is greater than go back i equal to i plus 1 j equal to j plus 1 let the iteration count you can set it. So, this is how the gauss seidel method works. Now you have a good idea how to solve a two dimensional conduction problem, in Cartesian coordinates this can be extended to spherical and cylindrical coordinates, even if I make a small change then add a heat generation this T 1 will be 100 plus 0 plus q v delta X square by k that will that term will get added and you can solve and you can solve and get the solution.

Now, if you see for a problem which is similar, to this how we can apply the finite volume method. What is the finite volume method is all about then we will close our discussion in steady state problem. Then how do we handle numerically the one dimensional transient and steady conduction basic ideas same finite difference. So, how the delta T and delta X must be chosen such that you have a scheme which is numerically stable with that we will close.

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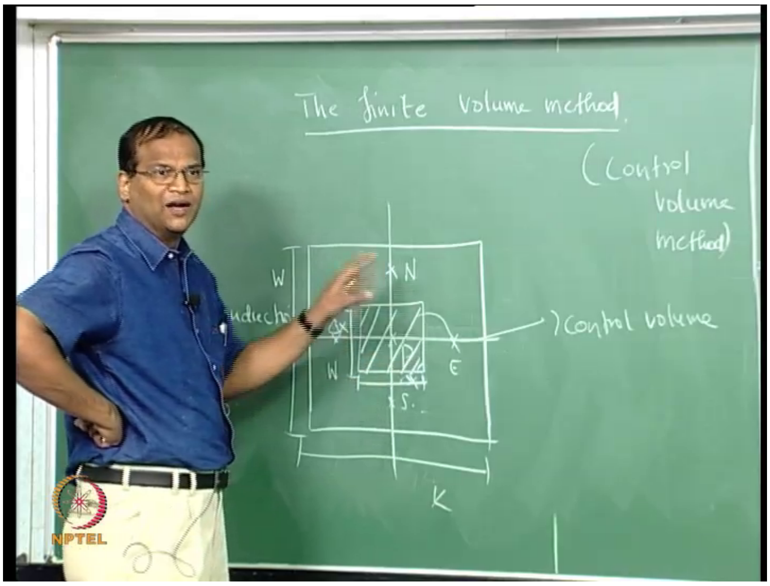

Now it is also called the control volume method, also called the control volume method there here we do not use the Taylor's series approximation we will basically look at a flux balance. Let us consider 2 D you take an area or a volume this is called the finite volume or control volume. For purposes of illustration I have taken a large control volume compared to area lies on the slab. So, the slab is like this this should be L and this should be W. L need not be equal to W if L equal to W you have square slab it is also can be a rectangular slab also. This control volume is centered around the node p node p is our node of interest; we want the find out the story of the temperature or velocity or whatever at the node p.

This node p has some neighbors' east west north and south. No I missed that it has 4 neighbors' for to the make the problem to make is tractable I will say this as delta X this is also delta X. That is the distance between east and p is delta X, distance between p and west is also delta X and north and p is delta X, and south and p is delta X. I have exactly taken half of this distance half of this distance half and half I have made a controlled volume. Now it is two dimensional steady constant property I have to find out it is only conduction no convection is taking place, no heat is generated it is steady no enthalpy is stored or depleted in the control volume now I have to apply the flux balance.

This just a starting point of a derivation of the governing equation, but then when you do that k d T by dx I will put a T P minus T W by delta X 3 I will put an approximate that is a control volume. That will be valid if the control volume is very small. I cannot put that k d T in a boundary layer dT by dx is T 2 minus T 1 put a thermo couple outside the boundary layer say T 2 minus T 1 by y 2 minus y 1 but, if you are able to put thermo couple within this so that it does not affect the boundary layer you are still getting reliable result you can use that logic, because q conduction will be equal to q convection at the first layer.

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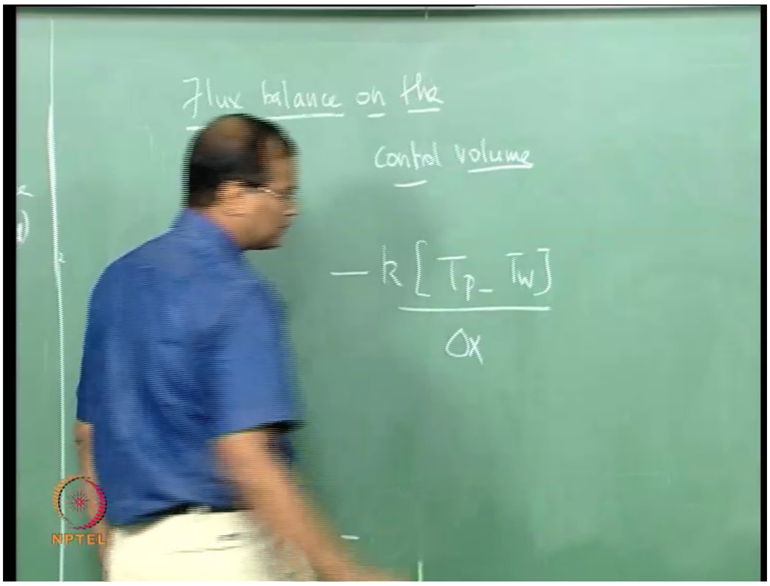

Now let us get back to this, now flux balance the control volume is infinitely deep in the direction perpendicular to the plane of the board or you can assume it to be 1 meter. You can assume it to be 1 meter, now here there is a conduction heat flux is entering here there is a conduction heat flux which is leaving, here there is a conduction heat flux is entering, here is a conduction heat flux. The net the algebraic sum of all these fluxes will be 0, because there is no nothing else is in this problem.

Let us start from the left minus k T P minus T west divided by delta X, when it is entering q when it enters the control volume it is positive when q leaves the control volume it is negative that is the sign convention of thermo dynamics but, w leaving the control volume is positive w entering a control volume is negative so we will stick to this minus k is already is coming because of the Fourier’s law of heat conduction into delta Y into 1 let us keep it delta Y into 1 I will then say delta Y equal to delta x.

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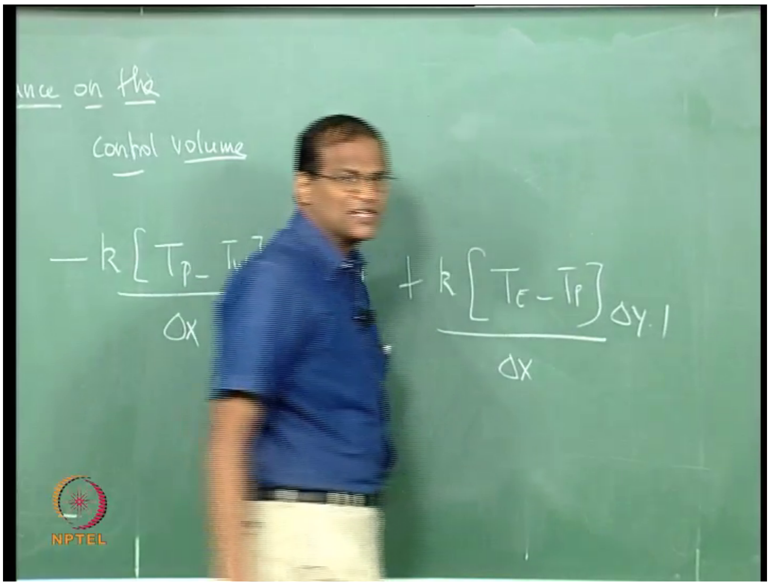

So, that you you are clear plus k into T E is minus T P is it is this clear this is the flux which is coming out of this I put minus k d T by dx into minus, because it is a flux which is coming out the control volume is losing that flux. So, it is a minus of minus that is why it becomes plus.

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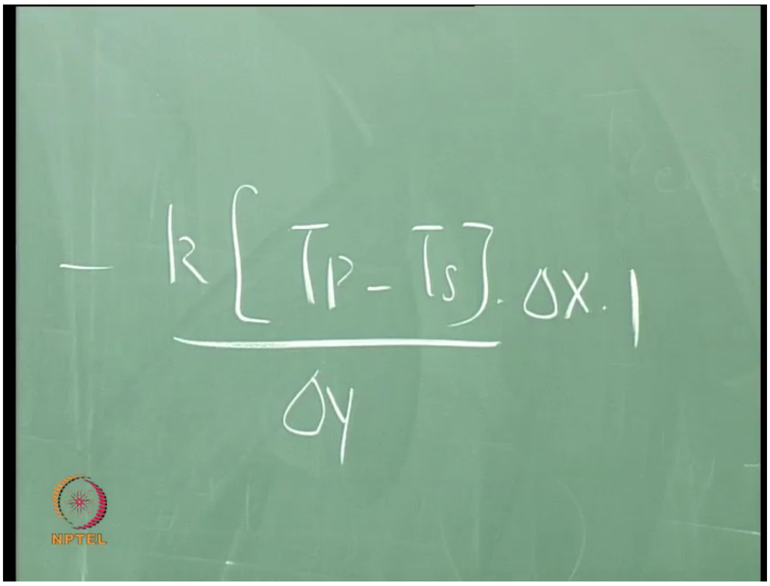

So, this is east and west is accounted north and south when it is entering from south minus k T P minus T S by delta Y is equal to delta X into 1 that delta X delta Y you know why it is coming right plus the algebraic sum of all the fluxes is equal to 0.

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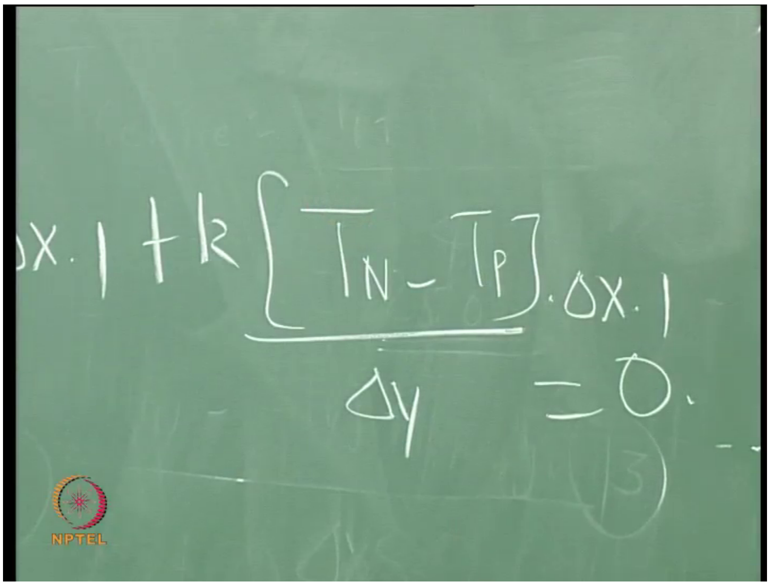

If you want algebraic sum of all the fluxes that whatever it generated, if you want generation q v into delta X into delta Y into 1 you can add. If it is not equal to 0 rho c p rho c p into delta X into delta Y into 1 into dou T by dou tau whatever you want you keep on adding that is all. Now what is the equation number we will continue from or we will start with one one.

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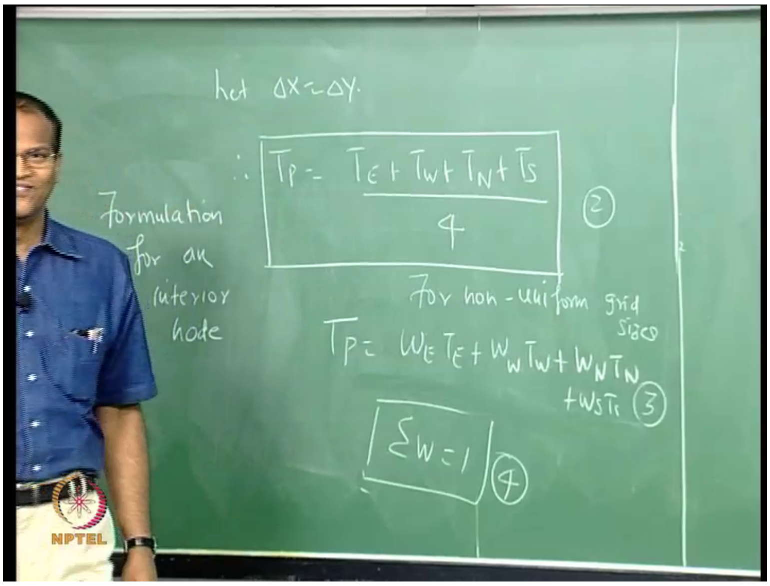

Now if you simplify what you get let delta X equal to delta Y so everything is cancelled. The answer is exactly as the same as what we got using central differences it should be the same we cannot get any other answer. There is no approximation except that dT by dx is approximating by this, because it is a elemental control volume it is I do not want to put dT by dx equal to this again Taylor's series expansion is not required, because I am using a small control volume so long as the control volume is small this will work. So, this basically actually we should say it is a control volume based control volume based or finite volume based finite differences, because I am using finite difference here. What the FBM people will do here is instead of taking instead of taking dT by dx as T P minus T W by delta X what they will do T they will put it as T is equal to a plus b x plus c x square.

Then they will get this a b c and this a b c they got elemental coefficient like that this a b c they will evaluate for all control volume they will assemble it will be a matrix and they will solve it and shape function of the element and all that they will do all that. So, this formulation will start here then they will start playing later on they will not use the Taylor's series, but here we are using the Taylor's series. So, for a constant if delta X is equal to delta Y and all that after sometime you do not know what is the difference whether it is finite difference or finite volume. So, it is finite volume based finite differences is that clear. So, you can but, this is more regress, but basically you do not start from some governing equation and all that it is like as though you have you have got the governing equation the governing equation basically will be differential form of this you are actually working on the integral form of this you approximated for dT square by dx square. You do not use the dT square by dx square here you are approximating only for dT by dx are you are able to see the similarities and differences between these two.

But, if you do not like if you do not like to put it like this and put it as a polynomial function or something and then it becomes finite elements that is it there is something called boundary element method all that. This is special class of techniques which has been developed. So, with this we will close our discussion, but only one thing is the formulation I have given you is only for the interior nodes you need to have a formulation for the boundary nodes also, just 5 minutes we will see the formulation how to do a formulation for boundary node and then we will go to unsteady and then close. So, this is east plus west plus north plus south by 4. If you have non uniform grids shall I call it 5, that weights can be immediately calculated once you finalize the grid size, once you decide your grid the weight can be calculated for any idea you can calculate the grid correct for uniform grid size all the W's are 0.25 is that clear? So 1 2 I give now it is.

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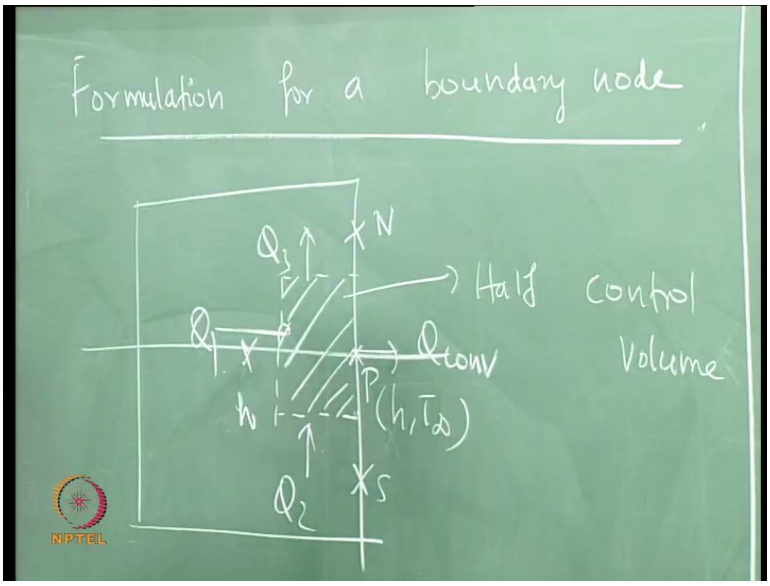

Formulation for a boundary node. So, you are getting a boundary here that means, you are having h and T infinity. So, you have your i i j here how do you think we can solve this no your i j is here, this is p west all are there except the east east is gone. Now you you can develop have a half control volume Q 1, Q 2, Q 3, Q convection.

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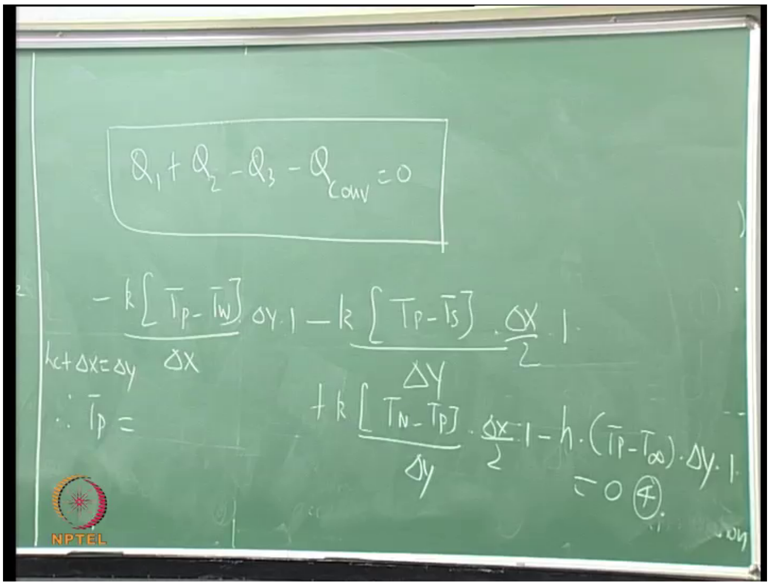

So, is it correct Q 1 and Q 2 are entering the control volume Q 3 and Q convection are going out of the control volume, therefore, Q 1 plus Q 2 minus Q 3 minus Q convection equal to 0 correct. Now you can write for Q 1 Q 1 is any problem which one I am I am not doing any mistake, what is this flux is based on this distance only no there is no problem. What is the problem I am evaluating the flux here, it is half way so it is this minus this divided by full distance is delta x. The other one is into divided by delta Y multiplied by delta X by 2. What about q3 plus minus h h into T P minus T infinity that will be positive is equal to very good that is all.

Now I will leave it as an exercise to you you can collect all the terms, you can write an expression for therefore, T P equal to, let delta X be equal to delta Y shall we write it can we shall we spend 2 minutes and finish it, cancel off the delta X delta Y is it minus h is minus or plus is that correct is it correct are we taking care of everything there is a T P coming here.

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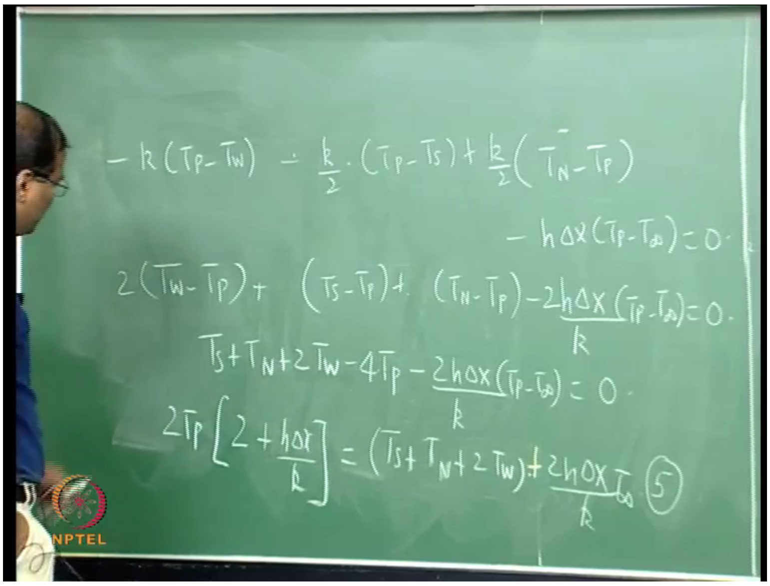

So, what do we do now 2 T P 2 plus no it is there where minus where no no I am I am I am having T p on this side plus plus 2 h is it 2 h. Let us look at this equation this is the control volume based finite difference formulation for a boundary node where there is a convective heat transfer coefficient. Now please look here, suppose h is 0 that means, it is and adiabatic condition this will vanish this will vanish this will become 4 T P therefore, T b will be north plus south plus 2 times west divided by 4. That means, it is as though an imaginary node existed on the two west was defeated here instead of the east the west is coming twice. So, that is why if it is adiabatic you can just extend the domain and make it another half so, that you use the east plus west plus north plus south by 4 which I told you yesterdays in yesterdays class method of superposition. That is image point technique that is what people call as image point technique

So, this h could be instead of h you can have radiative heat transfer coefficient epsilon sigma T to the power of, but it will become little more involved that T will be that you have to linearize the radiation and all that, or you have to take the radiation from the previous iteration and use it that is called the linearization of the radiation.

So, once you have got interior node boundary node, now you can formulate any problem as far as so long as it is steady. Of course, this will all become messy if you have rid spherical and cylindrical cylindrical and spherical coordinates, but the logic is the same you should be able to sit down with pen and paper if you want to code yourself this is the way you will start.

So, you can handle boundary conditions now very easily, you can incorporate adiabatic boundary condition and boundary condition and the third type of boundary condition using this fine.

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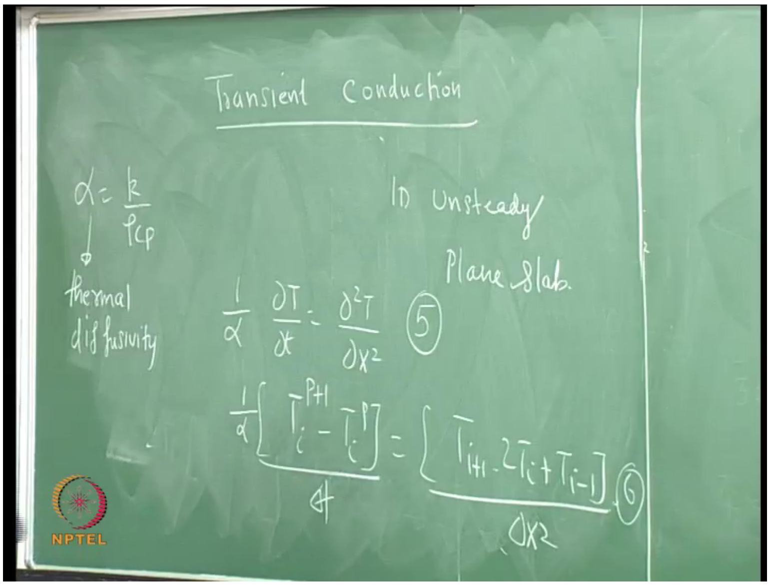

Now let us close this, let us us go to numerical scheme for transient conduction. Plane slab plane wall or whatever. So, your what was the equation number 5 or 6 or 5 alpha is the thermal diffusivity k by rho c p. Now you can use a finite difference approximation for this. T i it is l formulation the formulation is incomplete, because I am not telling you it now at what time the right side is evaluated can you answer that question.

The superscript p denotes the time interval and I denotes the nodal point, compared to the last two classes what you have studied the new thing that enters the time. So, right side I am not now telling you what at what time at what time step at what time step the right side is evaluated. If RHS is evaluated at time step T P then we have an explicit scheme.

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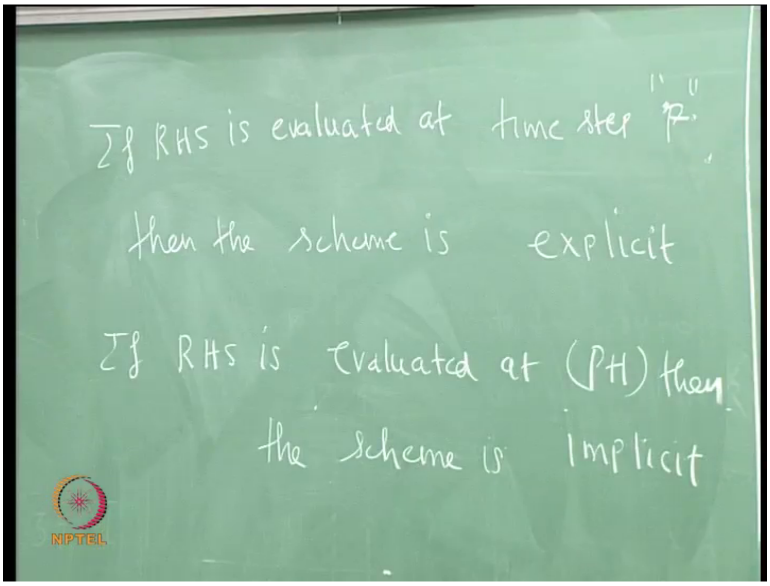

What does that mean, you start with an initial condition that means for all x you know the value of temperature. So, that is at T equal to 0 is known so you start that T equal to delta T the first time step for getting the first solution of the first time step you need to know the solution at 0. That 0 is available that is the initial condition is available to you right side is know T a of p is known therefore, you can find out T i of p p plus 1. This T i is any I this I can vary from one to n therefore, you can find out the temperature at naught at all nodal points in the domain at the time interval of p plus 1 if the information is available like p. so, it is explicit, because the unknown here is this the whole formulation is written for finding the temperature at any node I for an time step p i plus 1.

However if RHS is evaluated at p plus 1 it is implicit. You want p i p plus 1 T i p plus 1 T i plus 1 p plus 1 p i minus all are there. Therefore, it has to be simultaneously solved at each and every time step. So, that is the implicit scheme so you have to use tri diagonal matrix algorithm or something. And each and every time step you have to simultaneously solve you cannot simply march in time, you take a base solution time T equal to 0 and keep on marching in time by by just adding at T T plus delta T T plus 2 delta T without having to do without having to simultaneously solve at each time step that is the explicit, but implicit left side is unknown right side is left side unknown you are writing in terms of putting all whatever you want to solve is on is on the left hand side and all the other quantities on the right hand side.

In any equation in mathematics right hand side is known left hand side is unknown. Unfortunately when you put the implicit scheme right hand side also becomes unknown so several terms are there. So i minus 1 i i plus 1 all the things you want at p plus 1 therefore, you have a tri diagonal matrix algorithm which have got only three row elements in the matrix will be important then you assemble everything simultaneously solve at that time step then go to the next time step and so on.

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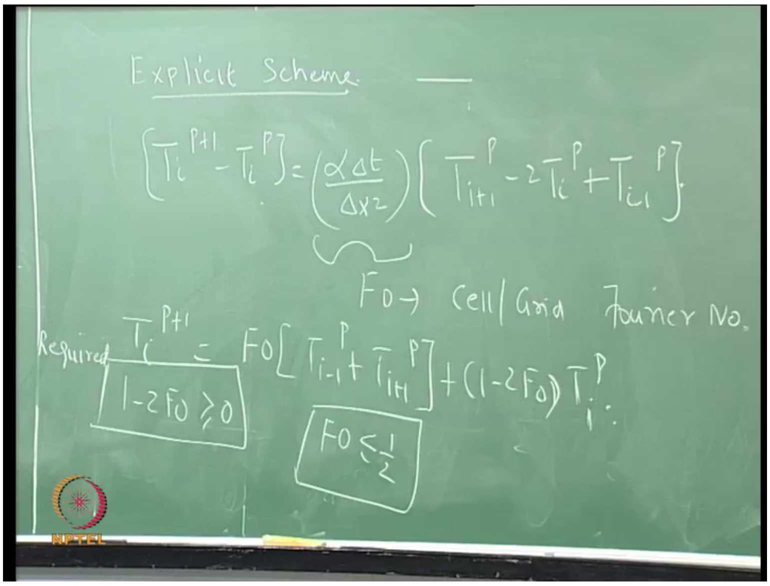

Now if you work on this let us look at the explicit scheme. So, this alpha delta d by del x square is called the cell or grid Fourier number. Now you can rewrite this as a correct yes or no, I just combined the first and third term put it in brackets second term I have combined with the term from left hand side. Now we can prove some thermo dynamic arguments that, 1 minus 2 F naught must be greater than 0 required. That is from the thermo dynamic arguments you can prove, that the coefficient which is multiplying the T i of p must be positive, otherwise you will get meaning it violates the second law of thermo dynamics you can prove it mathematically.

Therefore, if 1 minus 2 of Fourier number must be greater or equal to 0 Fourier number must be less than equal to half. Alpha T by l square is a general Fourier number which we used in unsteady conductance; this is the cell or the grid Fourier number for computation. So, the explicit scheme is not unconditionally stable it is stable only up to Fourier number less than or equal to half. If it is if you are setting it somewhere close to a half the half close to half then the errors will not grow, but the solutions may oscillate therefore, normally people will keep it Fourier number less than equal to 0.25 to damper the oscillation studies have shown that if Fourier number is less than equal to 1 6 or 0.166 you get the least truncation error in a scheme.

So seasoned practitioners of finite difference method will set a Fourier number between 0.1 and 0.15 or 0.15 or 0.1 do not try to go to very close to that 0.5. So this requirement must be worked out for each of these problems, for your problem it is like this the coefficient of T i plus p whatever multiplies that must be greater than 0 from that is where it start. If you have additional terms like heat generation or there is a pin which is losing heat then you have to find out how it will work out. Now I can just in just 2 minutes I can show you why it has to be like that.

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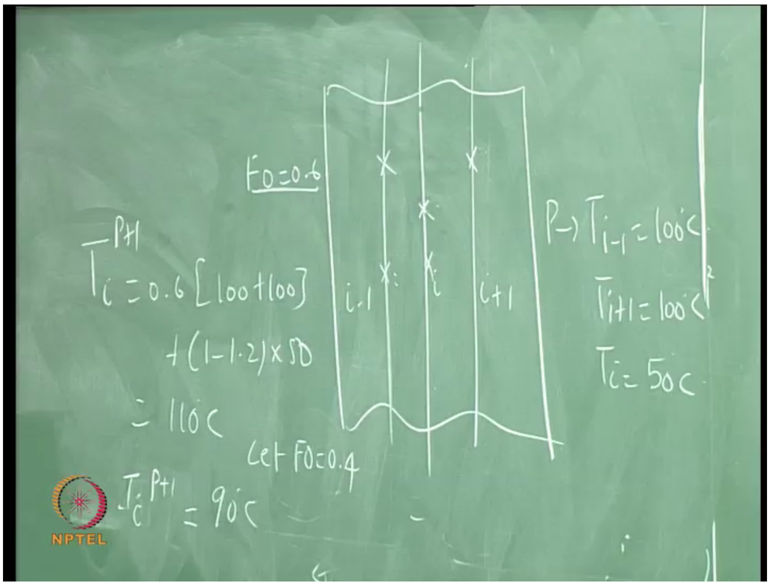

Now let us consider a slab, where it is initially it is hot initially it is cold from the surrounding heat is being applied. So, let us consider two nodes this is i this is i plus 1 this is i minus 1. We are starting at a particular time step P, is the situation clear that is these two temperatures are at 100, this temperature is 50 that means, heat is going into the system. Now I am using the T i of T plus 1. I want to calculate T i of I am having Fourier number 0.6 what happens to my calculation Fourier number so I have 0.6 of 100 plus 100 plus 1 minus 1.2 of 50 correct. If that P the temperature at T i at P plus 1, no cancelling it can be easily done. What is the temperature now 110 is it realistic the maximum temperature of the problem is 100 from 100 it is getting it cannot become 110 it cannot become hotter than this.

Now let F naught be 0.4 what happens T of i plus p plus 1. So, Fourier number greater than 0.5 gives meaningless results you are getting a center temperature which is exceeding the surrounding temperature, but the original physics was these two were hot this was trying to heat this it cannot get something cannot heat to some other thing more than itself. I am sun is at 6000 degrees it cannot a body to 7000 degrees centigrade correct. So, it violates the second law so that is what we say violation so this is 90. So, it is bearable 90 is also too high, but at least is less than 100 so Fourier number 0.2 0.3. So, these are quick proof of the point that Fourier number must be less than equal to half.

So therefore, when you do an in explicit scheme be careful why students prefer explicit scheme explicit scheme is simple to solve implicit scheme is difficult, but it is unconditionally stable, but once you solve you get very good solution with implicit. Now you know why the cell Fourier number is very important for transient conduction. Now I have written the explicit scheme you can also write the scheme for implicit and that is the right hand side will all have terms you have got P plus 1 P plus 1 p plus 1 lot of unknowns will be there therefore, it has to be simultaneously solved at each and every time step before you proceed to the next step.

We start at the time T equal to 0 all solutions are known initial condition is known, from the initial condition you will you start marching in time, but at each time the boundary condition should be available. So, it is called the initial value boundary problem I v b p problems, because you have got dT by d tau and also d squared d by dx squared. So, it supports two conditions in x and the initial condition at all times you should know that x is equal to 0 x is equal to l what the story with that only you can proceed is. So, this brings us to the end of numerical methods in conduction.