Conduction and Radiation

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Lecture No. # 06

Candidate Blackbody Distribution Functions Contd

So, we will continue with the discussion on candidate black body candidate black body distribution functions. In the last class we looked at some of the candidate black body distributions like Wien's distribution and the Rayleigh jeans distribution of both of which fail in different parts of the spectrum. In the sense that they did not agree with the experiments fully that is in certain portion of electromagnetic spectrum they agreed in certain portions of electromagnetic spectrum, they diverge or deviated from observed black body behaviour.

And the Rayleigh jeans had fatal deviation from the experiments, at shorter wave lengths this is basically called the ultra violet catastrophe, because it happens in the ultra violet portion and beyond, but there is controversy in literature regarding the you can Google this ultra violet catastrophe and see why this name has come and whether Planck really worried. I mean, did Planck tried to curve fitting and tried to see why Rayleigh and jeans is not matching or independently developed the quantum theory and then after some time he figured out that other people also got the incorrect distribution and so on. But, he to put it in historical perspective it is good to proceed with the argument that Planck looked at all this distribution found out they are not agreeing then he added 1 minus 1 and then in the denominator and then it agreed, then he found out the classical physics that is Maxwell Boltzmann's statistics does not possible to figure out the correct distribution therefore, it proposes quantum hypothesis.

And then we saw that 1 was in 1900 and 1 was in 1905 but, Planck's was in 1919 not 1 and after 1900 and 1905 these Rayleigh and jeans they published their findings. 1900 the same journal philosophy magazine lord Rayleigh had published. The same result after 5 years the same result is published in the same journal and, but in between Planck had published the correct result not in the same journal. I think it is not in the same journal, but it was known that by 1905 it was known that this is not the correct distribution, because it is deviating, but still they went ahead and published so there are so many I mean, lot of controversies surrounding the birth of quantum mechanics.

Now, in today's class will derive the Rayleigh jeans distribution first will derive the Rayleigh jeans distribution and then we there is some commonality between the Rayleigh jeans as well as the Planck's distribution hence many of the steps are the same, but the thecritical steps where we look at where we look at the average energy of the oscillator, then there is a deviation there will in tomorrow's class we derive the Planck's distribution. Then we make appropriate changes and finally, get the famous C 1 lambda the power minus 5 divided by E to the power of C 2 by lambda 2 minus 1.

In today's class will derive the Rayleigh jeans distribution it involves some very tricky mathematical steps and all this just pay attention you may not be able to follow. All the steps involved generally, because this is all lifetime work noble prize winning work and all that we are trying to do it in 1 hour, but you should be able to get the overall gist of the whole thing, but if you want more detailed derivations you are encouraged to look at not at radiation books no radiation text book will give all this you have to look at some books in modern physics alfredfrazeror some other books if you look at modern physics they will derive all these equations.

I am not sure whether you will understand fully, but present the material would have been presented in many of the physics text books. Now let us look at radiation field enclosed in a box it is a cubicle box it is of size a so the volume is a cube. We are looking at we are looking at stationary waves within this box and radiation field is enclosed within this box.

Now the goal is to find out what is the average, what is the radiation energy density within this box? If we get the radiation energy density within this box that is U then you know the relationship between U and I see if we get an expression for U then you will get an expression for I and that is that is basically the Rayleigh jeans distribution so the effort goes in determining the radiation energy density. When you are looking at waves, then you have to find out what are the possible number of what are the possible number of waves between two, between the frequency nu and nu plus G nu.

If you are considering a frequency nu then consider a small interval D nu about this nu between nu and nu plus G nu what are the possible number of it is like possible number of modes of vibration possible number of standing waves. If you get the possible number of standing waves for this cubicle volume multiply it by the average energy per frequency or average energy per frequency average energy for each wave multiply it by the total number of waves and divided by the volume of a container which is A cube. So, you have a spectral spectral radiation energy density, regarding the point from classical physics every harmonic oscillator has got 2 degrees of freedom 1 is a kinetic energy and 1 is the potential energy. And you know from and we know from boltzmann'sstatistics that the energy associated with the kinetic energy is half KT and the energy associated with potential energy is another half KT.

So, KT is average energy if you multiply by the number of waves number of possible waves between nu and d plus d nu then you have got the numerator the denominator anyway is a cube. Say you have got an expression for the radiation spectral radiation energy density in terms of nu, that is U U nu from U nu you can get I nu then the relationship between U nu and I nu is basically like is that correct, but I have to put a and d nu is positive d Lambda will be negative is not it.

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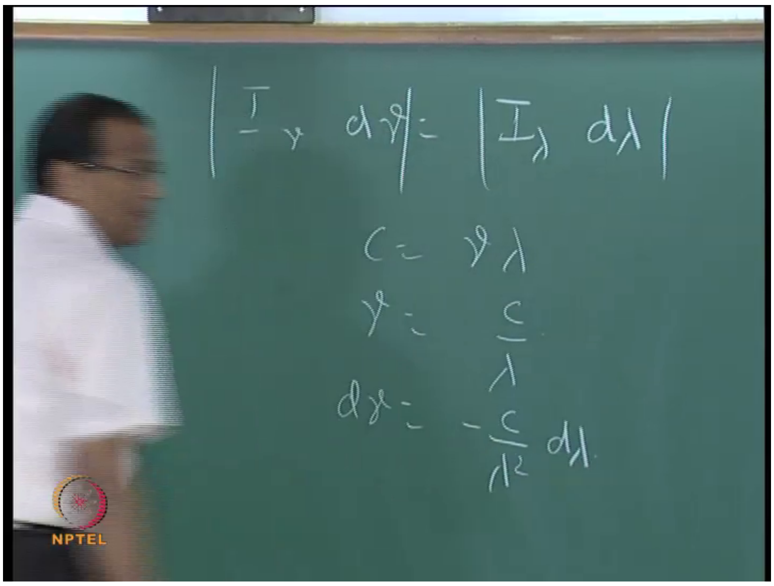

So, but C is equal to so nu is equal to forget about the sign I nu knowing fully well that I am ignoring the minus sign. Did I make a mistake Deepak instead of watts per meter watts per meter square per micrometer steradian I can also work with watts per meter square per steradian into what are the unit of frequency hertz.

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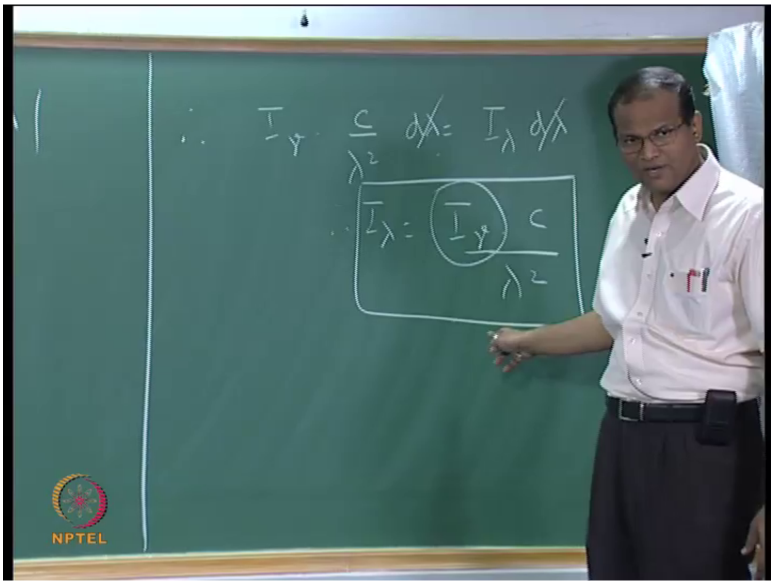

So, how do you convert watts per meter square per hertz into steradian into watts per meter square per micrometer into steradian, because why I am specifically doing this is it is easy for me to get I nu first and suddenly when I am doing the derivation suddenly I change I nu to I lambda you should not get confused. So, I am putting I am writing this upfront I will also do it when I do this derivation are you getting the point. So, this is a relationship between I I lambda I nu.

So, if you get I nu your home in fact, you do not even have to get I nu what is the relationship between I and U. 4. Q is 4 pi by IB by C. So, if you 4 by IB by C fine. So, if you get U Unu then you actually solve the problem from U nu you can get IB nu from IB nu you can get IB lambda. That IB lambda you know the result, what is the Rayleigh jeans C 1 lambda to the power of minus 5 divided by C 2 by lambda d now it is 11;10 by 11:50 we have to derive this so that is the story. Now so lord Rayleigh and jeans they must have spent years trying to get this because the result was not there at that time, we are trying to understand what other people have done, but they tried to create this so there is lot of difference.

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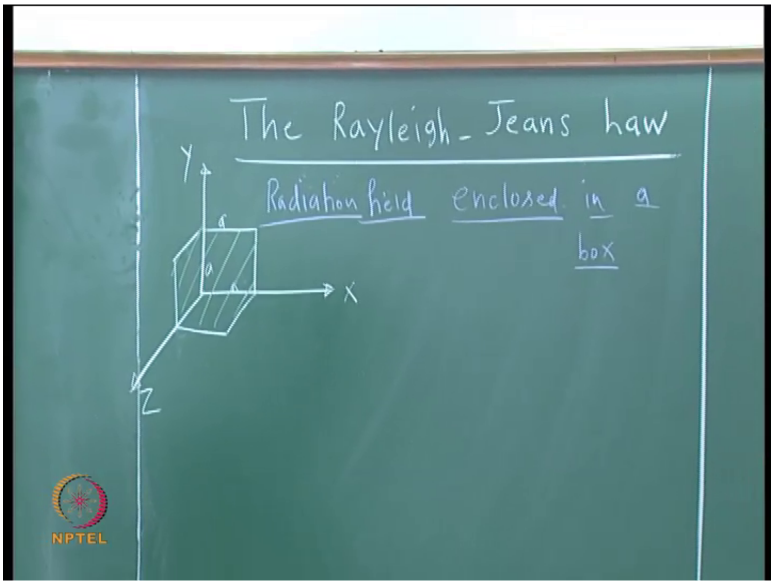

Now consider radiation field within this box. Now let us assume that the box consists of what what kind of what kind of waves we will have all the walls of the container closed impermeable. Which consists stationary waves standing waves? So, the now you have to find out the allowable frequency modes allowable frequency of the waves this can be obtained on basis of resonance theory and kinetic theory of gases.

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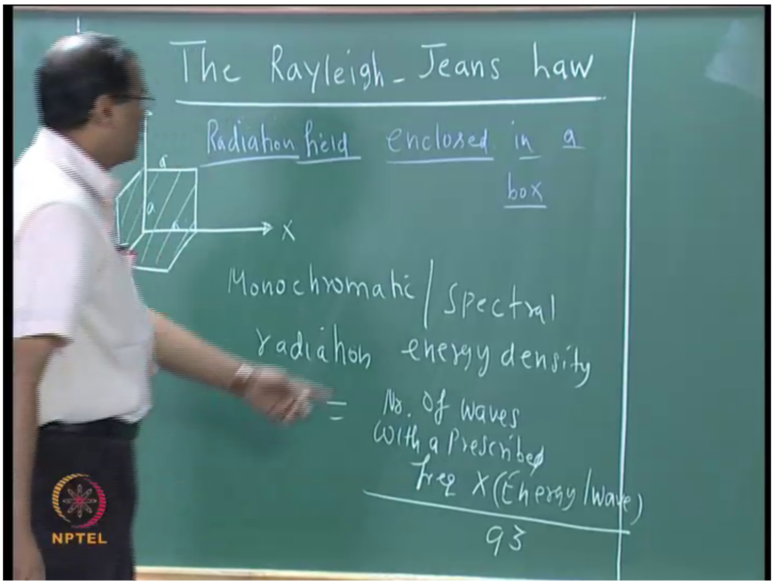

Now monochromatic this is an alternative terminology for spectral monochromatic is equal to number of waves into energy associated with the waves divided by the volume of the container correct. What is the volume of the container?

Student: A cube

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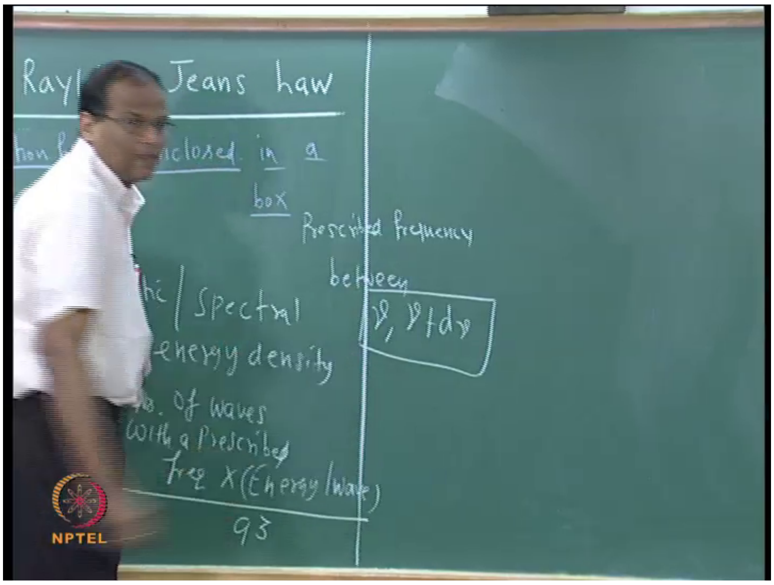

A cube very good what is the prescribed frequency between 1 gigahertz and 1.000001 gigahertz how many waves are there, something like that we are doing that is the d nu.

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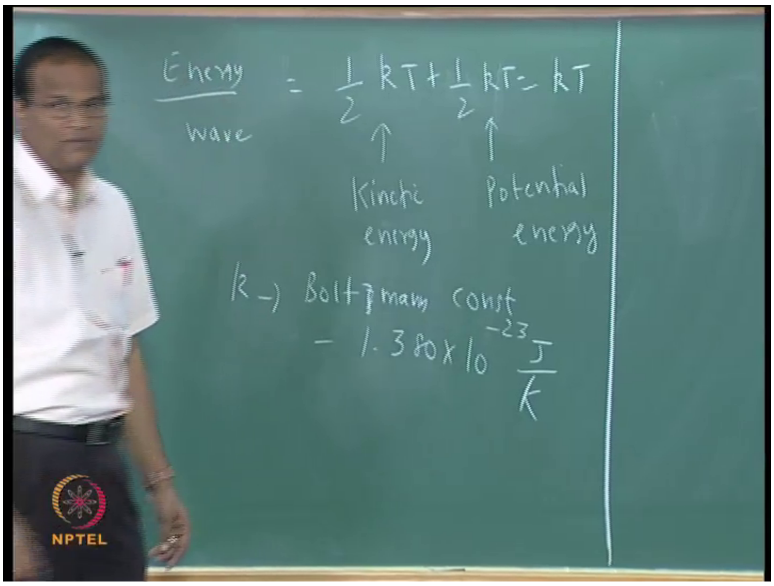

Now what is the energy per wave? half KT. This is for kinetic energy this is corresponds to potential energy. What is K K is the Boltzmann constant Boltzmann constant already existed, because Boltzmann figured it out first. K 2 to the power of minus 23 tell me the units joules per Kelvin correct ya joules per Kelvin into the Kelvin will become joules that is energy the unit of energy is joules.

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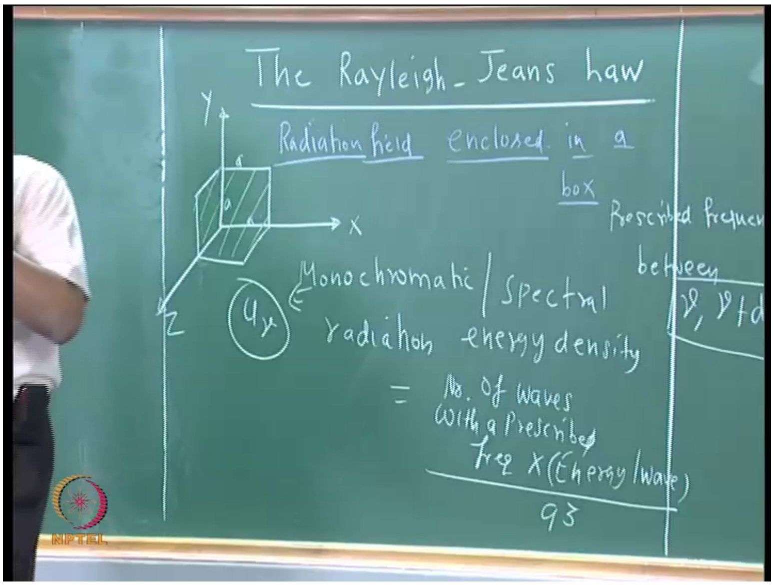

This is nothing, but your nu from U nu I know the story U nu can be converted to I nu I also know how to convert I nu to I lambda. There are 3 things involved in this U nu two two quantities in the numerator, and 1 quantity in the denominator. Denominator is volume a cube I have found out the other quantity in numerator is energy per wave that also I know. Now getting the Rayleigh distribution Rayleigh jeans distribution now boils down to getting the number of allowable number of waves which are which are possible between the frequency nu and nu plus d nu, this is not so simple to get. So, we have to go to the 3 d wave equation to get that number we have to go to the 3 d wave equation. So, we have to go through the mathematical development.

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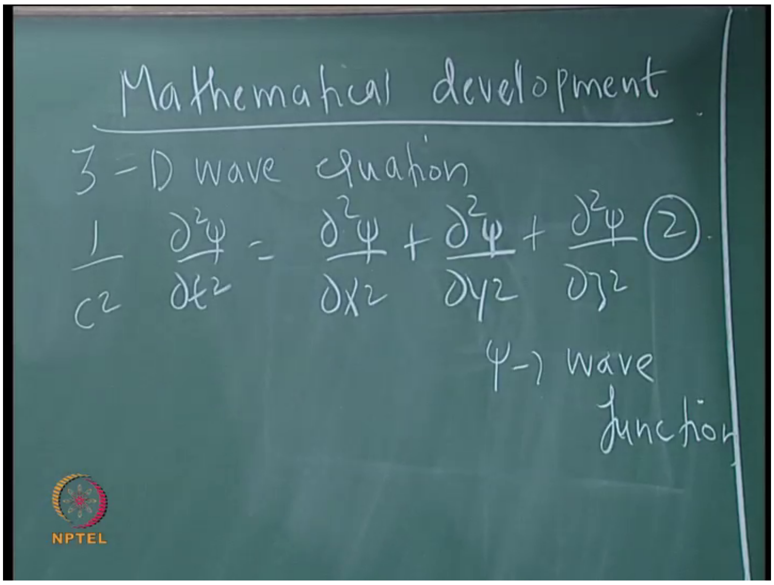

We have to use the 3 d wave equation is applicable to standing waves which are in which are moving around in the container. Now 1 by C square shall we start with 1 this 1 we will start as 1, plus psi is the wave function psi is the wave function. So, it is a electric for a electromagnetic wave it is for an electromagnetic wave psi represents the electric or magnetic field magnitude psi is the wave function. Do not worry I am not going to derive fully the equation the solution and all that and apply the boundary conditions and use the method of separation of variable, we have to get the most pertinent quantity of interest is the frequency will pull out that frequency that is a number of possible modes of number of possible what number of waves. The number count of number of waves of particular frequency if we get that then we can ditch the 3 d wave equation we do not have to worry about it fully because this is not that course on quantum mechanics or statistical physics that is not our intent, but we need to know how to get the number of waves in the prescribed frequency.

Whether it is 1, 2, 3, 4, 5, 6, 7, 8 or how can it be related to other quantities of interest. I would expect that the I would expect that the radiation energy density, because the radiation energy density should be independent of the container. If a bigger container gives me different radiation energy density Rayleigh Rayleigh jeans distribution it would dependent on the size of the container. Which is far from the truth I know that it is not correct. Therefore, I expect that the number of waves will be a function of some term a cube will be there. So, that the denominator I will remove the a cube volume, so that U nu becomes independent of A correct. So, when in my in my development in my development of this wave equation that a will come it has to come, because the boundary conditions are applicable at X is equal to 0, X is equal to A, Y is equal to 0, Y is equal to A,Z equal to 0 Z equal to A you are able to see the connection no.

The radiation spectral density should be a function of a density of the material is dependent on volume of material, or yes the mass of the material is dependent on volume of the material. We talked about intensive and extensive property in thermodynamics. If you suppose the air is so much in air, if you take some volume and calculate the density and you take a bigger volume which is twice this, you get twice the density no therefore, the U nu must be independent of a cube or a, the denominator has got a cube, energy is not a function of a therefore, the number of waves must be some a cube must come there. Now in this wave equation there is no a, but after all this wave equation is applicable to the box through the boundary condition the a will come in the solution. You should intuitively expect some A cube will get cancelled somewhere otherwise we are in trouble. Now what sort of equation is this? What sort of an equation is this?

Student: Thyson equation

Second order differential is it a PD or OD

Student: PD

It is a PD then it is a hyperbolic.

Student: PDE.

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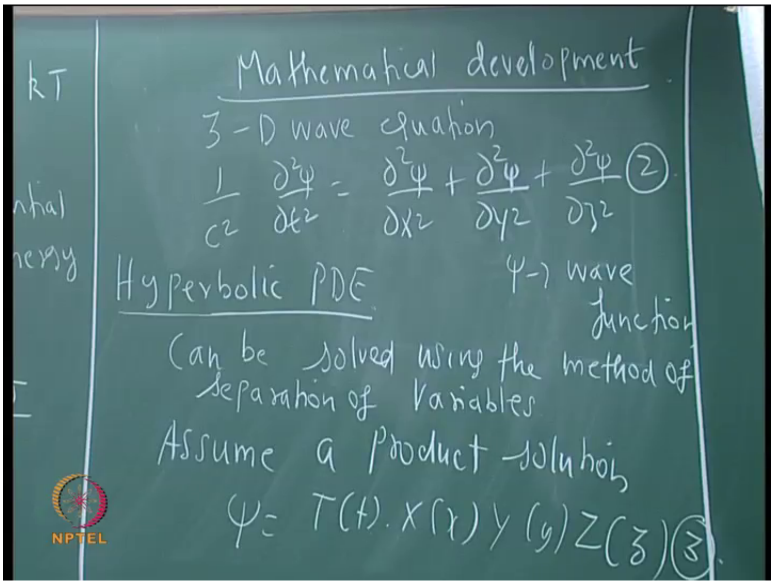

You have to get the equation of characteristics and find out B square minus 4 AC. For a second order partial differential equation, B square minus 4 AC less than 0 equal to 0 greater than 0 depending on whether it is less than greater than A equal to 0. You classify PDE's and elliptic hyperbolic or parabolic. This is not a mathematics class again if you have doubt go to Google or wiki pedia or take any maths book you can figure it out it is a hyperbolic equation. Now this can be solved using method of separation of variables it is a linear PD or non-linear PD. So, it is a linear PD or non-linear again 1 doubt arises what is difference between a linear PD and non-linear in the left hand side you got U dou E by dou X. That is a non-linear that is a non-linear term why it is called non-linear, because it can be taken as dou by dou X of U square by 2.

So, it is depending on U to the power of something U dou E by dou X is dou by dou X of of U square, 2 U DE by DX divided by 2, or in a differential equation U is what you want to find the dou E by dou X is multiplied by the U itself, that becomes if it is X DU by DX it is.

But if it is U DU by DX it is not easy, and then V dou V dou E by dou Y it becomes that saves so much of effort is spent in trying to solve equation. fine Now I can it can be solved using method of separation of variables, the first step in method of separation of variables is to assume a product solution assume a product solution. The psi equal to T of t X of x Y of y Z of z 3; I hope somewhere in earlier courses, you have solved some equation either heat equation or Laplace equation using the method of separation of variables. What are the boundary conditions for this problem?

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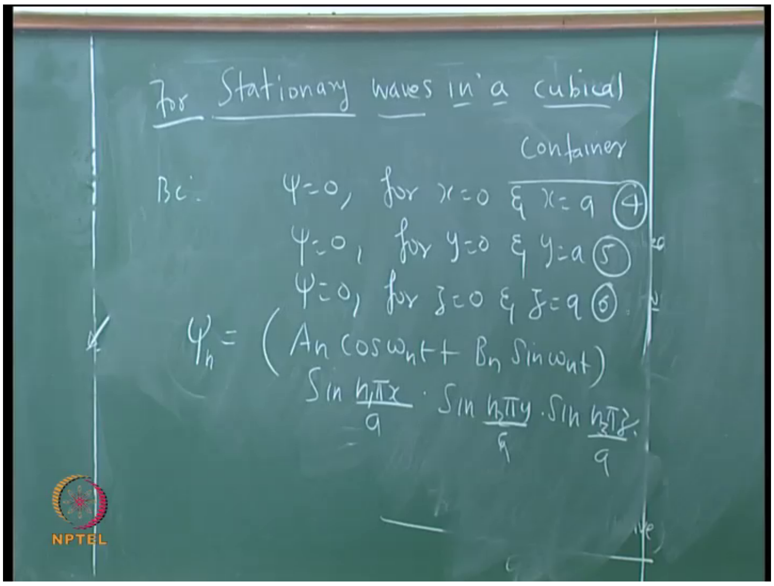

So, for stationary waves for what are boundary conditions it should be 2 equations, but we can keep it as only 1. How many boundary conditions how many boundary conditions does it support 6 2 on X 2 on X 2 on Y 2 on Z. How any condition are required with respect to time 2 either you say psi equal to 0 or T equal to 0 and write you tell the story about dou psi by dou T or if you know some other time you can usually we know the information about psi at time T equal to 0, that is called initial condition. So, since it is second order in time we require 2 initial conditions or 2 conditions on time and then 6 condition in space now similarly, psi equal to 0. We do not have to worry about the condition with respect to time, because we are not trying to solve this I will get an expression of frequency am looking at the frequency I will relate it to the number of waves in the cubical container y equal to a. So, what is the solution to this equation to guess ya solution will be psi will be equal to psi n will be equal to correct into of course, why should the n be the same. So, it should n 1 where n 1, n 1, n 2, n 3 are arbitrary integers they cannot assume fractional values this can be verified.

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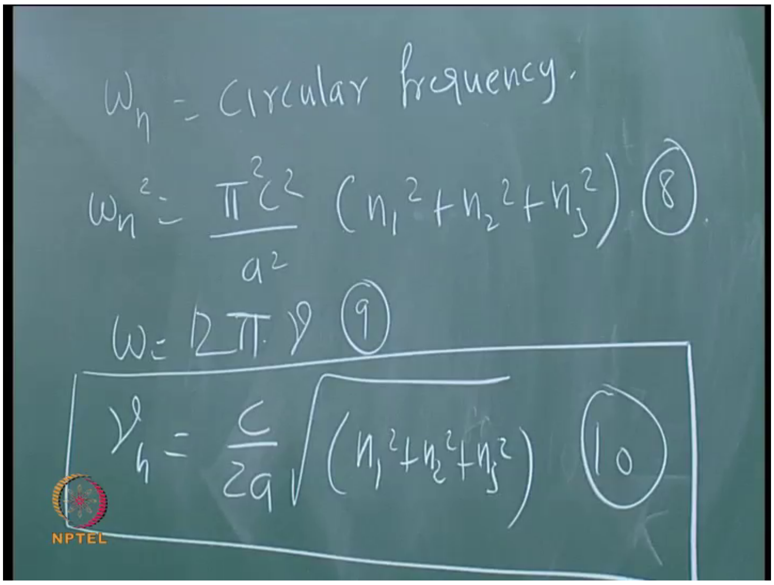

You must have studied the solution of wave equation so n1, n2, n3 are integers. What is omega n there is a wave with the omega n is multiplied by time. So, omega must be a frequency why omega means, what type of frequency circular frequency omega n is the circular frequency. So, what is the expression for the circular frequency omega n will omega n square? So, 7 8 it is called the circular frequency. Omega can be omega is equal to omega in terms of nu 2 pi correct yeah please get an expression for nu n C by 2 a C by 2 a root of n 1 square plus n 2 square plus correct. So, we have we have achieved some limited success so far, we look at the mathematical development of the 3 d wave equation, and then we wrote the general hyperbolic PDE, we got a general solution and then we got the we did not solve it fully, but whatever we wanted was the expression for frequency and circular frequency then we put it in terms of nu and then we got nu n is c by 2 a into n1 square plus n2 square plus n3 square 2 root of.

Now we are left at this stage how do we proceed further. We want the number so what is the challenge now the challenge is the challenge is is to come up with an expression for the number of these discrete frequency modes, that are allowed between 2 interval that are allowed in a frequency interval nu to nu plus d nu. And then you got the and then you you can go home your home, but now you got some n1 square n2 square n3 square it is more confusing than our starting point. So, something we have to in order to do this we have to look at some analogy. So, we are actually we have to do some number count the number of possible the number of discrete modes of of these waves which are allowed between frequency interval of d nu between nu and nu plus d nu. For that we will we will resort to geometry we will use some analogy. So, can you make it can you make it bigger.

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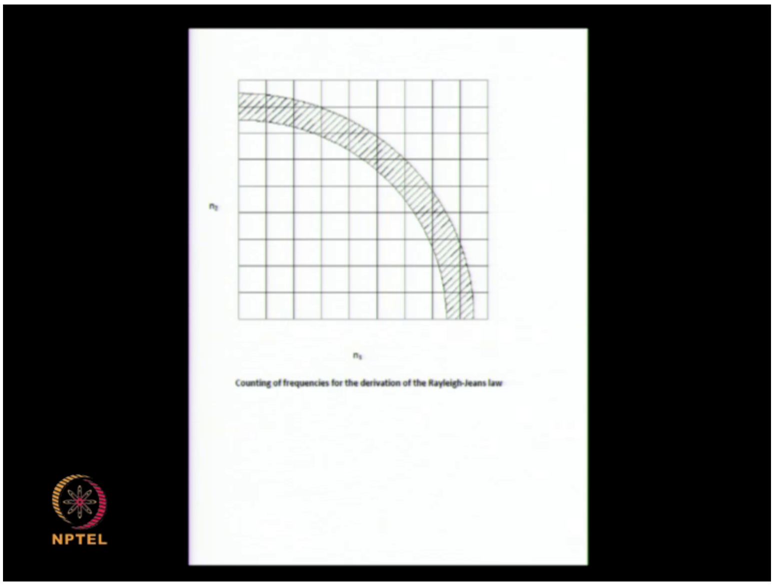

Consider consider a situation like this n1, n2, n3 can take on integer values and n 1, n 2, n 3 all have to be positive. fine Now let us consider 2 dimensional situation where you have got unit squares, where you have got the unit squares. I am looking at the I am looking at the differential area, I am looking at the differential area, 2 pi r dr in the first quadrant, and n1 and n2 are just like these 2 numbers we have n 1, n 2, n 3 I am doing a 2 d representation of this. So this 1, 2, 3, 4, 5, 6 now let us consider we are starting from 0 0 here this is all unit radius this is therefore, here they have come up to this is 8, that is it is between it is between 7.5 and 8.5 what will be the area.

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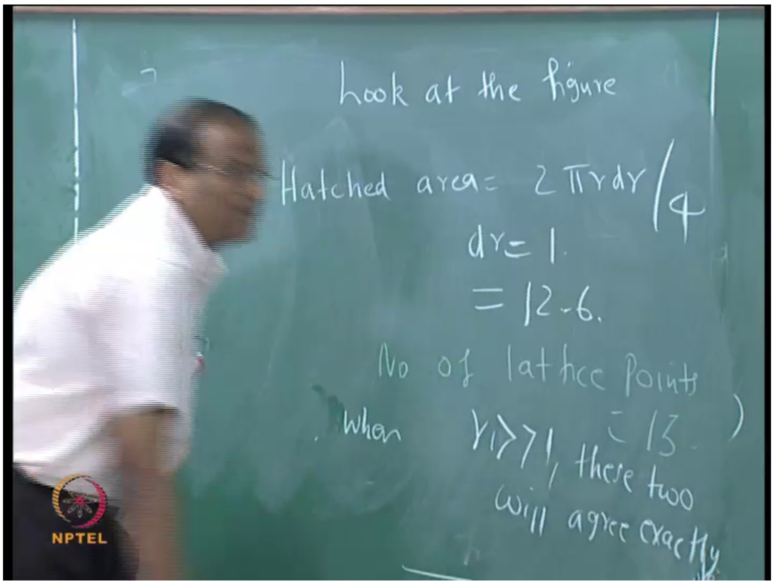

Look at the figure everybody has this sheet hatched area see this is not the only way of proving the Rayleigh jeans distribution, but I thought this very intuitively appealing some other physics books may deal with it differently, but I thought this is easier for all of us what is the hatched area. 2 pi r dr what is dr dr equal to is is everybody convinced that dr equal to one, because unit square I am taking so what is 2 pi r dr now tell me r equal to 8 please calculate 2 pi r dr by 4, because it is a quadrant everybody is when I say 2 pi r dr nobody is opposing me 2 pi r dr by 4 2 pi r dr is the full thing how much what is the what are you getting 12.56 what is this elementary geometry, what is the big thing the beauty is count the number of latice points which this hatched area cuts 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13. So, if n or r is sufficiently greater than 1 the area of the quadrant is exactly equal to the number, but what we want for spectral density is the number is the number of waves.

So the number of waves can be related to the area, but here we had only n 1 n 2, but in actual case we will have n1, n2, n3 instead of a quadrant we will have an octant of the sphere. The first octant if we are able to figure out 4 pi r square dr by 8. That is exactly equal to the number of latice point that number of latice point the number of waves which which are allowed between between the between the frequency of nu and nu plus D nu.

Is it so number of when r is much much greater than 1 these 2 will agree exactly. So, we are not home yet, but we are going in the direction.

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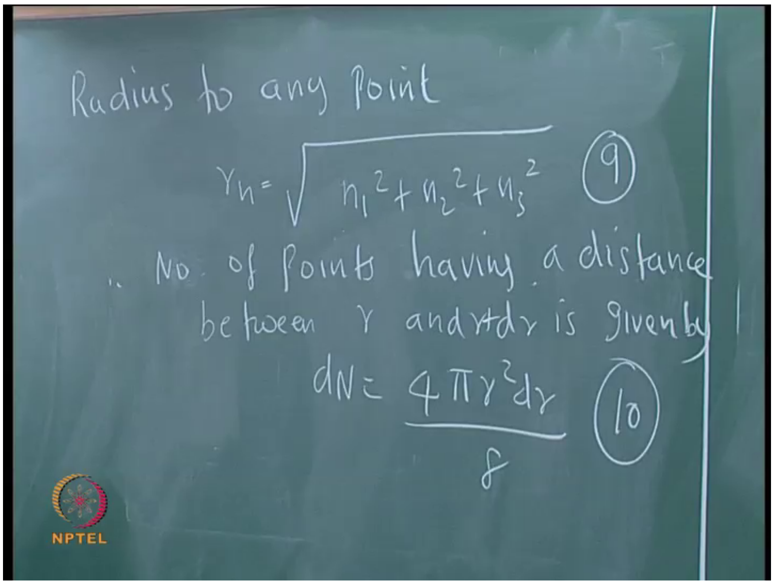

Now the radius to any point. So, the radius to any point r n is given by root of n 1 square plus correct. So, in your coordinate system n1, n2, n3 root of n 1 square plus just like root of x square plus y square plus z square put some equation number. So, what we are now trying to do is in order to get this. Let us think of these discrete frequencies in terms of a space lattice and we are counting the number of lattice points having unidimension that is what that is what I have done, but I have done it for the 2 dimensional case now the number of points having a distance therefore, given by equation 9 you are convinced in a 2 dimensional case the number of points which are between r and r plus dr can be easily counted by not doing a number count, but by just taking 2 pi r dr by 4. Therefore, the number of points lying between r and dr sorry r and r plus dr in a 3 dimensional in a 3 dimensional lattice space is given by 4 pi r square dr by 8 instead of 2 pi r dr by 4 we have 4 pi r square dr by 8.

So, the number count was analogous to the area elemental area, in the case of a 2 d is equal to the elemental volume in the case of the 3 d. Which 1 good which 1 d Nis elemental know we are still, yeah but we will take that elemental and divided by the elemental A cube and then we can proceed. That is a representation because 2 pi r dr is also elemental. So, when r is much greater than what this can be visualized as the number of lattice in the volume of the of an octant of the spherical surface having a differential thickness correct.

Since, the number of lattice intersections corresponding to the number of units itself if you say something like this you may get confused, but you have already understood. So, we can so this is the terminology used by those guys.

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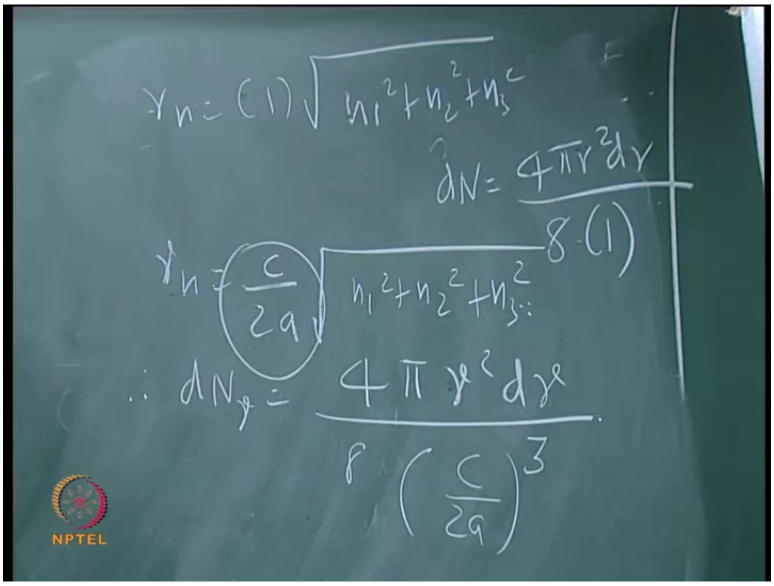

Now now from r n is equal to 1 into then we had d N is equal to 4 pi please remember that we divided by a 1 in the denominator, because it was unit radius, because the numerator has got units of numerator has got units of volume denominator also has to be by volume, because the left is just a number volume can never be equal to the number for illustration. I told you unit that means I am dividing by 1 into 1 into 1 please remember, but now but, now your r n is c by 2 a root of so r n is equal to c by 2 a, correct this what we got we already got this. Therefore, d N nu must be equal to 4 pi nu square d nu divided by 8 into what do you what should you divide by c by 2 a whole cube. When it was 1 into n square plus for a 2 d case, if it is r n is equal to 1 into n square plus n 2 square, the d a was 2 pi r dr. When r n is equal to 1 into n 1 square plus n 2 square plus n 3 square within root it is d N equal to 4 pi r square dr by 8 into 1. That 1 is corresponding to this, but now we already derived the circular frequency nu n is if is equal to c by 2 a root of n 1 square, plus n 2 square, plus n 3 square 15 minutes ago.

We derived this therefore, it will be 4 pi instead of r I will have nu square r square replaced by nu square dr is replaced by d nu I divide it by 8 and instead of 1 I have to do c by 2 a, because c by 2 a is multiplying the root is prefixing all.

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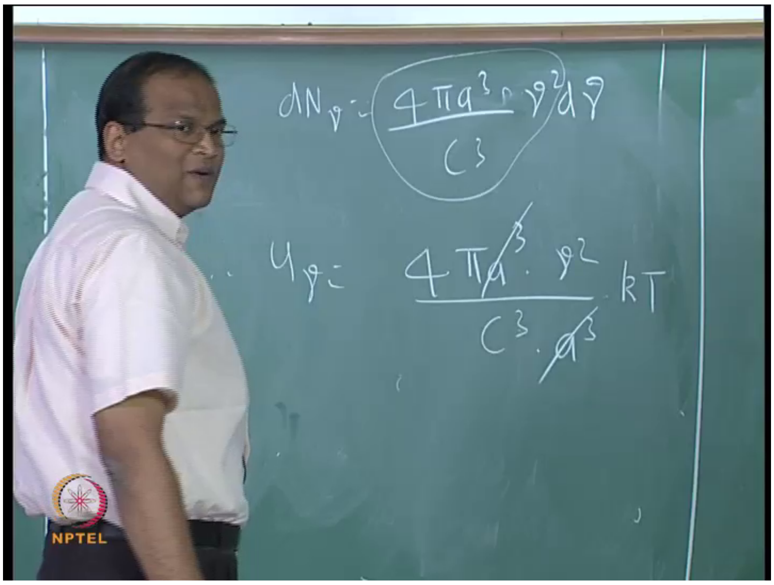

Now what is dn nu now now please get the expression for dN nu 4 pi a cube agree all these you can keep in brackets, they are not central to our they are only for our understanding. It would have been terrible it would have been terrible, if I told you the elemental number of number of possible wave modes is given by this how many of you would have understood. That is why I used the 2 d analogy we star[ted]- we did the number counting extended to 3 d instead of multiplying by 1. I multiplied by c by 2 a and all this story. Therefore, so I will take only this anyway my d nu was equal to 1. Now I will take this u nu is cube is gone a cube is gone.

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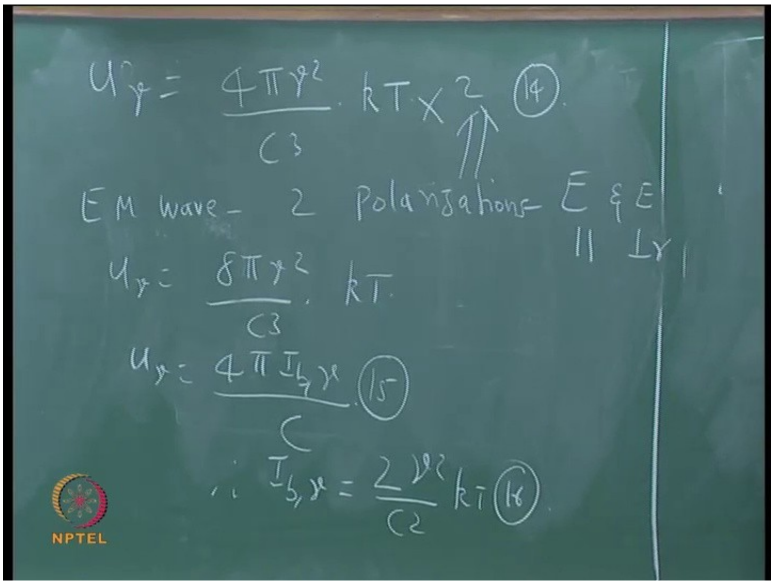

Now what is the equation number what is the equation 13 13 good. So, U nu equal to 4 pie nu square by c cube into KT, but 2 possible directions need to be considered it is in electromagnetic wave. What are 2 types of polarizations which are possible for an electromagnetic wave E parallel and E perpendicular, electromagnetic wave correct electromagnetic wave? What are these called E parallel and E particular polarization and horizontal polarization therefore, u nu must be into 2, because there is an addition degree of. So, it will be 4 pi nu square by c cube KD for horizontal polarization 4 pi nu square by c cube KT for the vertical polarization you add these 2.

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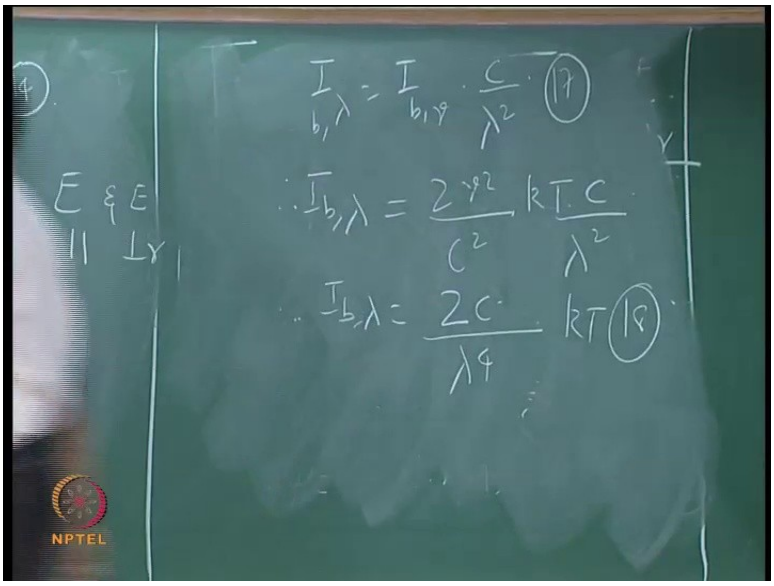

Now you got u nu equal to 8 pi nu square c cube KT. So, what is the relationship between u nu and I nu. Therefore, IB nu equal to can you help me out IB nu equal to 2 nu 2 why are you getting struck 2 nu square correct. What is the relationship between I lambda equal to IB lambda equal to IB nu into, correct c by lambda square or lambda square by c, do not make please do not let me down am not recalling with memory am also doing with you therefore, IB lambda equal to yeah tell me 2 2 nu square by c square, nu can be written as c by lambda c cube is it correct? if I made a mistake you just tell me fine very good.

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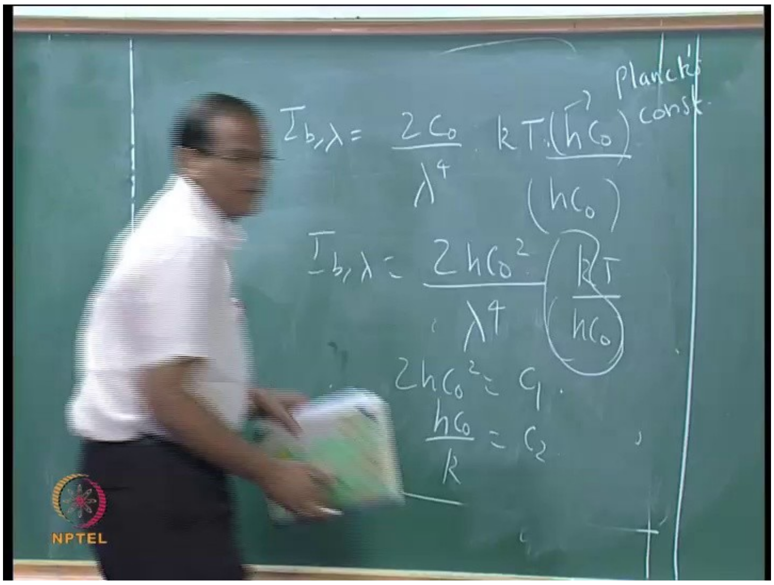

I can make it c naught I am multiplying dividing by. So, IB lambda why 2 h t naught square what is this remaining KT by h c naught which 1 it is the Planck's constant it is not, but people know the first radiation constant and the second radiation constant. The 2 h c naught square is the first radiation constant c 1. Now h c naught by K so now h c naught by k is c 2 now can you put the IB lambda.

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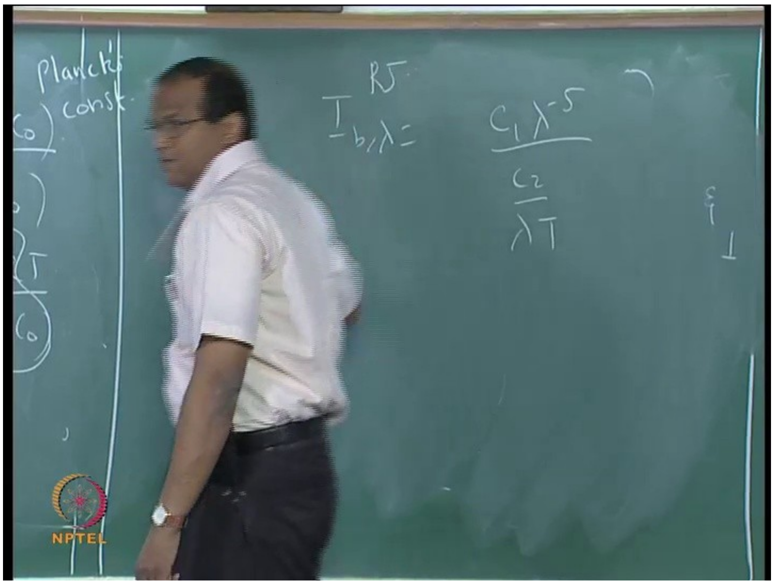

We are this is the Rayleigh jeans distribution of course, he did know that 2 X e he got some all this and, now this is called the first radiation constant, and this second radiation constant fine. So, I think you have to go through you to have to go back to your hostel and then go through this couple of times it is not so easy to follow this, but it is without any blemish its everything is correct in this and you will have to understand if something is bothering you can come and ask me. So, we first derived incorrect distribution and in tomorrow class we will derive correct distribution that is the IB lambda Planck. So, this is our number counting

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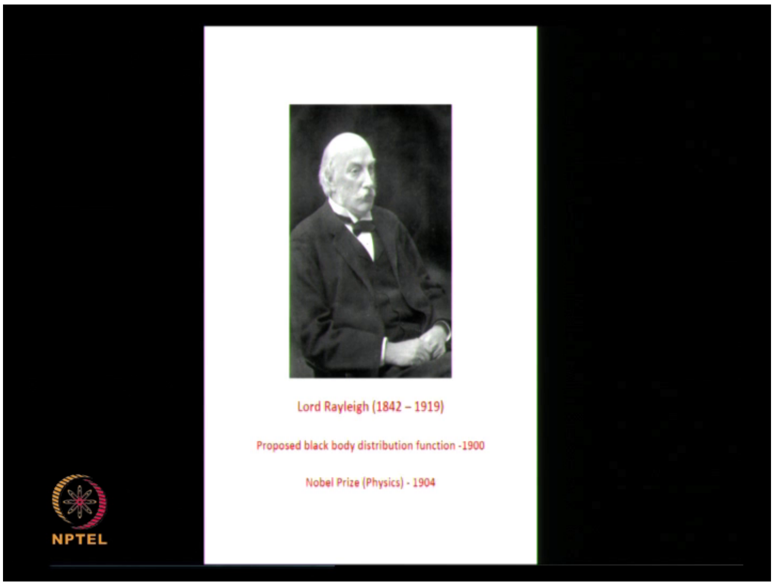

This is Lord Rayleigh 1842 to 1919 he proposed black body function 1900 noble prize 1904

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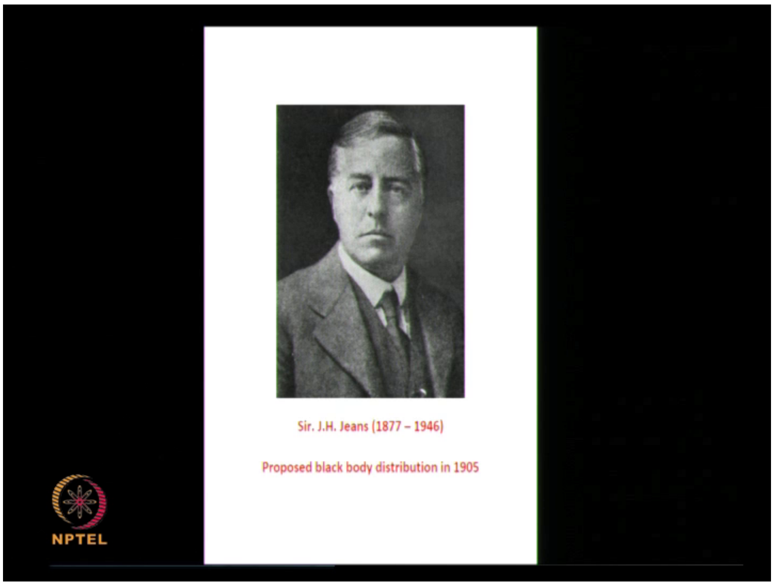

Sir Jeans 1877 to 1946 he proposed black body distribution in 1905 both of them published the same magazine for journal called philosophy magazine.

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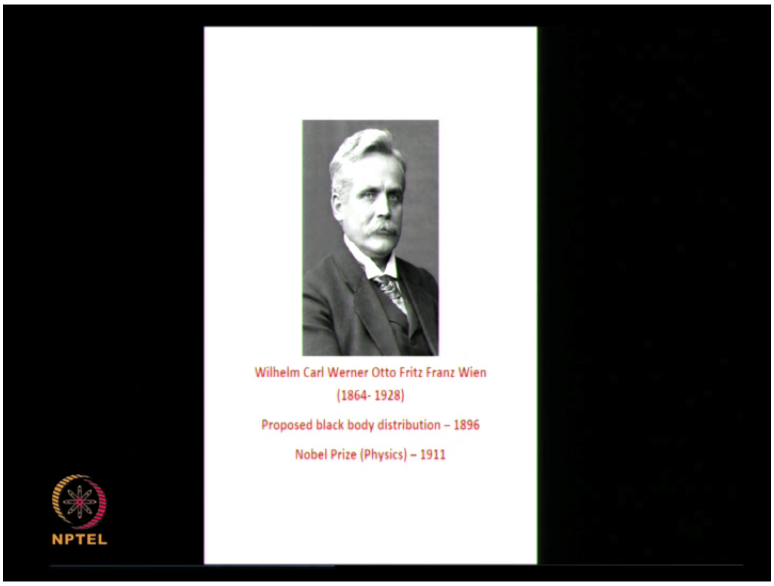

This is the guy with short name, so wines proposed black body distribution 1896 noble prize 1911 for radiation all these people are getting noble prize in radiation.

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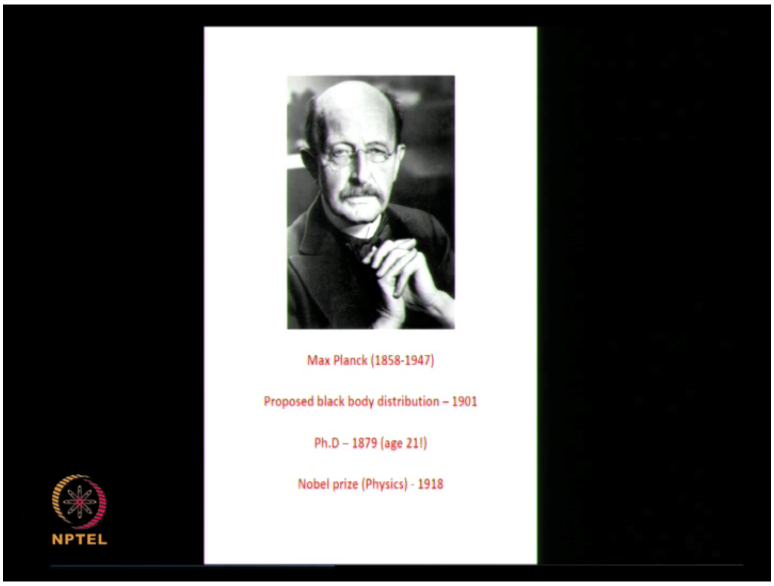

Then finally, max Planck 1858 to 1947 he proposed black body distribution 1901 he got p h d at age of 21 1879 he defended his p h d on second law of thermodynamics noble prize 1918 on that note we will close the class with picture of Planck Planck the ultimate max Planck. So, we will meet tomorrow; so we will see how this Planck figured out the distribution