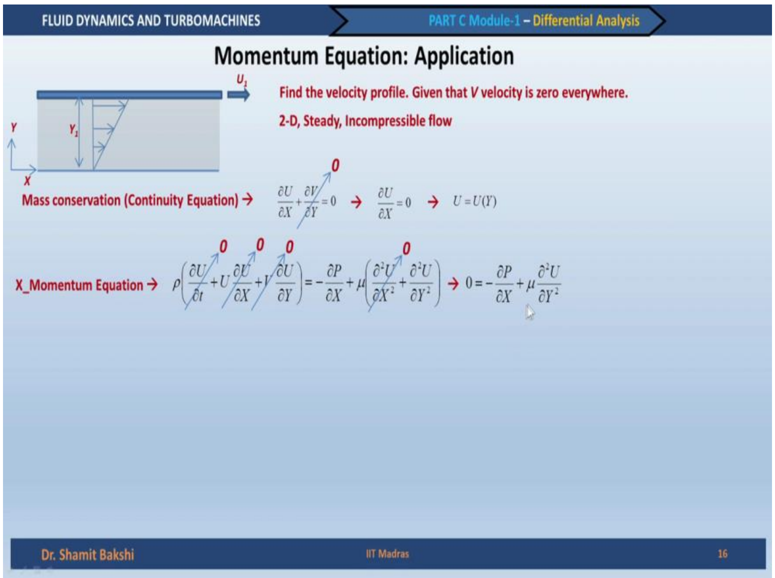
Fluid Dynamics And Turbo Machines.
Professor Dr Shamit Bakshi.
Department Of Mechanical Engineering.
Indian Institute Of Technology Madras.
Part C.
Module-1.
Tutorial.
Differential Analysis.

Good morning, so this is the tutorial session for the 3rd week. We in this week we have looked at the Naviar-Stokes equation, the derivation of the Naviar-Stokes and the definition of fluid rotation, vorticity, vorticity vector, stream function, potential functions, Euler and Bernoulli's equation. So some of the concepts we will demonstrate to some tutorial problem. So first we take up the application of momentum equation.

(Refer Slide Time: 0:55)



So this basically, it is a fluid, so we have a fluid here and there is a plate on top of the fluid, the plate is moved at a velocity U1. So we want to, so the statement of the problem is find the velocity profile. Given that V velocity is zero everywhere. So this is a two-dimensional two-dimensional flow, steady flow, not changing with time and incompressible flow. So let us see if we can start from the general form of the Naviar-Stokes equation and get to a solution of this velocity profile. So first thing is the mass conservation equation or the continuity equation, so for the 2-D steady incompressible flow it is the form Dell U by Dell X plus dell

V by Dell Y is equal to 0, so this is, in this case also we will see that this is very important to utilise the mass conservation equation.

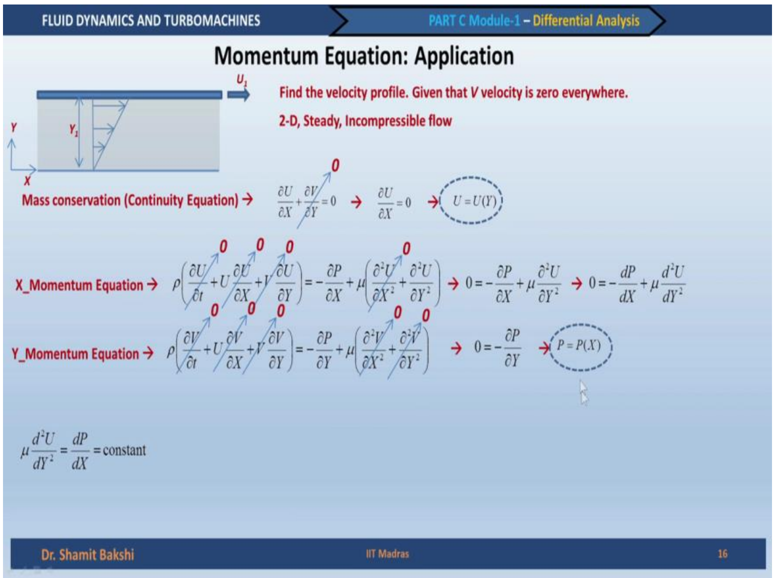
So as the V is identically zero everywhere, so V velocity is zero everywhere, it means that dell V by Dell Y is also zero. See one thing we have to keep in mind that velocity zero does not necessarily mean that gradient is zero but in this particular case because the velocity is zero everywhere, that means identically, then the gradient has to be zero. So we can drop out the dell V by Dell Y term. But Dell Y by Dell X is, sorry Dell U by Dell X is nonzero, so directly, but because Dell be by Dell Y is zero, so Dell U by Dell X is also zero. So what does that mean, it does not mean that U is constant because this is a partial differential equation, U is a function of both X and Y, so Dell U, if you take a derivative of U with that to X, it can be zero if U is a function of Y. So in general we can write from here U as a function of Y because suppose U is Y square, if you do a partial derivative of Y square with respect to X, it will become zero.

So in general we can write U as a function of Y. So in physically what it means is that, it means that U is not changing along X direction. So if you take, say at some point in X here, let us say X1 and X2, the U velocity and that means the velocity profile itself because this is applicable throughout the flow. So the velocity profile does not change along the X direction. This flow, this kind of flow has a different name also. This is called fully developed flow. When the flow in a particular direction does not change along that particular direction everywhere it does not change along that direction, then it is called fully developed, it has developed completely. Anyway we will come to that again while discussing pipe. So that means velocity is non-constant but it is variation is only along Y and not along X. Now let us see X momentum equation. So while dealing with this kind of a problem, what we can do, at least the way which is demonstrated here, we start with the full form of the X momentum equation and see what terms can be neglected. So we start with the full Naviar-Stokes equation, okay, X momentum equation, so then we see what terms are negligible here. So first of all this is a steady case, so this term goes out, so this is zero.

Then see we already got Dell U by Dell X is equal to 0 from here right, so if we use that criteria, then this 2nd term also vanishes, it goes out. Then see V is zero, so this term also goes out. See to begin with we do not do anything, we just, to deal with this kind of a mathematical problem with start with the full form and see how the terms dropout. So the 3rd term also drops out. Then what happens, then we have Delta 2 U Dell X 2 as zero because of course if Dell U by Dell X is zero everywhere, not at a particular point, everywhere this is zero. So that means if you take the 2nd derivative of that, that will also be zero. But again just to remind us that if Dell U by Dell X is equal to 0 at a particular point, that does not mean that Dell 2U by Dell X2 is zero but this is identically zero here, so this drops out.

So we see here that this term also, Dell 2U by Dell X2 drops out, it is also zero because Dell U by Dell X is 0, identically zero, that means zero everywhere, so Dell 2U by Dell X2 will also be zero. Now this reduces to this form, that means now you can write, you are left out with only 2 terms on the right-hand side Dell P by Dell X or minus Dell P by Dell X more specifically plus mu Dell 2U Dell Y2.

(Refer Slide Time: 5:40)

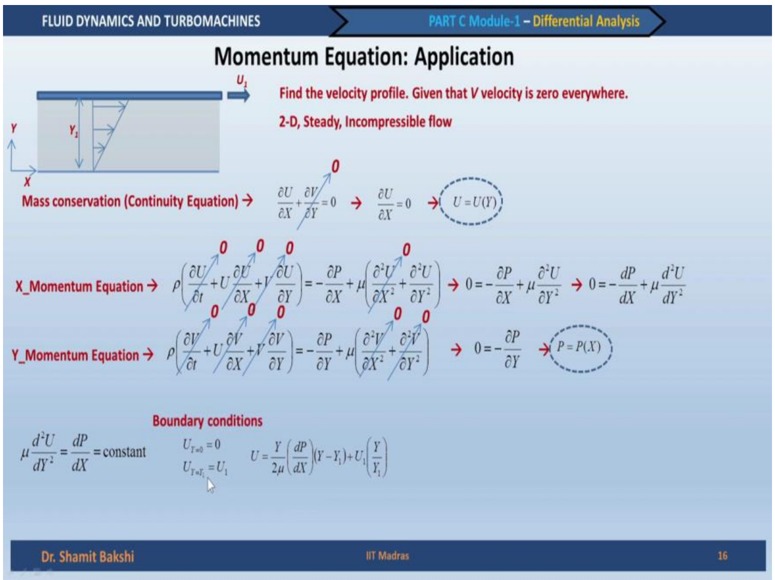


Before going further with this equation let us look at the Y momentum equation. So the Y momentum equation looks like this here, as shown here. Even the Y momentum, even though the V velocity is zero in this case it is important to consider this equation. We will see why. So because V is zero, all the terms actually disappears here are zero. Dell V by unsteady flow or V is equal to 0, both will tell you that the first term is zero, the 2nd term is also zero because V is zero everywhere, Dell V by Dell X is 0, this is also zero, this is also zero because V is zero everywhere. So all the terms are zero.

So that means what are you left out with, you are left out with a very important conclusion that Dell P by Dell Y is equal to 0. That means P is a function of X, so as you see here in this particular problem, U is a function of Y, that is the velocity changes in the Y direction and the pressure is only changing in the X direction. Okay so the U velocity is fully developed, it does not change along the X direction, the P, the pressure is actually only changing in the X direction. So this information is important to reduce this equation now because if P is purely a function of X and U is purely a function of Y, then this can be now written as in terms of not partial derivatives but total derivatives. So we can write this as d minus dP by dx plus mu d2Y dy2 is equal to 0.

So this equation now let us see we can, if we can solve this equation. This equation essentially means that mu d2U dy2 on the left inside is equal to dP by dx and this should be equal to a constant. How can we say confidently that this can be, this can be, this will be a constant? Because see on the left-hand side we have purely function of Y or its derivative, 2nd derivative of something which is purely a function of Y. On the right-hand side we have P which is purely a function of X. A function of Y and X, they can only be equal if both of them are actually constant. So we can easily write this as constant. Now by doing this, after doing this it is easy because this is a constant, we can easily integrate this equation. So of course this comes from the fact that U is only a function of Y and P is a function of X, that we already said that we can write it as constant because of these 2 factors.

(Refer Slide Time: 8:24)



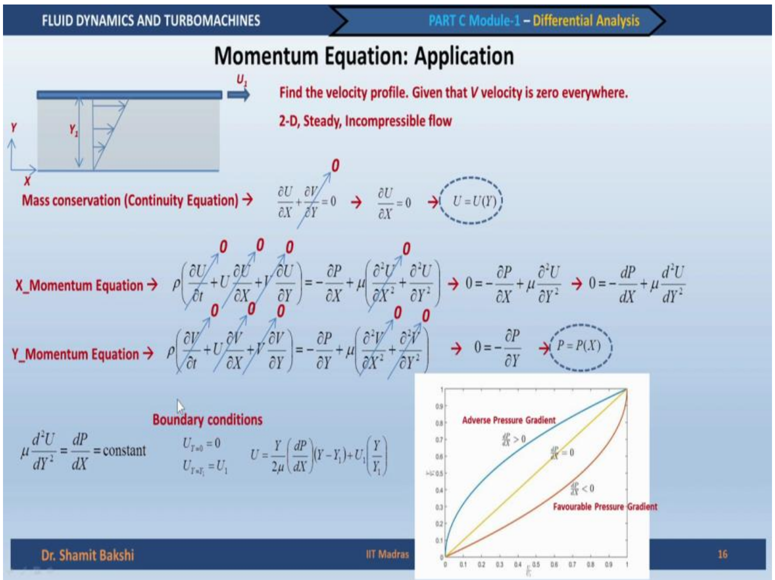
Now we apply the boundary conditions. The boundary conditions are that at Y equal to 0, no-slip condition is valid, U is zero and at Y is equal to Y, again no-slip condition is valid. That means this is equal to U1. That means the velocity of the fluid touching the lower surface of the plate has the same velocity as the plate. So that is also no-slip condition. So in both ends here we apply the no-slip conditions, that means there is no slip between the fluids and the plate with which or the solid object with which it is in contact with. So by applying that condition and we can easily integrate this equation, it will come in this form. If you just do this integration, this is basically double derivative of a function is constant. Which this constant in this particular case is given as dP by dx.

So if we do that particular integration, we get with this boundary condition we get this as a analytical solution to this problem. So analytical solution means U is known as a function of Y directly. So in these 2 form. Now what we can do is, so you can apply the boundary condition and see that it is automatically satisfied. For example if you if you put Y is equal to Y1 here, this term becomes zero and Y is equal to Y1 here becomes U1. So U becomes U1 and if you put Y is equal to 0, so if you put Y is equal to 0, then what happens, then this term becomes zero because of this Y and this right-hand term also becomes zero. So this also satisfies this boundary condition. So this is a quick check for anything so that to make sure that we have applied the boundary condition appropriately. So now if we go forward with this solution and try to actually see how the velocity profile is, this gives us some interesting insight into the problem.

So for example we start with the first case where dP by dx is zero. That means this drops out, that means there is no pressure gradient, only only the plate is pulling the fluid. The plate is only, with the help of shear it is actually making the fluid flow, there is no pressure gradient. So then you have a very linear profile like this, so this is basically the solution. On the other hand if you have Dell P by Dell X, let us see is greater than zero, what it means, that means if you go along X direction, pressure is increasing. So dP by dx is greater than zero means that because if X is, as X increases, pressure is also increasing. So if the pressure is increasing along the direction of the flow, then it will actually try to oppose the flow. So that kind of a pressure gradient is called an adverse pressure gradient.

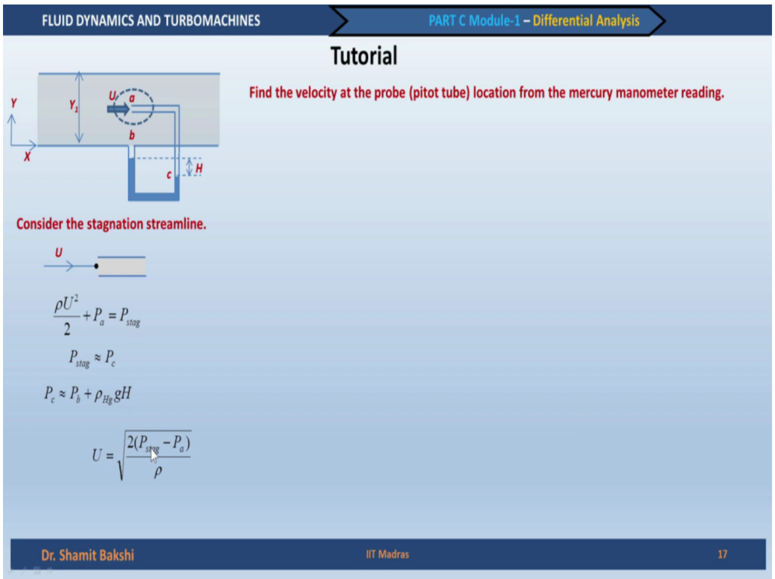
So dP by dx greater than zero is an adverse pressure gradient. And we get a velocity profile like this. So this velocity profile actually means that here on the X axis we have plotted U by U1 and on the Y axis, on the vertical axis, we have plotted U Y by Y1. So as you can see here it means the boundary conditions at Y by Y1 is equal to 0, U is zero, U by U1 is zero and at Y by Y1 is equal to 1, U by U1 is equal to 1 in all the 3 flows, in all the 3 cases of dP by dx. So we were talking about the adverse pressure gradient, so what it does is basically it tries to oppose the flow and you get a velocity profile like this. We will see in the next chapter that basically this adverse pressure gradient is something which is responsible for flow separation.

(Refer Slide Time: 12:15)



So we will see that later. And there is a 3rd situation which is called a favourable pressure gradient, that is dP by dx is less than zero. So it is less than zero, that means it is negative, that means if you go along this direction, then the pressure is reducing. So flow will have naturally a tendency to flow in that direction, the fluid will have a natural tendency to flow in the direction where in which the static pressure reduces. So this helps the flow, that is why it is called a favourable pressure gradient and we will see that this favourable pressure gradient will never cause a flow to separate. That means detached from the surface. So we will discuss that, this will be useful while we deal with the boundary layer flow in the next chapter. But overall this flow this problem demonstrates how systematically we can apply a particular flow situation to the general form of the continuity and the momentum equation.

(Refer Slide Time: 13:10)



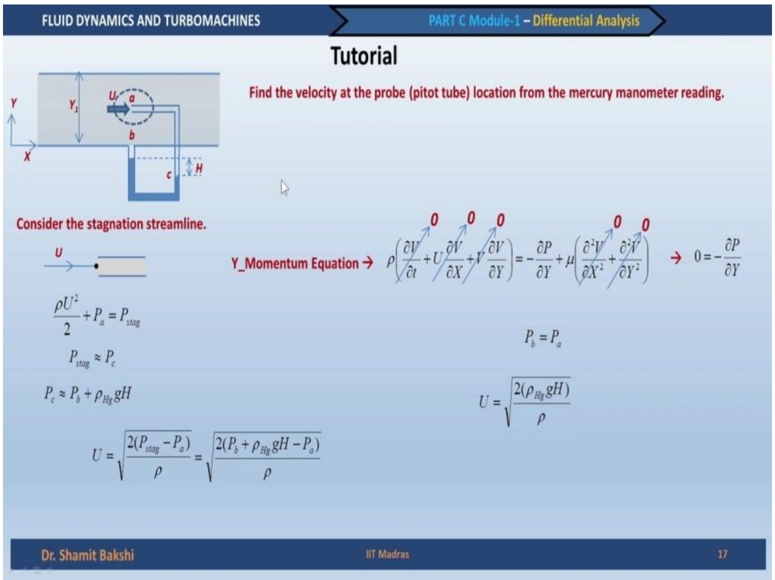
This is the first case, the 2nd case in the tutorial is this particular problem. This one demonstrates the Bernoulli's, the application of Bernoulli's equation. So let us say what, let us see what is the statement of the problem. So find the velocity at the probe pitot tube location from the mercury manometer reading. So let us explain the problem here. So let us say this is a flow again between 2 parallel plates or through a channel, we want to find out velocity. So this is actually a probe to measure the velocity. How it measures the velocity, what we can do is if we put a particular tube here, which is called as pitot tube, so if we put this tube here, so the flow will stagnate in front of it. And then if you have manometer, a mercury manometer in this particular case, it will, this stagnation pressure will actually push the mercury head down and then on the other side if it is connected to the flow itself, this will go up and by noticing this reading, this edge reading we can find out the magnitude of the velocity at this location.

So how to, how does this velocity relate to this edge. So let us see that if we can derive it using the Bernoulli's equation. So for that we know that Bernoulli's equation is applicable along the streamline. We consider the stagnation streamline. What is a stagnation streamline? So let us look at this part of the flow carefully. So this is the tube now, so this is your tube, the pitot tube, the tip of the pitot tube, the flow is coming along the streamline and stagnating here, so this dot shown at the tip of the pitot tube is the stagnation point. If you remember our derivation of Bernoulli's equation, what it said is the sum of the dynamic pressure head and the static pressure head, that is constant. Now what happens is the velocity here, the particular velocity here in front, little front of the pitot tube is U but as it comes here, it becomes at the tip of the tube the velocity is zero.

It has to be zero because the flow cannot enter the pitot tube. If it enters the pitot tube this mercury will come out of here. So this does not have enough pressure actually to enter into the pitot tube, it just reduces a stagnant layer or a stagnant medium inside the entire tube. But the flow, what it does is, because it stagnates at the tip, so it increases the pressure within this layer, within this fluid inside the tube. As a result of this the high pressure will push the mercury level down. So this is the stagnation pressure line, sorry the stagnation streamline, we can apply the Bernoulli's equation directly here. So as we can see the static pressure at A, so if we take this as a point A rho U square by 2+ static pressure is equal to the stagnation pressure here because the sum of these 2 has resulted in the stagnation condition.

So if we go forward now, if you see this stagnation pressure here is almost equal to PC we said is almost equal to PC because there is a level difference between the C and the height A, we have neglected the weight of the fluid in this particular tube, that is okay because this height is generally very small or this density is also low if it is just an airflow. So this is more or less same. Now PC can be also written as PB plus this particular head. So if you look at the pressure here, what is it supporting? It is supporting this, if you draw a straight line along this, this is supporting the pressure at PB plus the liquid height here which is very small and the density is also less, so this pressure PB and the essentially this PB and the mercury head, the difference in the mercury head between the 2 limbs of the manometer, that is H. So rho HG is the density of the mercury and rho HG GH is basically the PC.

(Refer Slide Time: 18:32)

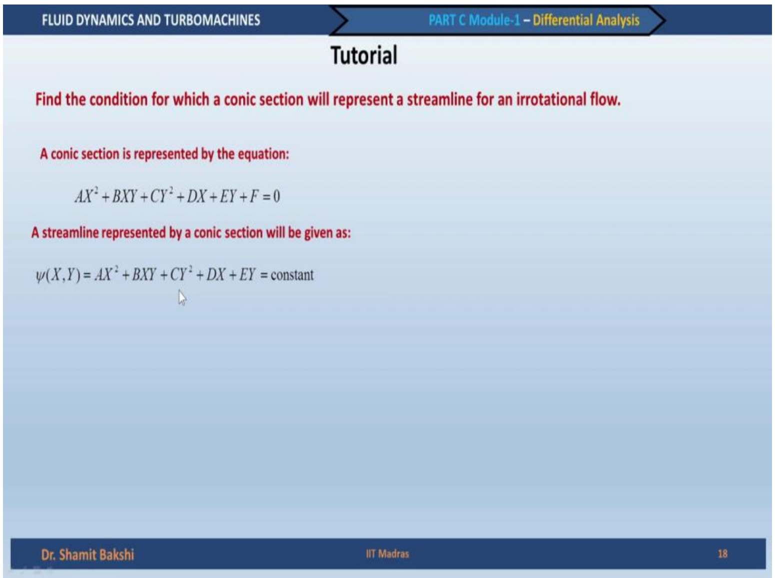


So now this means that P stagnation is basically PB plus this. So we can plug-in the P stagnation value of this into this equation here, if we do that, what we get is basically an expression for velocity. So the velocity which we are trying to measure using this pitot tube is equal to square root of 2 multiplied by P stagnation minus PA by rho. Now what is the P stagnation? P stagnation we saw here that P stagnation is this, so if we plug-in P stagnation as PB plus this into this equation what we get, we get this expression. We get this as PB plus this minus PA is equal to rho. Now we cannot proceed further because this is PB looks like here is unknown, what is PB? Let us understand what is PB. So basically this PB is the wall static pressure, the static pressure at the wall which is measured by, which is sensed by this limb of the manometer, this left-hand side limbo of the manometer, PB is basically the wall static pressure. What we have here is PB and what we have here is PA, so how does this PB and PA relate to each other? So this is a very important thing, unless we resolve this, we cannot get the velocity. Let us see how we can get that.

So for getting that we can take help of the last tutorial which demonstrated the application of the Naviar-Stokes equations. If we remember the Y momentum equation was something, was this and for this we said all the terms actually vanishes. What are we left out with? We are left out with Dell P by Dell Y is equal to 0. So this is a very important thing. So you see the pressure variation along Y direction is not there. So if pressure variation is not there in Y direction, that means the wall static pressure and the static pressure at the Centre of this channel, that is PA, both are same. So PA is equal to PB. Once you do that, once you use that information that PB is equal to PA, you get an expression for velocity.

So velocity is basically, so what you measure in this condition is basically the height of the mercury in the manometer, you know the density of the mercury, acceleration due to gravity and rho is the density of the fluid which is flowing through this channel. All these are known to you, you can get the velocity.So this demonstrates 2 things, one is the application of the Bernoulli's equation and secondly a probe which can be used to measure velocity in a flow, is called a pitot tube.

(Refer Slide Time: 20:51)

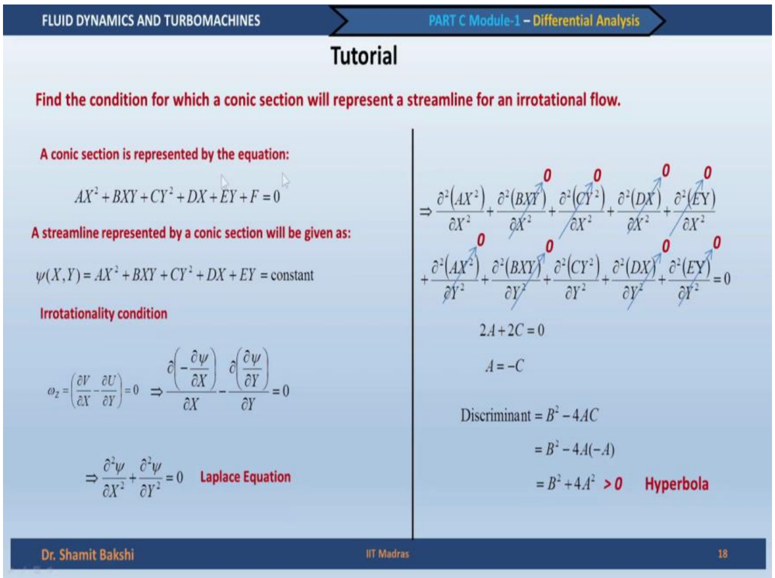


Then we go to the 3rd problem, the 3rd problem relates to the streamline, the application of the concepts of stream functions. So the question which is asked here is find the condition for which a conic section will be present a streamline for an irrotational flow. So I think you are familiar with this term conic section, if you are not, just to give some indication, the conic section is actually represented by this equation. So this is basically a general equation, if you look at it, it is, it has AX square plus B XY plus CY square plus DX plus EY plus F is equal to 0. So this is actually an equation of a curve. How do you get that curve, you take a cone and you take the section of the cone. If the, if you take, it is if it is plane parallel to the base of the cone, then it will form a circle. If you take a plane at an angle with the base of the cone, then the intersection of the cone and the plane will not form a circle, it will form and ellipse or it will form parabola or hyperbola.

So the when it forms a ellipse, a circle, a ellipse or a parabola or hyperbola, depends on how this particular coefficient relates to each other. There is something called discriminant, so this discriminant is defined as B square - 4A into C, it does not depend on this, so it depends on this particular coefficient. If that is equal to 0, it is a parabola, if it is greater than zero, it is a hyperbola, less than zero details an ellipse, like that. So we will see, this is just some basic our high school mathematics. So we see, if we suppose we have a streamline which has equation and we are also saying that this is, the flow is irrotational, so what is the condition that this equation can represent a streamline often a rotational flow. So this is basically the question.

So streamline represented by the conic section will be given as, okay so suppose now we this was just the conic section, equation of a conic section, the streamline represented by this curve will be given like this because psi X, Y is actually a constant. Along a streamline, for a particular streamline psi X, Y is constant and the function is given by this equation. So this F can be taken on the other and we can get this particular equation of the streamline.

(Refer Slide Time: 23:50)



Now the 2nd thing is we have to apply the irrotationality condition to this. What is the irrotationality condition, so this is a two-dimensional flow, irrotationality condition only says that omega Z is equal to 0, that means Dell V by Dell X minus Dell U by Dell Y is equal to 0. Okay, so if we apply this condition now, it means that Dell Dell X of minus Dell psi by dell X and minus of Dell Dell Y of Dell psi by Dell Y. So that means the psi function or the stream functions satisfying the Laplace equation. This we discussed while introducing stream function also that if a flow is irrotational, then the stream function, of course it is 2-D because stream function we are defining for a 2-D only, so the stream function if it is an irrotational flow, the stream function will automatically satisfy, will satisfy the Laplace equation. So if we apply that particular criteria to this function, what do we get?

We get something very interesting, we introduce this function into this equation, what we get is of course this is Laplace equation. So what we get is something like this, that means this is a first term Dell Dell Dell2 Dell X2 of psi, this one is the first. We plugged in term by term were doing the derivative and this is the 2nd term. So if you, so this implies, this equation implies that this should be equal to 0. Although this looks like a big equation but actually what we have done is we have plugged in this value into this. Once we do this, what we get, we get see this one XY, B XY, you take a double derivative with respect to X, the first time it will be BY and the 2nd time it will be zero, it will disappear. Same here because this is just a function of Y. Same for this because this is just X to the power 1. This is also, this will also disappear.

Dell Dell Y2 of X2 will also disappear, this one will also disappear because this is Y to the power 1 and we are taking a double derivative and this term again this is a function of X and we are taking double derivative with respect to Y, it will disappear, this will also disappear. So finally we are left out with these 2 terms, what it gives us is 2A plus 2C is equal to 0. So if this is the condition which it has to satisfy. Now let us see what when will this condition be satisfied. So that means A is equal to minus C, simple and then if we find the discriminant of this conic section, this is B square -4 AC, right, so if you plug-in this expression which is always true, if the flow is an irrotational flow. So if you plug-in that criteria here, what do we get? We get this as B square minus 4 A into minus A which is B square plus 4 A square and this is always positive because B square is always positive, of course these are not imaginary quantities, so this is always, the discriminant is always positive. So this will always represent a hyperbola.

So a conic section will always be a hyperbola is with represents a streamline for an irrotational flow. So this is the 3rd problem and the last problem in this tutorial section, this brings us to the end of the part C or the 3rd chapter or the 3rd week of the first module of this course on fluid dynamics and Turbo machines. In this chapter we have looked at the differential analysis of fluid flow, we have started with the derivation of the general form of the Naviar-Stokes equation. Of course we have taken a two-dimensional, we have gradually gone into more into a two-dimensional and incompressible flow. Then we looked at different important concepts like fluid rotation, vorticity, stream function, potential functions, Euler equation, Bernoulli's equation, and the applications of this equation to some specific situation like measuring, measurement of fluid velocity, application to a particular situation of the fluid flow and also some problems on stream in potential functions.

So this brings us to the end of the 3rd week of this course, the next week we will take up the flow, 3 different types of flows, the flow over a flat plate, the flow over an object like a cylinder and also the I think pipe flow, so the pipe flow. So these 3 types of flows we will deal with and where we will see, again those those are practical cases where this integral and differential analysis will find very good applications. These last 2 chapters, this 2nd and 3rd chapter were more about analysing mathematical procedures for analysing fluid flows. In the next chapter again we will see some practical situations where this analysis will find application. Thank you.