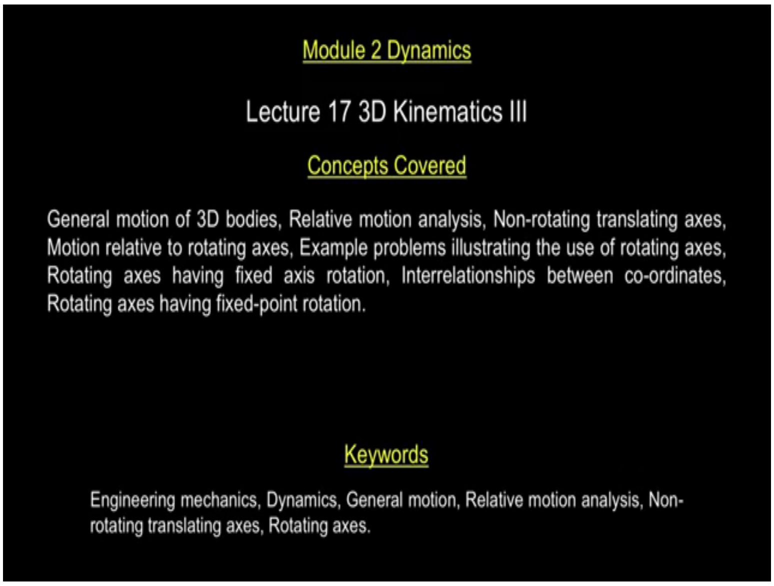
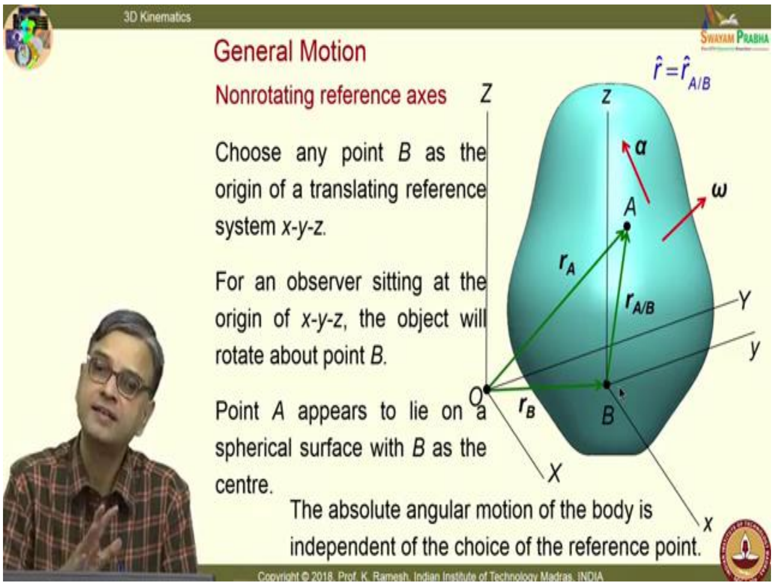
Engineering Mechanics
Prof. K. Ramesh
Department of Applied Mechanics
Indian Institute of Technology, Madras

Module - 02
Dynamics
Lecture – 17
3D Kinematics III



(Refer Slide Time: 00:30)

See in 3D Kinematics one of the important concepts that you learn is fixed point rotation, which we have seen in detail in the previous class. You know in 2-dimensional analysis we have looked at a, non rotating translating axis, attached to the body and we also saw a rotating axis attached to the body, we will now extend it for 3-dimensional problems.



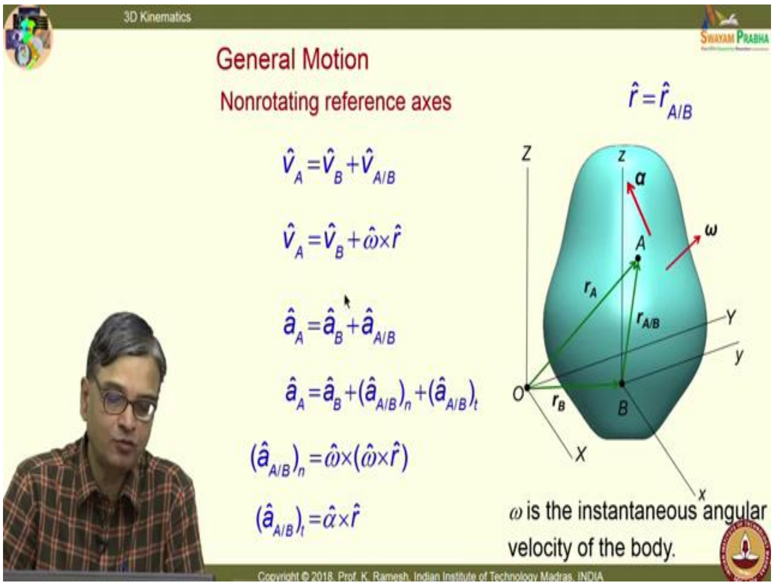
(Refer Slide Time: 01:07)

So, we move on to general motion and I am not calling this as plane motion. So, it can be any motion space and I have a 3-dimensional body, this body has some angular velocity and some angular acceleration, we have never put any restriction on this. And, what you need to notice is, these could be of different orientations in a 3-dimensional problem.

Similar to what we have done in 2 dimensions choose a point B, as the origin of a translating reference system, it is a non-rotating reference axis. So, choose a point B and attach small x, small y and small z to this point and we have already discussed in detail, if I attach a non-rotating axis, what way the person would perceive in a 2-dimensional system. In a 2-dimensional system he said that, he would perceive that it is rotating in a circular fashion.

Here you are talking in terms of 3 dimensions, here also the object will rotate about point *B*, instead of a circle, you would see that as a sphere because, we are looking at 3 dimensions, that is only difference. So, similar to the earlier one, we put the position vector r_B, position vector r_A and we are considering a rigid body. So, the distance between points *B* and *A* remains fixed and that is given as the position vector *r* or you can also qualify it that as position vector r_A with respect to *B*.

So, the main difference is, point *A* appears to lie on a spherical surface with *B* as the centre. So, that is the only difference between a 2-dimensional analysis and a 3-dimensional analysis. So, the important concept is, if you sit on a body and translate with it, you would see that it is rotating and no matter whether I see from point *A* or point *B*, I would still perceive same rotation, there is no change in that, whatever we have discussed in 2 dimension, naturally extends here, all the vectors will have 3 components, instead of a circular motion, it will have the point A would lie in a spherical surface.



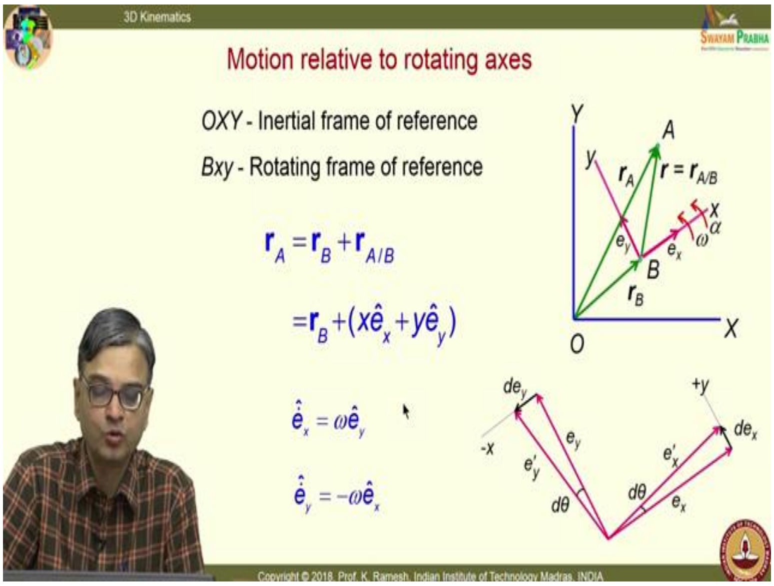
This we have also discussed in detail in 2-dimensional analysis, whichever point I take, I will still have the sense of rotation preserve, sense of rotation does not change. And it is a very subtle concept, whatever we have discussed in 2 dimensions, you could say extended for 3 dimensions.

(Refer Slide Time: 04:31)

And whatever the expressions that we have developed, we had $\hat{V}_A = \hat{V}_B + \hat{V}_{A/B}$ and the expressions what we have learned for circular motion again a place, here the vectors have 3 components. And, ω is the instantaneous angular velocity of the body and r is the position vector, $\hat{r} = \hat{r}_{A/B}$.

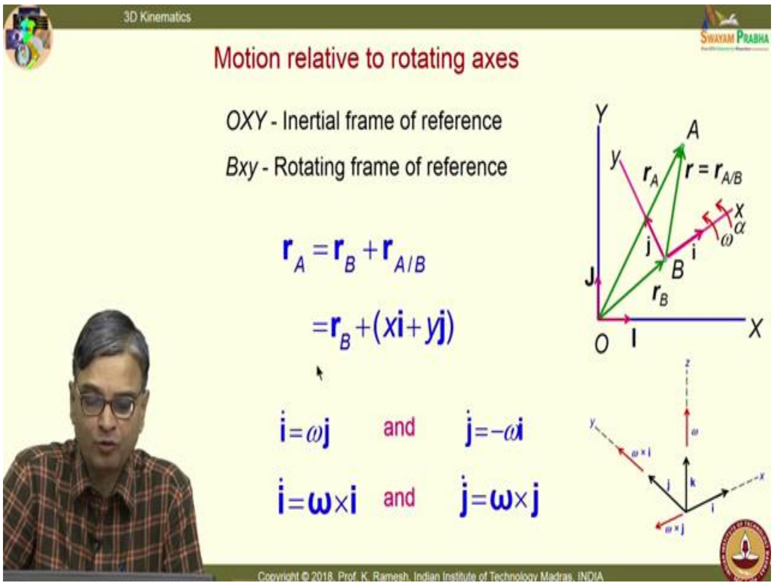
So, similarly you can also write for the acceleration and I here again I can write comfortably the relative quantity as consisting of a normal component and a tangential component, here again you use equations developed for simple circular motion. Only thing is the components are having the vectors will have 3 components in a quasi and you have to do little more computation, when you do the mathematical part of it.

So, I have the normal acceleration as $(\hat{a}_{A/B})_n = \hat{\omega} \times (\hat{\omega} \times \hat{r})$ and tangential acceleration as $(\hat{a}_{A/B})_t = \hat{\alpha} \times \hat{r}$. So, whatever you learn as circular motion used in different ways, at different parts of the course, the basic equations remain the same, you use it appropriately depending on the context. So, I am in a position to find out the velocity as well as acceleration.



(Refer Slide Time: 06:19)

Now, we move on to rotating axis. So, here I have an axis like this to emphasize that this is not a translating axis, I show that it is having a rotation. All the symbols are very similar, there is no difference in the symbols and here again you have this unit vectors and unit vectors we will write it as *i* and *j* later. And instead of looking at like this, if you look at as product of ω cross the position vector, then it is easy to employ and use, all these we have done in detail. So, I can afford to go a little faster. So, you write the basic expression like this and write it in terms of the unit vectors.



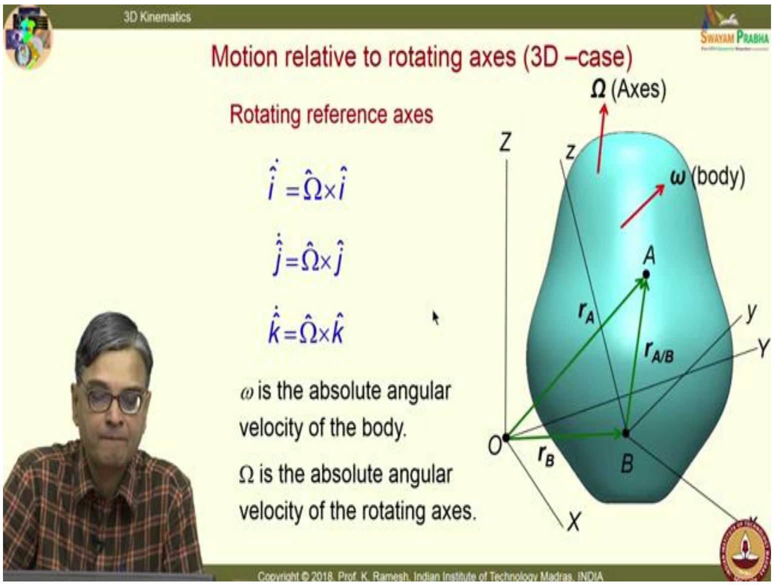
(Refer Slide Time: 07:25)

Then for convenience we replace it as *i* and *j* and I said that I can have velocity as well as acceleration, angular velocity as well as acceleration and whatever we have discussed, we have derived the expression for velocity, we have also derived expression for acceleration. They remain same and you also know $\dot{\mathbf{i}} = \omega\mathbf{j}$, which could be written differently as, $\dot{\mathbf{i}} = \boldsymbol{\omega}\times\mathbf{i}$ and $\dot{\mathbf{j}} = \boldsymbol{\omega}\times\mathbf{j}$, this kind of writing is very easy for your mathematical simplification.

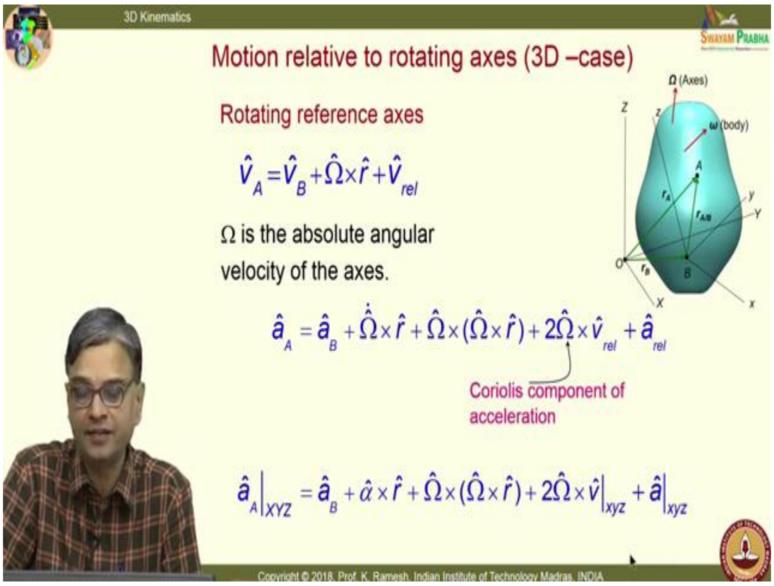
We have also used this when we discussed fixed point rotation, there again you will have *i* dot *j* dot *k* dot depending on which one is having this as non-zero and you know on 2 dimensional analysis, we have put this as ω and I have also returned on these expression in terms of ω.

(Refer Slide Time: 08:37)

In 3 dimension you know people also give, this as a $\hat{\Omega}$ (capital omega). So, I have body rotating with some angular velocity and you have the axis rotating with an angular velocity $\hat{\Omega}$. Just for convenience, see we solve problems with several omegas and I said that you should be alert in using the particular omega; it is again a notational change.



There is nothing, different from the conceptual point of view. So, I have a rotating reference axis attached and you distinguish there could be some part of the 3-dimensional body will be rotating with some angular velocity, the axis angular velocity, keep it as $\hat{\Omega}$, you reserve that as a symbol. So, it makes your life simple, when you write those expressions and substitute the relevant quantities.



I have the position vector smart and I also see $\dot{\hat{i}} = \hat{\Omega}\times\hat{i}$, just repeating those expression $\dot{\hat{j}} = \hat{\Omega}\times\hat{j}$, $\dot{\hat{k}} = \hat{\Omega}\times\hat{k}$.

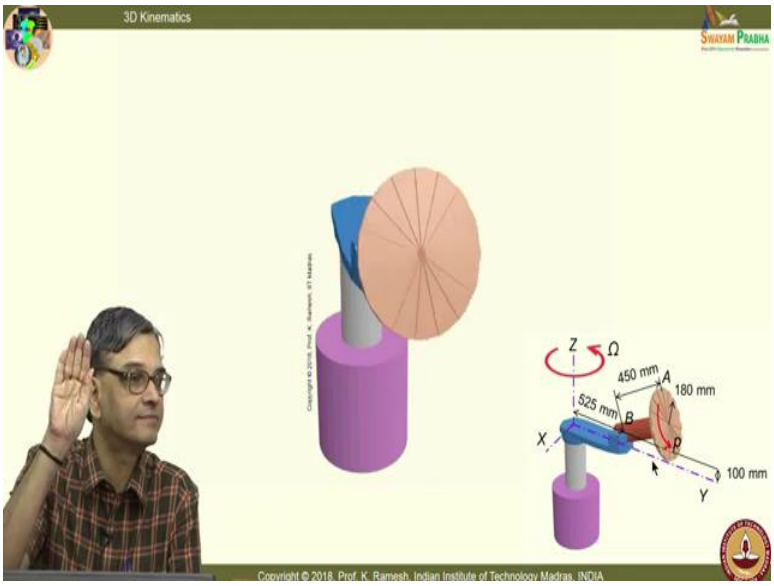
(Refer Slide Time: 10:09)

And you have a similar expression like what you have got in 2 dimensions, from a notational point of view we write it as a $\hat{\Omega}$ and each of these vectors will have 3 components. So, you have the angular velocity of the axes and I have the angular acceleration is written down like this.

So, this is the Coriolis component of acceleration and I can also visualize, I have written it for acceleration, you can write it for velocity also. Just to emphasize, I choose my rotating reference axes. So, that I am in a position to easily measure the velocity V_{rel}, which is nothing, but, V_{xyz} here and a_{rel} which is nothing but, a small *xyz* here, another left-hand side I have referred with respect to the fixed axis, capital xyz. The difference is in the 2 dimension we had only *xy*, here I have *xyz*. Otherwise, there is absolutely no change.

The mathematics will be little involved and a problem complexity can be little higher, other than that it is just an extension of what you have learnt in 2-dimensional analysis, I have written down, this form of expression for acceleration, you can write it for velocity as well. So, when I say V_{rel}, it is V_{xyz}.

(Refer Slide Time: 11:58)

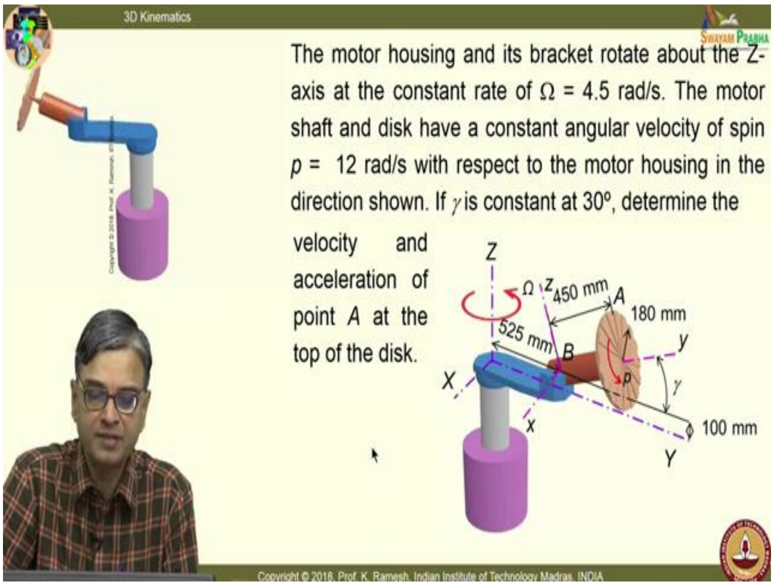


So, let us solve some interesting problems and just observe the animation. So, that you recognize what are all the motions that are available to the mechanism. So, you perceive and write them clearly in this sketch, one is I can see that it is rotating about this vertical axis, it is very clear.

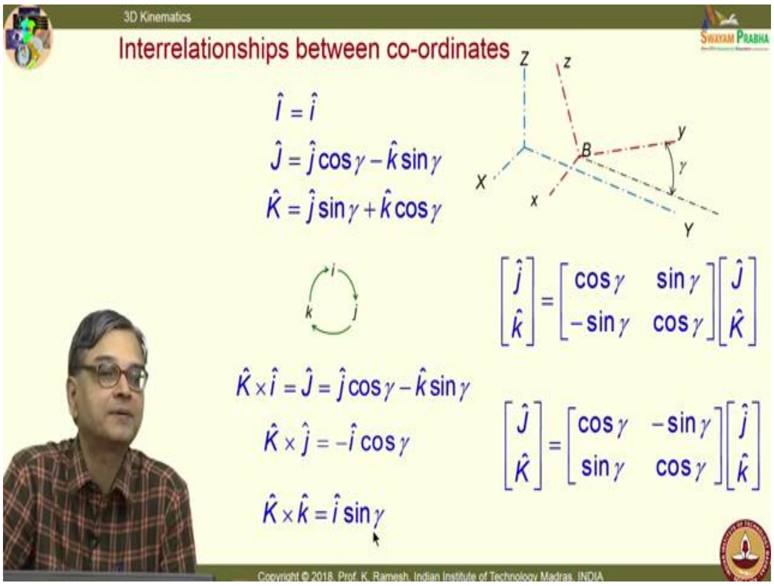
So, I have a rotation about *z* axis, then observe that this disc is not stationary, this is rotating which way, this is rotating anti clockwise, that is also very clear and you know this also has a provision, when I have this motor, this motor can be kept at any angle, there is also a provision like this for this and you are given the capital *XYZ* in the problem statement itself and you are also given small *xyz* for the problem under consideration.

(Refer Slide Time: 13:07)

So, write down the problem statement. So, the motor housing and it is bracket rotate about the *Z* axis at the constant rate of $\Omega = 4.5$ rad/s and you have the spin of this disc is 12 rad/s and you are also given what is the rotating frame of reference, it has made your life simpler, it is listed in the problem statement and this axis located at an angle gamma, if gamma is 30°, determine the velocity and acceleration of point A at the top of the disk.



So, it is a straight extension of what you have done as a 2 dimensional analysis to 3 a dimensional analysis and you could see very clearly, when I have a motor, when I sit on point B, I would perceive the rotation very comfortably and I can find out, what is V_{rel} and a_{rel} very easily. So, even though you do not decide what is the coordinate system that you should choose, even though it is given in the problem, from the problem you can understand, that you are in a position to find out a_{rel} and V_{rel} comfortably with this. Leaving that apart everything is in 3 dimension, that is all the difference ok. So, let us get into the solution of the problem, this diagram also we will be repeated again and again.



(Refer Slide Time: 15:01)

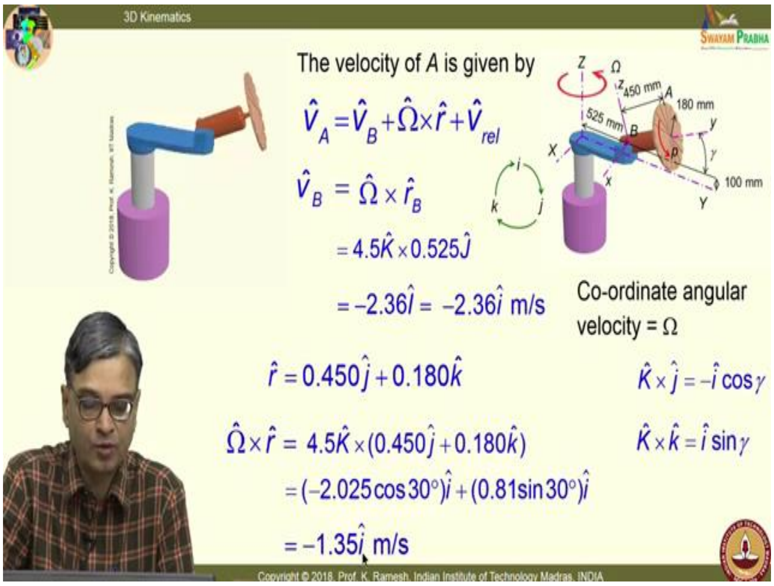
So, that you will have an opportunity to fill in the blanks and one of the important aspect in this problem because, we have not done that in detail in 2 dimension, I thought I would at least solve one problem where you have a coordinate axis is inclined and this it is better that we get the interrelationship between coordinates up front. So, that the mathematics becomes much simpler.

So, the way the coordinates are aligned the, this plane that small *y* is a plane is actually merging with the *Y* is at plane, it is only at a height of 100 millimetre that is given in the problem. So, the x axis is small x is aligned to capital *X*. So, I have $\hat{I} = \hat{i}$, I have not shown the unit vectors here, for the capital *XYZ* you have capital *IJK*, small *xyz* I have small *ijk* and I would like you to fill in this matrix, do you know that you can write, the small unit vectors in terms of capital J and K in a matrix fashion ok, please fill in that and then check your, check your matrix with mine, I will just show that. It is easier way of writing things ok; I can get small j and k from capital *J* and *K* because, what I have is the axis y and z are rotated about the x axis by an angle gamma.

So, you can easily write this as a rotation matrix, $\begin{bmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix}$. I can also find an interrelationship between capital *J K* and small *j k*, you can do the reverse office also. So, you just change γ to - γ, I get this expression. So, the idea is when you have access which are displaced like this, it is better to find the interrelationship upfront, before you start doing the mathematical calculation, then you can figure out which way you would simplify the mathematics.

So, then I can write I have written what is the relation between I capital I to small *i*, *i* will follow the same trend, capital *J* to small *j* and *k*, capital *K* in terms of small *j* and *k* and you know in the problem you have the rotation about the capital *Z* axis. So, you will have necessity to get $\hat{K} \times \hat{i}$ and capital $\hat{K} \times \hat{j}$, $\hat{K} \times \hat{k}$. So, I also get those relationships upfront. So, that I can do the mathematical simplification easily and when I look at $\hat{K} \times \hat{i}$, small *i* and capital *I* are same. So, I get this as capital *J* and capital J is $\hat{J} = \hat{j}\cos\gamma - \hat{k}\sin\gamma$. So, you can do that for the $\hat{K} \times \hat{k}$.

See normally $\hat{K}\times\hat{k}$ if I say, it will be 0, but here it is $\hat{K}\times\hat{k}$ fine. So, you have to be alert you should not switch off your mind and solve the problem, that turns out to be $\hat{K}\times\hat{k}=\hat{i}\sin\gamma$. Once I have these quantities, then the problem the solution to the problem is fairly straightforward.



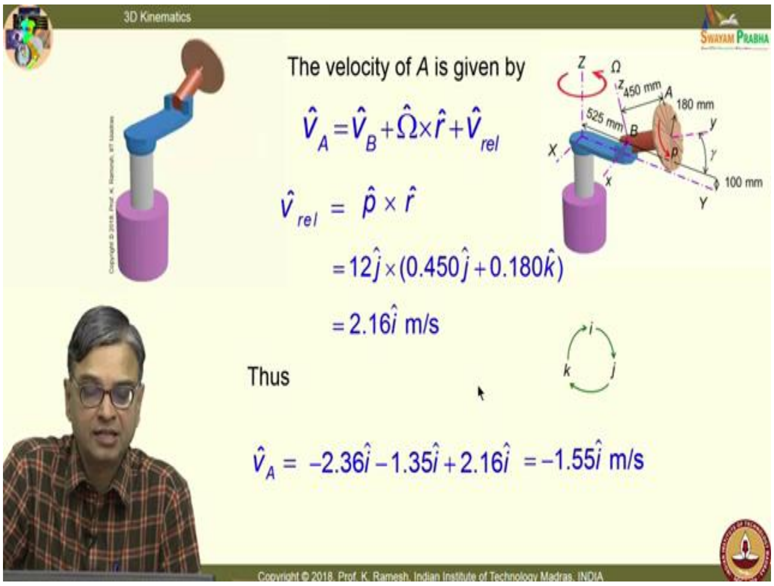
(Refer Slide Time: 19:20)

So, I have this coordinate angular velocity is $\hat{\Omega}$. So, I write each term, what we have done in the 2 dimension, similarly you do that. So, you write $\hat{V}_A=\hat{V}_B+\hat{\Omega}\times\hat{r}+\hat{V}_{rel}$, find out each of these quantities V_B is the coordinate velocity. So, which you can find out from the diagram easily so, it is $\hat{V}_B=\hat{\Omega}\times\hat{r}_B$ and so, this is the values in numbers. So, I have $\hat{K}\times\hat{j}$. So, $\hat{K}\times\hat{j}$ becomes I. So, I can write this as -2.36*i* m/s. So, essentially, I am writing all these quantities in terms of the rotating frame of reference, small xyz.

So, I know what is V_B, then I do $\hat{\Omega}\times\hat{r}$. So, the r is you are given this value $\hat{r}=0.450\hat{j}+0.180\hat{k}$ and you know $\hat{\Omega}$, then you also have this interrelationship which between the coordinates. So, when I use this, I get $\hat{\Omega}\times\hat{r}$ because, I am going to have this, that is why put what is $\hat{K}\times\hat{j}$ and $\hat{K}\times\hat{k}$ and so on and so forth, all that you have. So, I can easily simplify and this turns out to be an expression like this. So, it just algebra is very similar, you have to be alert and handle the coordinate system carefully. Then we go and find out what is V_{rel}. So, $\hat{\Omega}\times\hat{r}$ is -1.35*i* m/s.

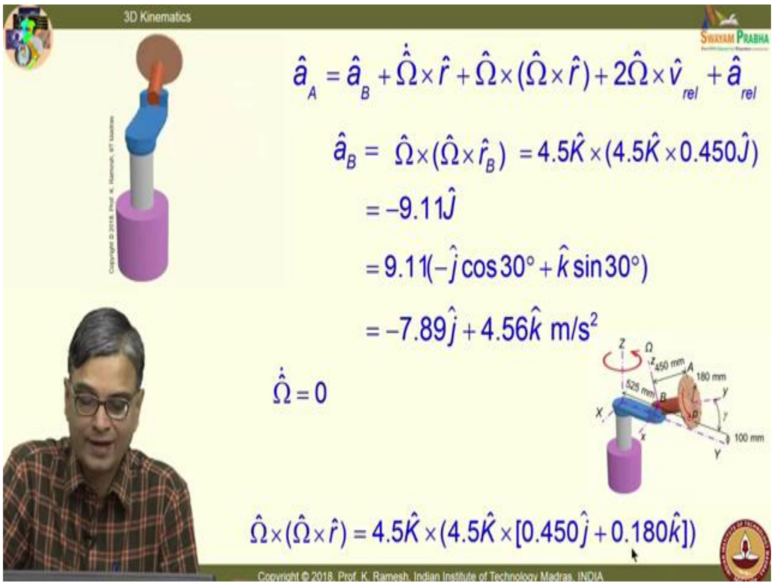
(Refer Slide Time: 21:25)

And V_{rel} you can find out based on the spin velocity ok. So, $\hat{V}_{rel} = \hat{p} \times \hat{r}$. So, thankfully the problem statement itself has different symbols for omega, it makes you less burdened in solving the problem ok.



So, I know V_{rel}. So, when I substitute for all of these quantities, I get the velocity of point A comfortably as -1.55*i* m/s. So, one of the challenges in this problem has been how to handle the coordinate system, other than that it is simple and straightforward to handle.

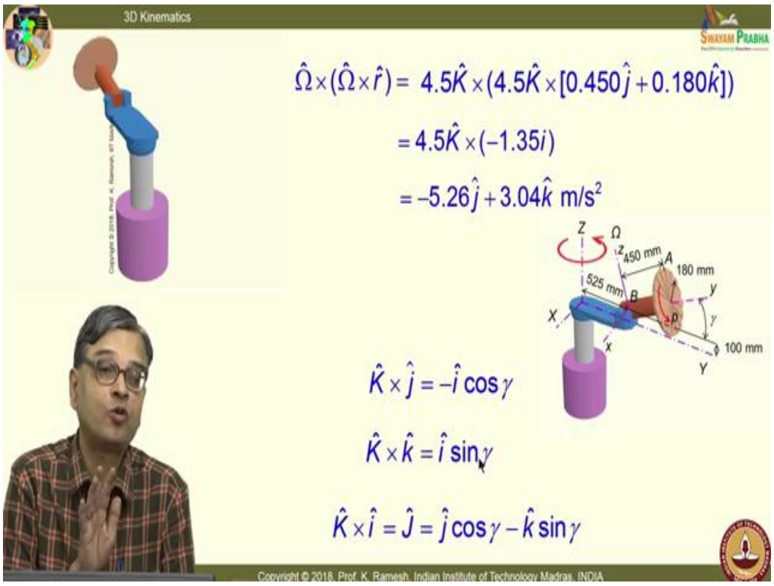
(Refer Slide Time: 22:24)



Similarly, we move on to acceleration. So, have a mnemonic way of remembering this expression, you have to remember this otherwise you cannot solve any problem and find out the coordinate acceleration. So, this is $\hat{a}_B = \hat{\Omega} \times (\hat{\Omega} \times \hat{r}_B)$, we have already written down all these quantities earlier. So, I get this as $4.5\hat{K} \times (4.5\hat{K} \times 0.450\hat{J})$. So, that turns out to be $-9.11\hat{J}$ and I always expressed the final expression in terms of the rotating frame of reference. So, I write it in terms of *j* and *k*.

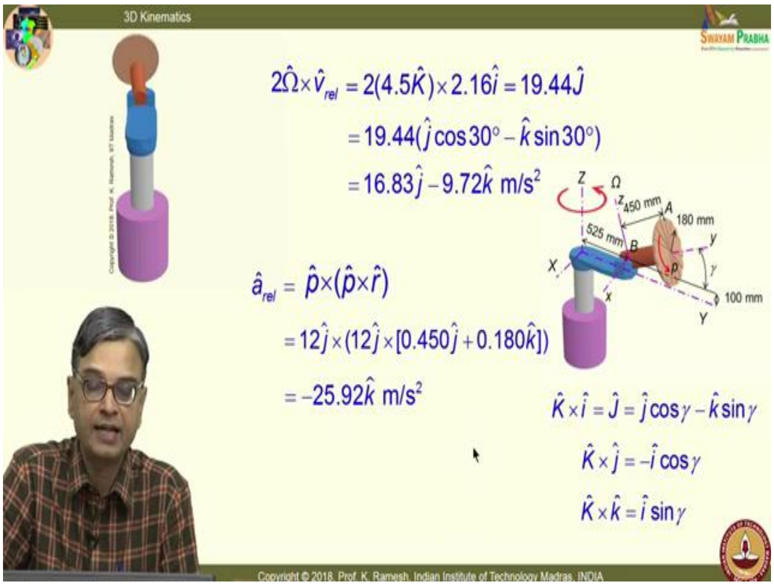
So, similarly you do it for each of the terms, I have a B obtained like this, then I have $\dot{\hat{\Omega}}$, this is having a constant angular velocity. So, $\dot{\hat{\Omega}} = 0$ and I have $\hat{\Omega}\times(\hat{\Omega}\times\hat{r})$. So, you have this expression.

(Refer Slide Time: 23:53)



And you can simplify that quite comfortably because, we have all the interrelationships available $\hat{K}\times\hat{j}$ and then you also need $\hat{K}\times\hat{k}$ that is also available. So, I express everything in terms of small *ijk*. So, you look at this interrelationship and finally, you get $\hat{\Omega}\times(\hat{\Omega}\times\hat{r})$. So, it is very straightforward and simple provided you handle the interrelationship between coordinates systematically fine.

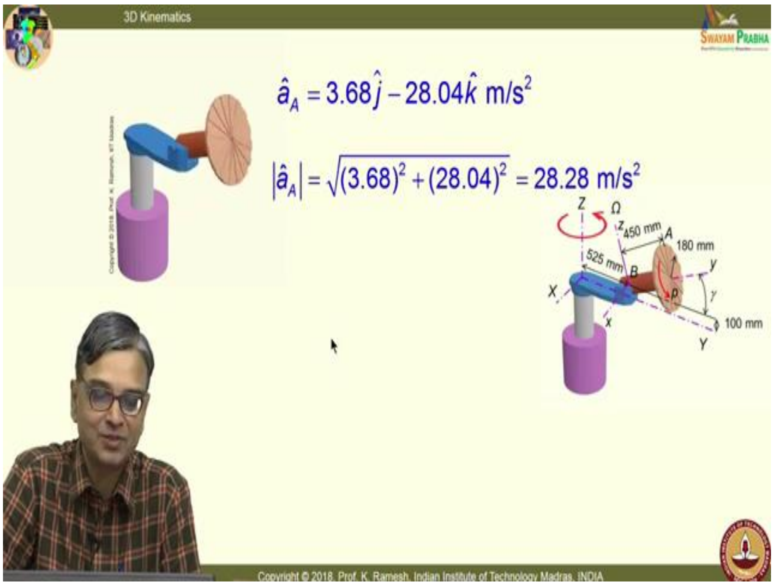
(Refer Slide Time: 24:39)



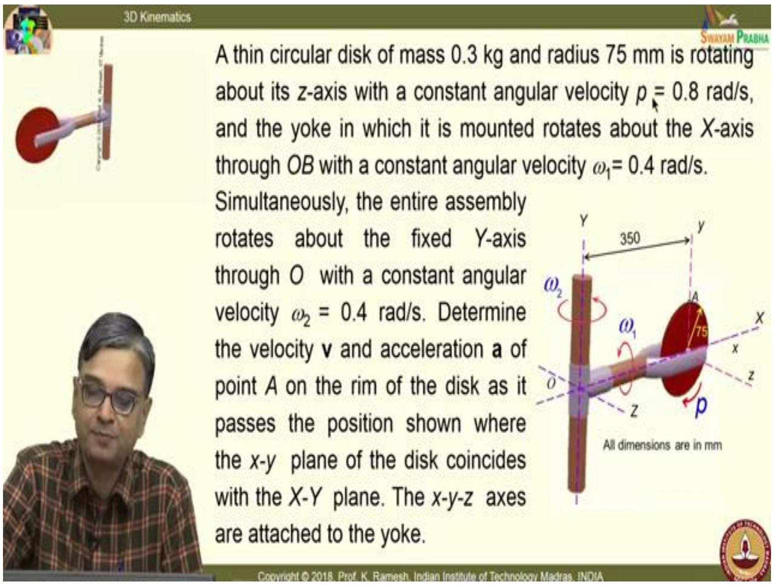
And I have this Coriolis component, which can also be written terms in terms of small *ijk*. So, I get the final expression like this and a_{rel} you can learn write it in terms of spin velocity and there is only constant spin velocity is given. So, *z* is at find out this. So, when I put all these quantities, I get the acceleration at point *A*.

(Refer Slide Time: 25:31)

And we can also find out the magnitude, it turns out to be 28.28 m/s^2. So, this is the straight translation from 2 dimension to 3 dimension, only thing is the coordinates where little difficult to handle but, you had solved a similar problem in 2 dimension also. What is the problem you solved? You solve the problem when this gamma was 90^o ok, that is the problem you solved, when I change the gamma it looks a very complicated problem and thankfully, I had a good animator who could make this visually possible ok.



(Refer Slide Time: 26:19)



Then we move on to another interesting problem, it is a very interesting problem, a first observe what is the kind of motion this has, what are all the motions because, once you understand the motions first, then you can attempt to make a solution. Let us see what are the things that you can see easily; I have this rotating about this axis that is the simplest one to see.

So, I have label this as ω_2, it is rotating about this axis, no difficulty at all anti clockwise and what way this arm is rotating, that is also anti clockwise, that is also easy to see, it is also rotating in anti clockwise motion also, it is rotating in clockwise motion you can see, that it is rotating like this ok, rotating clockwise and look at what way this disk is rotating, this disk what way it is rotating, at this position, you have to worry only about what happens at this position.

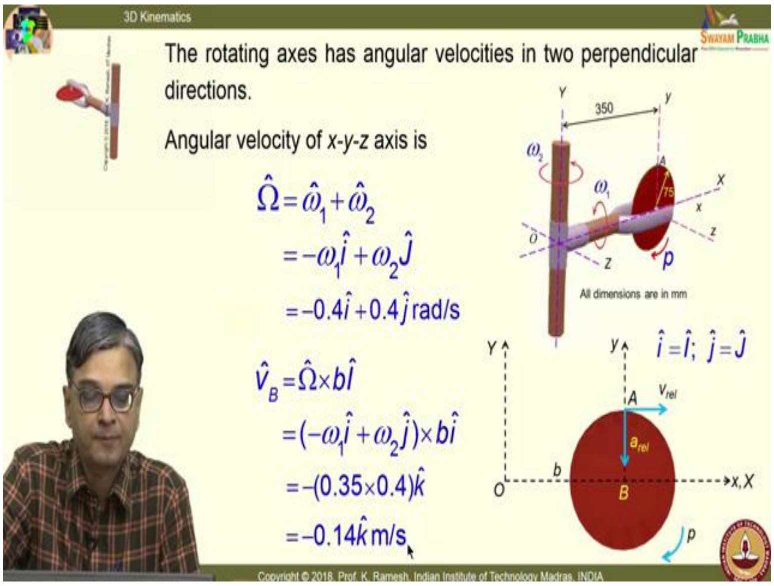
First observe this mechanism that is rotating clockwise. So, I have given the spin angular velocity in the symbol p. So, this is a very interesting problem, see we have looked at when I select the rotating frame of reference, it can just rotate or it can rotate in space which we have looked at it. Now, you have learnt in 3-dimensional kinematics, I can have fixed point rotation. So, here I am going to select my rotating frame of reference, which is also given which is attached to the arm at the centre of the disk. So, this rotating frame rotates with this sum as well as it has 2 angular velocities, ω_1 and ω_2, which are perpendicular like this.

So, the rotating frame of reference has the fixed-point rotation now. So, you can also have that, fine that is why I chose this problem and if I sit on this disc here, I just see the disc simply spinning and it is easier for me to get the V_{rel} and a_{rel}. So, to remove your mental block, I particularly chose this problem, a rotating frame of reference can have a very complicated motion.

There is no harm in selecting an axis like this as long as I am able to get a_{rel} and V_{rel}, if you understand this rest of it is very simple. So, you have to appreciate the motions in the body, these are all you know of fictitious problems where you get trained to look at different types of motions. See to effect this, it will be a very difficult circus to device the gadget to have all these motions packed in a small fashion and you are also given the values of these velocities, the spin velocity p is given as 0.8 rad/s, both ω_1 and ω_2 have 0.4 rad/s.

So, you are asked to find out the velocity v and acceleration a of point A on the rim of the disk as it passes the position shown and in order to have simplicity because, the problem is complex, the problem statement itself dictates, what is the capital *XYZ*, what is the small xyz, even if it is not given you are sufficiently trained how to choose the rotating frame of reference, The choice is I should get a_{rel} and V_{rel}, mean V and acceleration a_{rel} comfortably with the quantities specified in the problem, that is possible only if I sit on this disk and move with it ok.

(Refer Slide Time: 30:52)



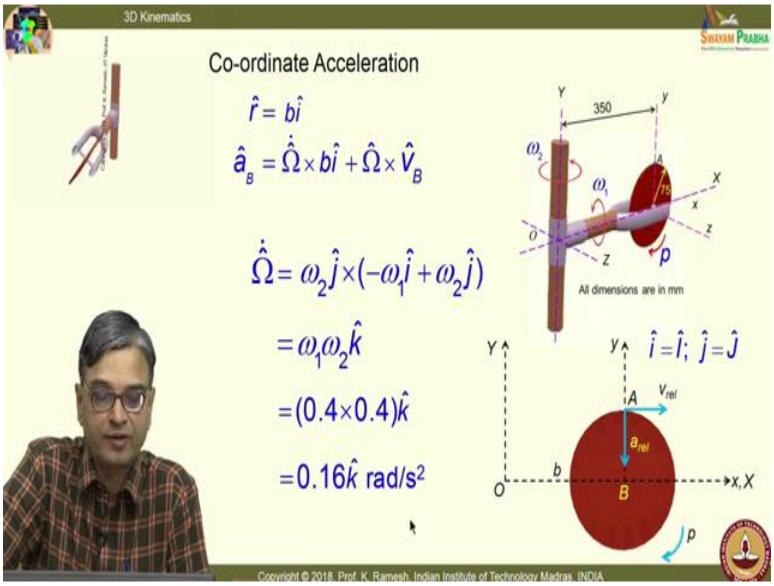
Once you understand this, rest of it is straightforward. So, we have fixed point rotation. So, the angular velocity of *x y z* axis, we have already seen that I can have a vectorial addition. So, I put the symbol $\hat{\Omega}$ denoting that this is the angular velocity of the rotating frame of reference, I have ω_1 and ω_2 and you should also note that all the axis, there is no confusion and axis, that is why I chose the problem because, the problem is complex from a different perspective, I wanted to make other quantities as simple as possible. So, that you get the idea behind it. So, I have this as $-\omega_1\hat{i} + \omega_2\hat{J}$ but, $\hat{i} = \hat{I};\ \hat{j} = \hat{J}$ So, I can rewrite as, $-0.4\hat{i} + 0.4\hat{j}$ rad/s and you also know how to get the acceleration.

So, whatever you have learnt in fixed point rotation, those concepts you have to stretch here and then do it systematically. So, I have this disk and this is same as small *x* as well as capital *X* and I have this capital *Y* axis here and I have the small *y* axis here, I have the point *B* marked at the centre of the disk and I have the point *A* and this has a spin velocity clockwise and I can easily find out the V_{rel} as well as a_{rel}.

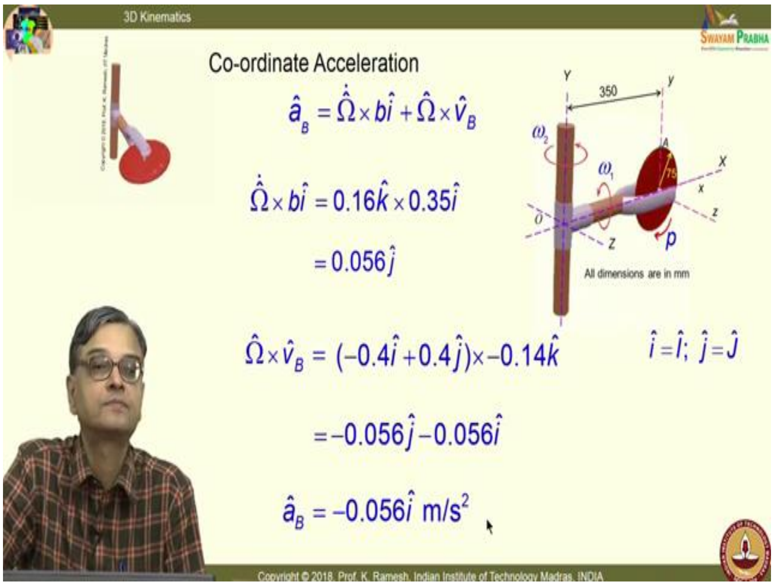
So, you choose the rotating frame of reference, such that you can measure these quantities or determine these quantities comfortably and I have V_B as the coordinate velocity. So, this is the position vector I have this as b i and b is also given, it is given as 350 mm. I know $\hat{\Omega}$ when I say $\hat{\Omega}$, I must recognize that it has combination of ω_1 and ω_2. So, I have the expression translated into numbers and finally, I miss some substitutions, please verify this, verify them and if there are any typographical error, please bring it to my attention. I get V_B as -0.14*k* m/s.

(Refer Slide Time: 33:54)

And we would also determine, when we find out V_{rel} we will also find out a_{rel} and I have



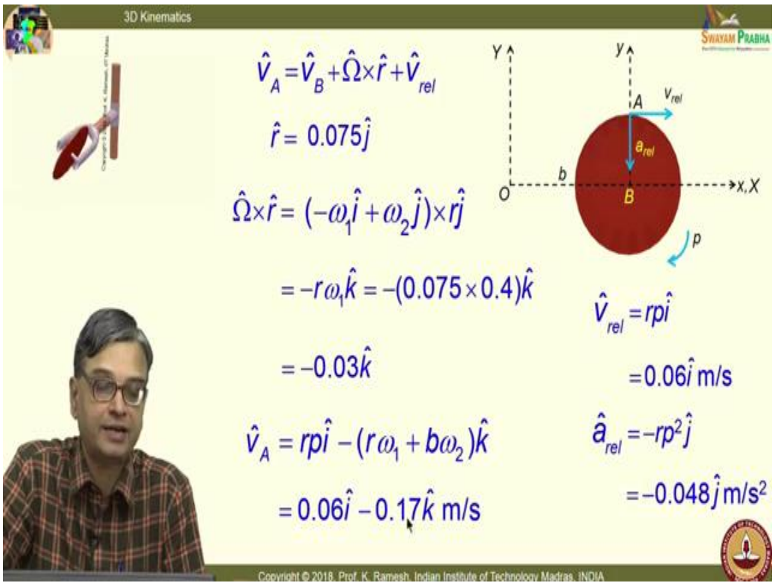
to get the coordinate acceleration. So, r as bi so, I have this as $\hat{a}_B = \dot{\hat{\Omega}} \times b\hat{i} + \hat{\Omega} \times \hat{V}_B$ and I should recognize that I will have $\dot{\hat{\Omega}}$ and this I can write it in terms of $\hat{\Omega} \times \hat{V}_B$, which we have always been doing to simplify the calculations and how do I write $\dot{\hat{\Omega}}$ go back to your fixed-point rotation. So, I have to have $\omega_2 \times \omega$, $\omega_2 \times \omega_1$ is what I want but, I can also write it in terms of this because, this j cross j will go to 0.



So, essentially, I get, $\omega_1\omega_2\hat{k}$. So, you should recognize even though these angular velocities are constant because, of the directional change you still have $\dot{\hat{\Omega}}$, it is not 0. So, it is a very nice problem combining the fixed-point rotation and rotating frame of reference. So, once we have solved a problem like this, then you have many recipes for you to attack a problem. So, I have this value substituted and I get this as 0.16*k* rad/s^2.

(Refer Slide Time: 35:39)

So, all the other quantities I can put $\hat{\Omega}\times\hat{v}_B$ and so on and so forth. So, I have this as 0.056j and I get $\hat{\Omega}\times\hat{v}_B$, the values are substituted and this turns out to be $-0.056\hat{j}-0.056\hat{i}$. So, I finally, get $\hat{a}_B=-0.056\hat{i}\ \text{m/s}^2$ so, I have been able to get the coordinate acceleration.



(Refer Slide Time: 36:22)

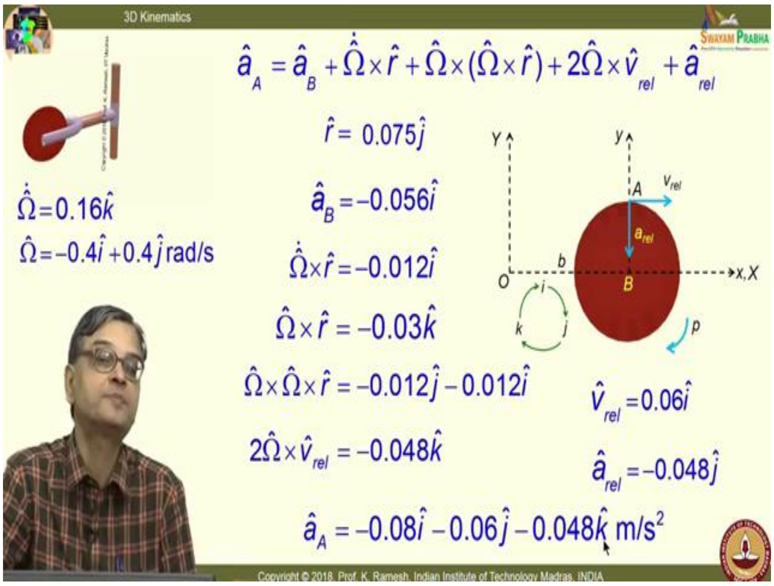
Then I have this $\hat{V}_A=\hat{V}_B+\hat{\Omega}\times\hat{r}+\hat{V}_{rel}$ the standard expression and $\hat{V}_{rel}=rp\hat{i}$, I can also write V_{rel}, 0.06i m/s and $\hat{a}_{rel}=-rp^2\hat{j}$. So, the choice of your rotating frame of reference is selected. So, that you comfortably evaluate V_{rel} and a_{rel} without any difficulty.

But I can have a very complicated motion of the rotating frame of reference, it has a fixed-point rotation this time and you can write what is r, it is given in the problem statement what is the size. So, I can get $\hat{\Omega}\times\hat{r}$ and substitute these values, I get this as -0.03k m/s. So, I have V_A as in terms of expressions, when I substitute the values, I have this as $0.06\hat{i}-0.17\hat{k}$ m/s. I have missed some steps, please fill in those blanks, substitute and then verify that the final expression is correct.

So, we have been able to find out what is V_A, in a similar fashion we will also find out the acceleration, you have to recognize that the rotating frame of reference can have fixed point rotation, there is no mental block in selecting that kind of a axis of reference, that is the message in this problem.

(Refer Slide Time: 38:24)

So, I have this expression for acceleration and I know what is *r*, $\hat{r} = 0.075\hat{j}$ and we have already determined what is a_B and we have also determined $\dot{\hat{\Omega}} = 0.16\hat{k}$ and when I put $\dot{\hat{\Omega}} \times \hat{r}$, I can get the value like this, -0.012 and I have $\hat{\Omega}$ expression is also given. So, I can write what is $\hat{\Omega} \times \hat{\Omega} \times \hat{r}$, $\hat{\Omega} \times \hat{r}$ is this quantity, $\hat{\Omega} \times \hat{\Omega} \times \hat{r}$ turns out to be like this, then I can also write we know what is V_{rel}, we know what is a_{rel}, V_{rel} is so much and a_{rel} also we had determined it earlier. So, I have all these quantities. So, I can easily compute $2\hat{\Omega} \times \hat{v}_{rel}$, as well as put the value for a rel. So, $2\hat{\Omega} \times \hat{v}_{rel}$ turns out to be -0.048*k*.



So, when I put all these quantities, I get the acceleration finally, as $\hat{a}_A = -0.08\hat{i} - 0.06\hat{j} - 0.048\hat{k}$ m/s^2. So, it is a very nice problem which combines rotating frame of reference and fixed point rotation and if you are systematic, there is no difficulty in solving this problem, it is fairly straightforward but, you have to be extremely systematic, like what I have shown intermediate steps, if you develop this kind of a habit, it is also easy for us to grade your paper. Rather than you write the final expression with, if the final expression is correct there is no difficulty, if there are mistakes how to give partial marks it becomes so difficult for us to do that ok.

So, in this class what we have looked at is extension of problems in 3 dimensions, we have looked at non rotating translating axis attached to the 3-dimensional body, then a rotating frame of reference attached to a 3-dimensional body. First, we solve the simple problem, where there was the axis where little difficult to handle.

So, we have also learned how to handle those interrelationships comfortably, then we looked at another very interesting problem, wherein we combined fixed point rotation and rotating frame of reference. The rotating frame of reference can in general have even fixed-point rotation, there is no necessity that it should be a simpler axis, it can have a complicated rotation.

Thank you.