Computational Fluid Dynamics Using Finite Volume Method

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Lecture - 21

Finite Volume Method for Diffusion Equation: Unsteady diffusion time-stepping schemes and Truncation errors of the FV schemes

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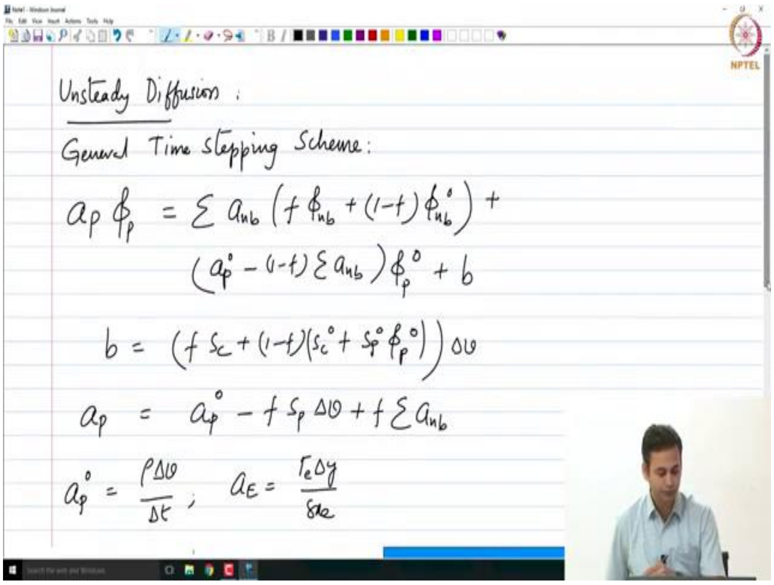

Good morning. Let us get started. So, we were discussing unsteady diffusion and we have also derived a general time stepping scheme right, derived a general time stepping scheme that is in terms of a factor called f right. Can you help me write down this general time stepping scheme? This was $a_P\phi_P$ equals $\sum a_{nb}$ f times ϕ_{nb} plus?

Student: 1 minus 1 minus.

1 minus f times.

Student: ϕ_{nb}^0.

ϕ_{nb} is 0 plus we have?

Student: (Refer Time: 01:06) a_P^0.

a_P^0 minus sigma.

Student: 1 minus f 1 minus delta.

1 minus f 1 minus f times $\sum a_{nb}$, is it?

Student: Yes.

Right.

Student: Yes sir.

Times ϕ_P^0.

Student: (Refer Time: 01:22).

Plus b right, that is what we have and the term b itself is f times S_C plus f times S_C^0 plus 1 minus f times.

Student: (Refer Time: 01:36) 1 minus (Refer Time: 01:38).

1 minus f times?

Student: S_P^0, S_P^0 (Refer Time: 01:42).

So, this was how much? f times S_C plus?

Student: 1 minus f 1 minus f.

1 minus f times?

Student: S_C^0; S_C^0.

Student: Plus S_P^0.

Plus S_P^0.

Student: ϕ_P^0.

ϕ_P^0. So, essentially 1 minus f operates on both of them right. On both of them or?

Student: Yes sir, both of them.

Both of them ok, times?

Student: Delta.

ΔV right that is what we have. Essentially, goes to the other time level that is 0 and what about a_P? a_P was how much?

Student: (Refer Time: 02:11) a_P^0.

a_P^0.

Student: Minus f.

Minus f times?

Student: S_P times ΔV.

S_P times ΔV plus f times.

Student: Summation a_{nb}.

$\sum a_{nb}$ right, that is what we have ok. So, we have calculated all these things and of course, a_P^0 is $\rho\Delta V$ upon Δt and all our a east, a west all of them have their usual definitions that is $\frac{\Gamma_e \Delta y}{\delta x_e}$ right and so on right.

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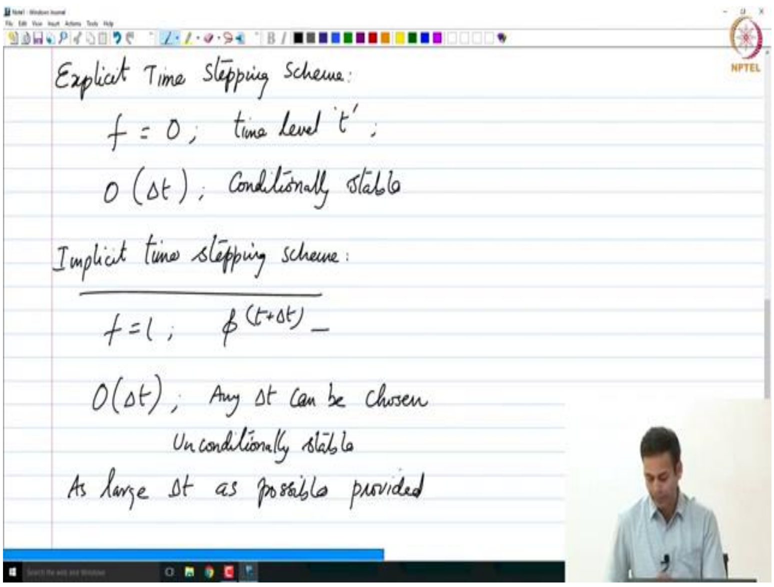

So, we also said that, we looked at two different schemes by setting the value of f; one was an explicit scheme. Explicit Time Stepping Scheme, this we obtained by setting a value of f equals 0 right, where the values of ϕ at time level t prevail over the entire time step right that is what we said and also, other properties that we looked at were the explicit scheme was only order Δt in terms of the truncation error right which we did not prove. We said this will be of order Δt and is the scheme stable for any Δt that we choose?

Student: No, no.

No right. We kind of derived heuristically by looking at the coefficients of ϕ_P^0 right, that if it is only kind of conditionally stable. So, if we choose a Δt that is some factor times order Δx^2, then only the scheme produces positive coefficients for ϕ_P^0 term right. As a result, you would not get wiggles and the solution, you would get would be would converge right.

We also looked at another time stepping scheme by setting f equals 1 that was the implicit time stepping scheme by in which we said f equals 1 right. In this case, the values of ϕat $t + \Delta t$ are assumed to prevail over the entire Δt right. Essentially, the unknown values are assumed to prevail and then, we also said without proof that this is also only order Δt accurate in terms of temporal accuracy. But what about the stability? Can we choose any Δt that we wish?

Student: Yes.

We can actually choose any Δt right. So, this is any Δt can be chosen because this is unconditionally stable that is what we said. We did not prove it again. So, we only said this is unconditionally stable. From a heuristic perspective, we never got these coefficients which can become negative right. We do not have those coefficients ah; so, unconditionally stable.

However, you cannot choose any Δt that you wish because your accuracy may suffer right. Essentially, as long as we can take as large Δt as possible provided your accuracy is guaranteed right; provided accuracy is guaranteed ok.

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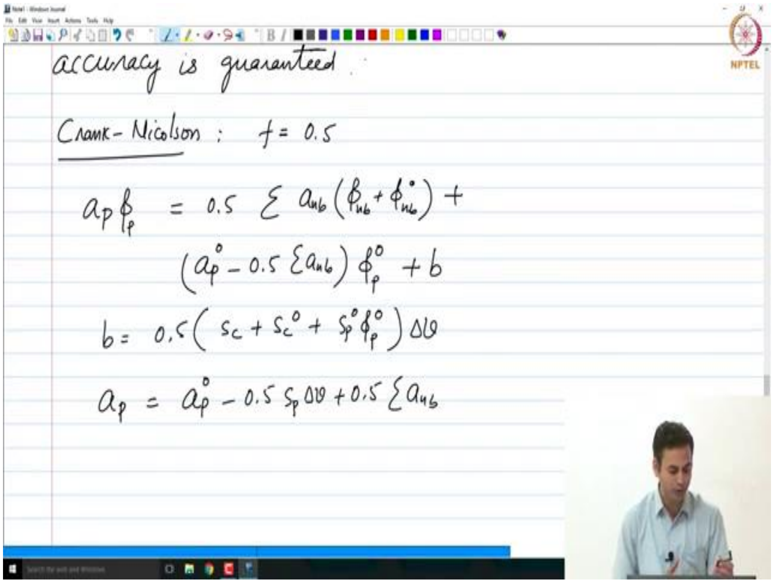

So, if accuracy is not there, then you would have to kind of reduce the size of Δt right fine. Now, we are going to see one more scheme. So, essentially, the idea is these two time stepping schemes are only order Δt. So, as you refine the time step, your solution accuracy would only improve by one order right. It is not going to be square.

So, as a result, these are not very promising schemes. If you want to reduce the temporal error, so we are going to see something known as a Crank Nicolson scheme which is kind of intermediate between these two ok, in which you would set f equals one half ok. f equals 0.5 into the general time stepping scheme and that would give you the Crank Nicolson scheme ok.

Now, let us kind of substitute these back into the original equation. So, into the general time stepping equation that is this one and then, see what would be the corresponding equations ok. So, maybe you can substitute and again, tell me what would these be. So, this would be a_P ϕ_P equals what would be the values on the right hand side? You would get a half here and a half from the first two terms right; you have f times ϕ_{nb} and 1 minus f, both of them will be half half right. So, that would be a?

Student: (Refer Time: 07:09).

Yeah. So, that is essentially 0.5 times $\sum a_{nb}$ times what?

Student: ϕ_{nb} (Refer Time: 07:20).

ϕ_{nb} plus ϕ_{nb}^0 right, it is correct ok. So, that is what we have and then, plus what else we have? Plus what about the a_P^0 term? That would be a_P^0.

Student: Minus.

Minus 0.5 times?

Student: $\sum a_{\mathrm{nb}}$.

$\sum a_{\mathrm{nb}}$ times ϕ_P^0 plus some b right. Now, this b is of course you have the source terms is there which is basically this term right. So, you have a 0.5 coming from S_C and S_C^0 and S_P^0 ϕ_P^0 would also have a one half multiplying them right. Essentially, you have one half multiplying this entire thing. So, this is what? This will be one half times S_C plus S_C^0 plus S_P^0 ϕ_P^0 times ΔV. Is that correct? Yeah ok. So, that is that is the value. What about a_p? a_p was how will be how much now?

Student: Same (Refer Time: 08:35).

Same?

Student: a_P^0.

a_p would be a_P^0.

Student: Point (Refer Time: 08:40).

Minus 1 half times?

Student: S_P ΔV.

S_P ΔV. Oh sorry, this is minus S_P right. This is correct.

Student: Minus.

Minus 0.5 into S_P ΔV and then plus 0.5 times?

Student: $\sum a_{\mathrm{nb}}$.

$\sum a_{nb}$ right ok. Is that correct? Just plug in f equals one half into the general equation ok. So, this is correct. So, these are the discretized equations for Crank Nicolson scheme and of course, you are a_P^0 would remain the same, that is ρ times ΔV by Δt ok.

So, what we say is that this scheme uses a linear interpolation between values of ϕ at t and $t + \Delta t$. As a result, you would get this one half factor in there ok. And so, what about will if we kind of make some comments about this scheme, what do you observe here? Is the right hand side completely in terms of known values or is it completely in terms of unknowns or what is it?

Student: (Refer Time: 09:40) both; both.

It is both. So, do you need to solve a system of linear equations at every Δt?

Student: Yes, yes.

Yes, you need to. So, this is more like a implicit scheme, but then you have a half of the explicit component also coming into play, that would go to the b term right that is also there.

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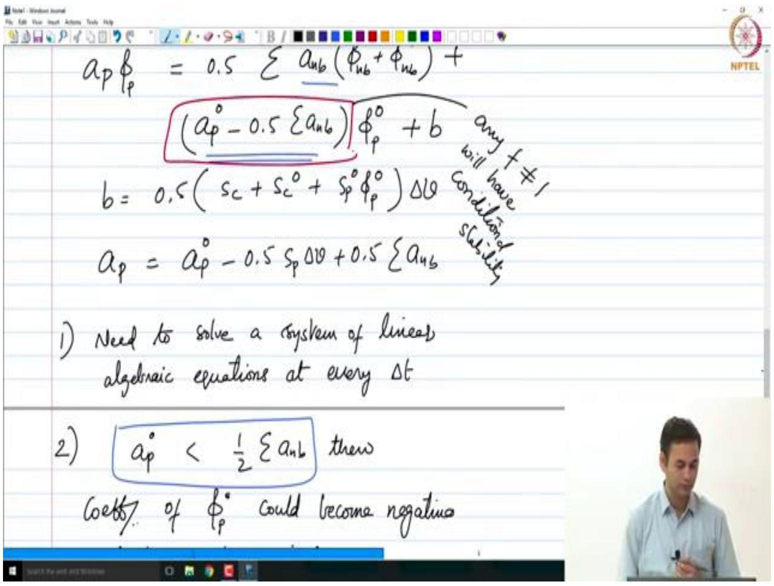

So, if we look at the comment. So, then this is; so, need to solve a system of linear algebraic equations at every Δt ok. Now, what about the coefficients? Would all of them be positive or would some of them may become negative? So, a_{nb}'s are always positive right; a_{nb}'s are always positive, what about the coefficient for a ϕ_P^0? Can this become negative?

Student: No.

No, ϕ_P^0 can the coefficient become negative under certain conditions. It can become right because a_P^0, if it falls below half of $\sum a_{nb}$, this coefficient can become negative. This is similar to what, we saw in the explicit case right. So, if your a_P^0 falls below one half $\sum a_{nb}$, then the coefficient of ϕ_P^0 could become negative ok, which is not good. Because if it becomes negative, then you will get into this oscillatory solution, which would eventually lead to divergence right.

Because ϕ_P^0 is the previous time level value and an increase in the previous time level value would actually decrease the value at the current time step for the same cell and this is not good. This is again from a heuristic perspective ok. Now, this is rather kind of conflicting right, we kind of have a value of f equal to half.

So, we are saying we are solving a system of linear equations; but then, we have now a condition on the stability right. So, this is not like the implicit fully implicit scheme right. This is kind of a half implicit half explicit. So, this could become negative which is not desirable, then you may have a question, then why do we actually solve for the scheme right.

It is not offering any advantage; you still have to solve for a system of linear equations. It turns out without proof; again, I am saying here which we will do it later or you would do it later, which would be the order of accuracy for time good, for this scheme comes out to be order Δt^2 ok.

So, this is second order accurate in time which the previous two schemes were not ok, because the previous schemes are only first order accurate in time. So, second order accurate in time that means if we can choose the Δt wisely, then we can make the error go down quickly compared to the other schemes right.

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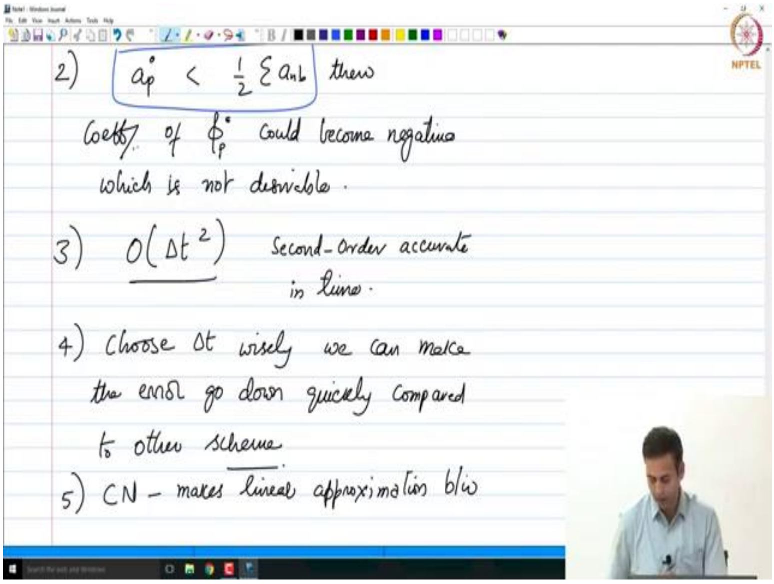

So, ok. So, that is the advantage here. The truncation error, you would make would be would go down at the rate of Δt square. As a result, you would prefer this, as you refine Δt. So, that is the advantage that Crank Nicholson offers on top of the other two methods; the explicit and the fully implicit methods. Now, if you go back to the equation, we had this condition. Right now, we said there is this condition; this has to be satisfied so that the coefficient does not become negative.

Now, under what circumstances can we get rid of this coefficient? Only for fully implicit, let us say you come up with your own scheme. You say f is why it should be 0.5, I would come up with my own scheme, I will say f equals 0.3 right that is of course possible; it is not people would not work with that as such. But in that case also would you still expect a coefficient here or no?

Student: Yes.

Yes right. Essentially, for all the values of f that is other than 1, you would have a conditional stability only like the explicit scheme ok. So, that is one thing to note. So, any f not equals 1 will have conditional stability right ok, will have only conditional stability that is what we look at it from here, fine. So, this makes a linear approximation between the two time level values.

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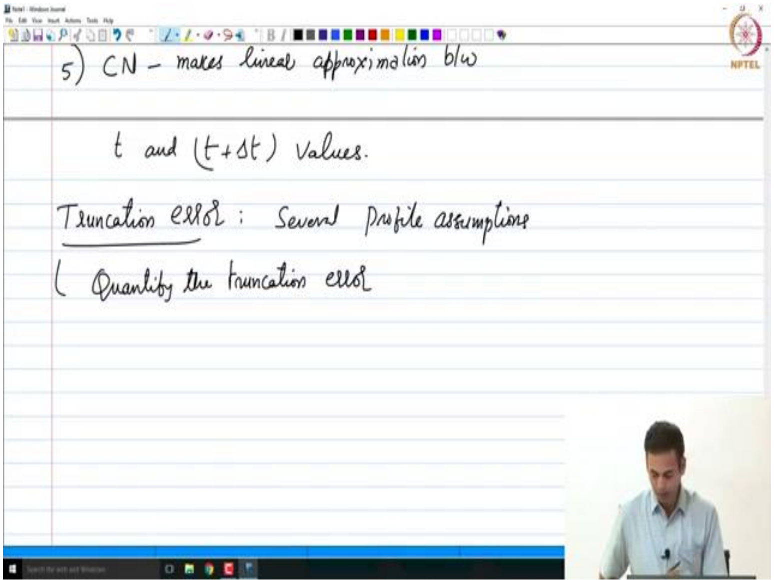

So, Crank Nicolson makes linear approximation between t and s $t + \Delta t$ ok. Questions till now? Now, how difficult or easy this is to code? So, if you have a fully implicit solver, is it very straightforward to do this thing right; it is. Is not it? Because you just have to take half of that amount and then, put the other half into the b terms and so on right; whereas, if you have an explicit thing, you cannot do it easily; you still have to solve for the linear system ok.

So, in terms of the implementation Crank Nicolson would be pretty much the same as what we have right for the implicit scheme. So, you do not have to code something extra, once you have in code for implicit schemes ok. Questions? No? Clear, understood? Fine, all right. Then, let us move on. So, the next thing, we are going to look at is we will evaluate the truncation error that we have encountered in our or made in our schemes ok. So, that is basically the spatial and the temporal truncation error analysis.

Because up till now, we said we have made several profile assumptions, everywhere right, without actually looking at what is the order of accuracy of the finite volume scheme that we have developed ok. So, now we kind of revisit that. So, we will look at the truncation error that is made. So, essentially, we made several profile assumptions. So, we will see, we will kind of quantify the truncation error ok.

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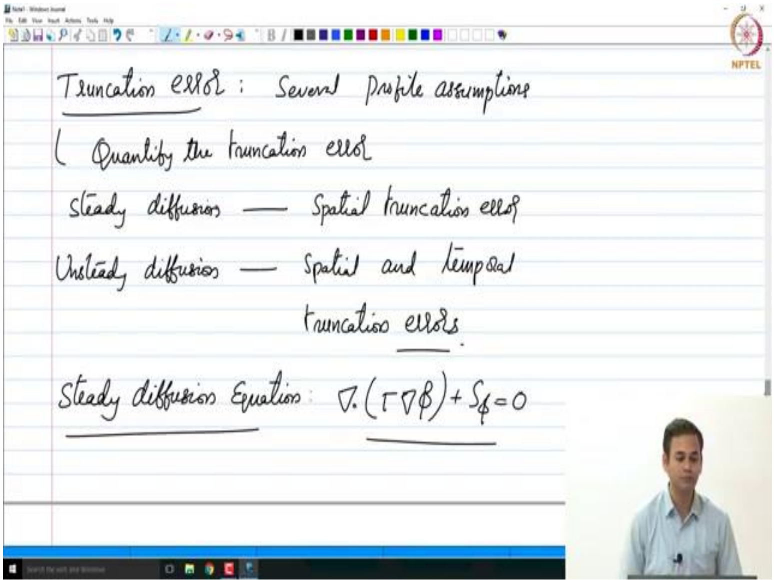

For both steady diffusion equation which will give us the spatial truncation error and we will look at the transient or the unsteady diffusion equation, which would give us both the spatial and temporal truncation errors fine. So, let us look at the steady diffusion equation; that means, we are looking at the spatial truncation error that we have incurred while making certain profile assumptions in the discretization ok.

So, if you look at the steady diffusion equation, that is $\nabla \cdot (\Gamma \nabla \phi) + S_\phi = 0$right. So, what are the several assumptions we have made in discretizing this equation using finite volume method? If you go back.

Student: Linearization; source term linearization.

Source term linearization ok, that is one thing; but source term linearization is something that we have already defined it. So, before that what was it? So, we had to linearize the source, but we assumed that the source term value is representative; the mean value in the cell is representative by the cell centroid value right. It is a representative of the mean value. So, that is the first approximation we have made right. Of course, we have made a we have linearized the term as well right; but before that we have done the representation by a cell centroid value right.

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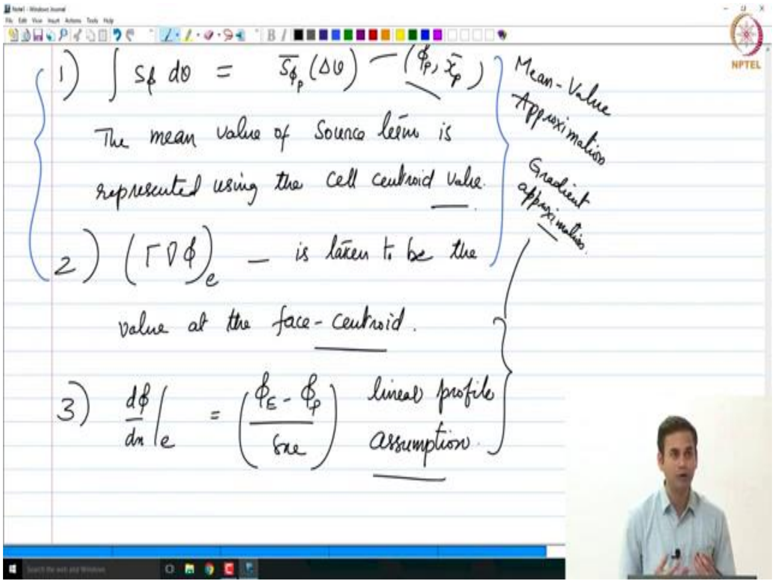

So, that is the source term that is if you have integral $S_{\phi}dV$, we said this is represented using some $\overline{S_{\phi}}$ times ΔV right. So that means the mean value of source term. Now, what is the mean value? An average value that is average value or the entire cell right.

So, that is represented by the left hand side right; if I have a mean value that is represented by the left hand side and the right hand right hand side says that this is the average value is evaluated using at the cell centroid that is $\overline{S_{\phi}}$ this is at P right. If we have cell centroid P, this $\overline{S_{\phi_P}}$ right, that is we are calculating at $\phi_P, \overline{x_P}$ right. This is evaluated $\phi_P, \overline{x_P}$ right like similar to what we are done in the assignment.

So, the mean value of a source term is represented using the cell centroid value right, that is we calculated at x_P right, that is what we have done. And after this anyway, we have introduced a linear model right which we saw that there is no effect of it in at the time of convergence. Because the linear model recovers the original source term that we have right at the as we converge our iterations fine. What are the other assumptions, we have made in the same discretization?

Student: Face centroid value.

Face centroid value. So, we got this other term which is basically $\nabla \cdot (\Gamma \nabla \phi)$, we did a Gauss divergence theorem and then, in the integration, we have replaced it with a summation saying that the value of gamma grad phi on the face could be represented using the face centroid value ok. So, there is an analogy between what we are saying now for the diffusion flux on the face and the source term that we have just said now right. There is an analogy because we are talking about either face centroid or cell centroid right.

So that means, the $\Gamma\nabla\phi$ on the face e is taken to be the value at the face centroid right. So, the face centroid value, if we can calculate that remains the same on the entire face that is an average mean value, which we have taken an integrated converted our integral into a summation right.

Because we had this integral $\Gamma\nabla\phi \cdot \overrightarrow{dA}$ that we converted into $\sum(\Gamma\nabla\phi \cdot \vec{A})$ right that is what we have done. So, this is a face centroid value. What are the other things that we have done? Have you done any other assumptions; profile assumptions?

Student: Linear profile assumption.

Linear profile assumption $\frac{d\phi}{dx}$. So that means, after this we have done the $\frac{d\phi}{dx}$ and $\frac{d\phi}{dy}$, all those things we said that this is evaluated using ϕ_E minus ϕ_P upon δx_e. So, this is a linear profile assumption. So, we assume that the ϕ varies linearly with respect to x right. So, these are the three profile assumptions, we have made about the variation of the dependent quantities right all right. Then, if you look at it, if we want to categorize these three, what about the first and two items that we have listed here, are they one and the same?

It is basically saying that if you have a value on a particular face or on a particular cell that we are represented using the face centroid value or the cell centroid value right. That means, the mean value is an approximation by the cell centroid value right that is what we are saying. So, essentially one and two are one and the same right. They are the same and the third one is a profile assumption right that is about calculating the gradient of the dependent variable phi ok.

So, we are first going to look at the first one that is basically this is the mean value approximation or average value in the cell and then, we are going to look at the second one, this is basically your gradient approximation and see what are the truncation errors that we would get from these two terms ok. In order to do the analysis, I would assume a 1D uniform grid ok. We will talk about a non-uniform grid little later ok, I will make you comment on it.

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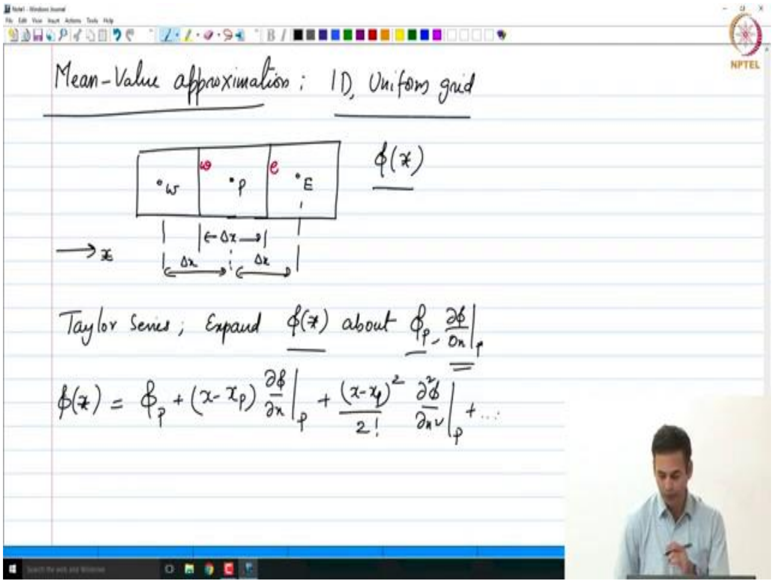

So that means, first let us look at the mean value approximation. So, I am considering a one-dimensional uniform grid ok that would be that would be what? That would look like this ok. So, we have a P cell, east and west and we have the faces that is the little e and little w ok. I will also consider the dependent variable to be some function $\phi(x)$ ok, it varies with x and this is the P cell is the control volume, we are looking at ok.

Now, because this is uniform, this distance is let us call it as Δx and the distance between the cell centroids also would be Δx right. It is a uniform mesh everywhere. Now, let me express $\phi(x)$ about the cell centroid value that is P about ϕ_P. So, I want to use a Taylor series and expand $\phi(x)$ that is within the control volume or between the control volumes, expand $\phi(x)$ about ϕ_P right. Now, ϕ at P is a particular value right and $\phi(x)$ is a continuous function right.

Similarly, all the derivatives at the cell P would be particular values right. If you were to write $\left.\frac{\partial \phi}{\partial x}\right|_P$, this would be a particular value right ok. Now, if I were to write this, how do I write $\phi(x)$? That would be ϕ_P plus at any location, let us say x that would be Δx that would be x minus x_P right. If it is going to the positive x side, you would have positive Δx; if it is going to the negative x side, you will have minus automatically taken care of ok, times what?

Student: (Refer Time: 26:51).

$\frac{\partial \phi}{\partial x}$ evaluated at?

Student: P.

P plus x minus x_P whole square by 2 factorial $\frac{\partial^2 \phi}{\partial x^2}$ evaluated at p and so on right, that is our Taylor series expansion for variable $\phi(x)$ about ϕ_P about the cell centroid value all right. So, now, let me integrate this on the entire control volume. The idea is essentially we want to calculate a mean value for this $\phi(x)$ right. Now, $\phi(x)$ could be something like a source term right. It could be a source x ok. Now, what would be the integration for the entire cell? We are going from west face to the east face right.

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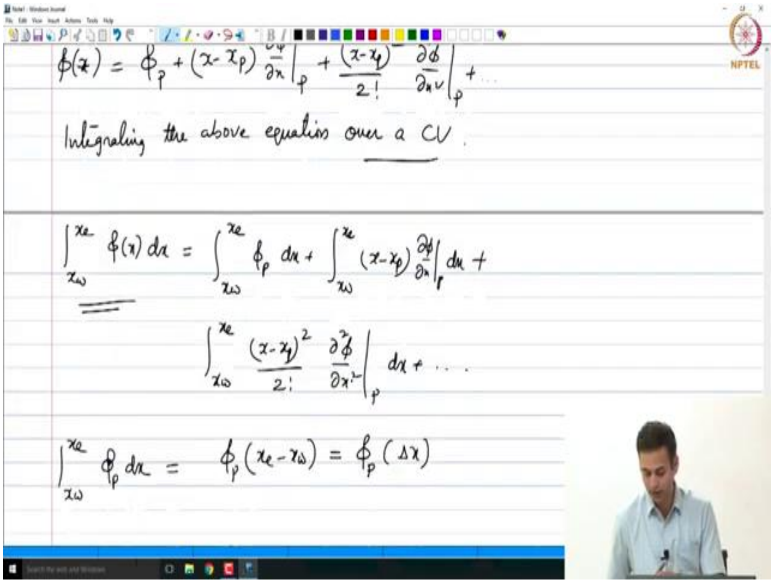

So, integrating the above equation over a control volume that means, what would be our x tends? Integral x_w to x_e, $\phi(x)dx$ right equals; on the right hand side integration would be x_w to x_e, $\phi_P dx$ plus x_w to x_e, $(x - x_P)\frac{\partial \phi}{\partial x}\Big|_P dx$ plus what else? Integral x_w to x_e, $\frac{(x-x_P)^2}{2!}\ \frac{\partial^2 \phi}{\partial x^2}\Big|_P$ dx and so on right, that is what we have right.

We have just integrated the Taylor series about the control volume all right. So, then, let us leave the left hand side as it is and consider each and every term on the right hand side ok. So, what will be the first term on the right hand side? Integral x_w to x_e $\phi_P dx$. What will this be?

Student: (Refer Time: 29:04).

ϕ_P times?

Student: x.

x; x is nothing but x_e minus x_w that is ϕ_P times?

Student: Δx.

Δx right. ϕ_P is constant. So, ϕ_P times x is what you would get and substitute the limits that is x_e minus x_w. x_e minus x_w is?

Student: $2\Delta x$; $2\Delta x$.

$2\Delta x$, is it why?

Student: (Refer Time: 29:28).

This is x_e, x_e is here, x_w is here, is only Δx right. It is not the capital E, its only the little e right. We are the control volume goes from x_w to x_e right, not from x_W to x_E right ok. So, this is only Δx right; yeah ok. What about the second term?

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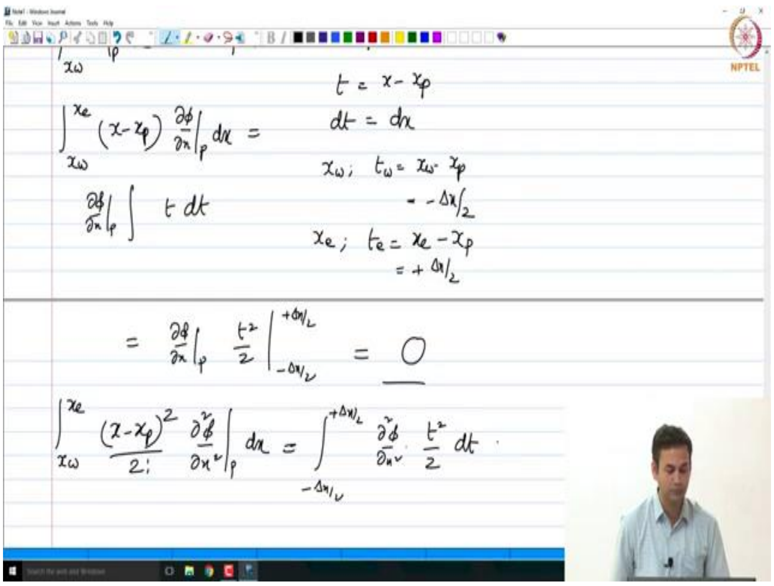

Integral x_w to x_e, $(x - x_P)\left.\frac{\partial\phi}{\partial x}\right|_P dx$, what will be this? Can you have a change of variables here? We call may be some t as $(x - x_P)$ ok, that is have a change of variable. Let us call t as $(x - x_P)$ and dt would be equal to dx. What will this be then? Integral $\left.\frac{\partial\phi}{\partial x}\right|_P$ is anyway constant right, times t dt; what will be the limits?

Student: $x_w - x_P$ minus x p.

$x_w - x_P$ or so essentially, if you have x_w, then t would be t_w would be how much? $x_w - x_P$ that will be how much?

Student: Minus delta.

Minus Δx.

Student: By 2.

Divided by 2 right. x_w to x_P ok; similarly, x_e would produce t_e that will be $x_e - x_P$ that will be?

Student: Δx.

Plus $\Delta x/2$ right. The changing the limits from x_e to x_w to about x_P you would get half on each side, $-\Delta x/2$ to $\Delta x/2$ right because now here x p 0 ok. Very good. So, what is this value now? What will be this entire integral? $\left.\frac{\partial\phi}{\partial x}\right|_P \frac{t^2}{2}$ right going from $-\Delta x/2$ to $\Delta x/2$. What will this be?

Student: (Refer Time: 31:35).

This will be 0. Is not it? Just have an even function going from minus you know this will be 0 ok, this will be 0, fine. What about the next term? That is x_w to x_e, $\frac{(x-x_P)^2}{2!}\left.\frac{\partial^2\phi}{\partial x^2}\right|_P$ dx. If you do the same analysis right, what will that be? That will be $-\Delta x/2$ to $\Delta x/2$, you will have $\left.\frac{\partial^2\phi}{\partial x^2}\right|_P \frac{t^2}{2} dt$ right. Would it be? This one yeah, just change of variables fine ok.

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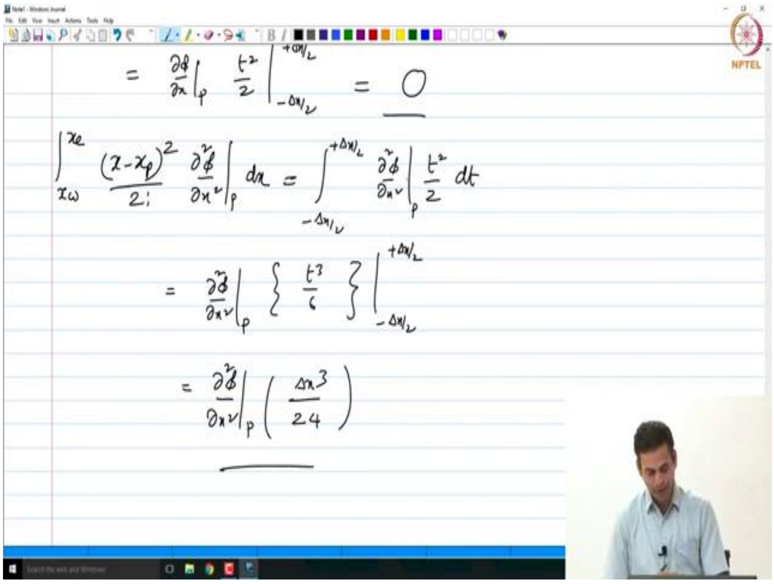

Now, this will be how much? This will be $\frac{\partial^2 \phi}{\partial x^2}\Big|_P$ evaluated at the cell centroid P times this will be $t^3/6$. So, this will be $t^3/6$ going from $-\Delta x/2$ to $\Delta x/2$. So, how much would this be? Sorry, this will be $\Delta x/2$ to the cube. You will get 2 cube 8, 48 right in the denominator and you have two 148th terms. So, that will be 1 upon 24 right. So, this will be $\Delta x^3/24$ right that is what you would get fine.

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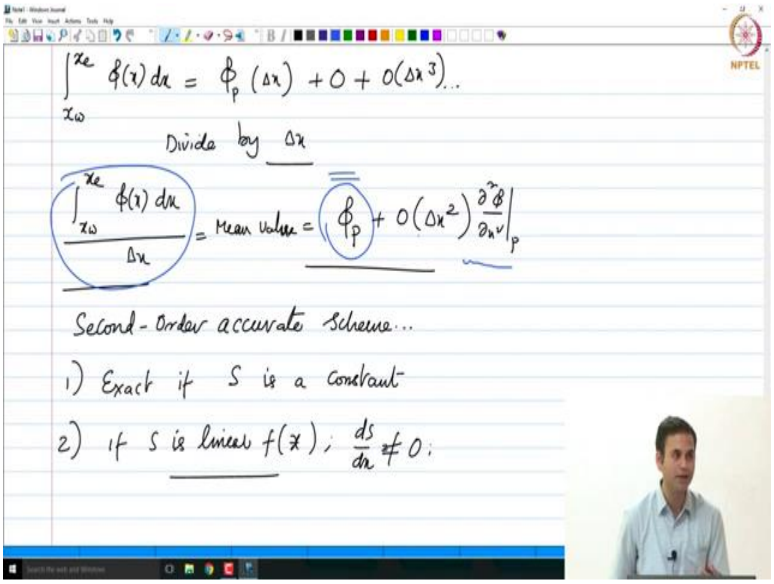


Now, let us go back and put it back into our equation. So, if you plug it in back, what will be the complete equation you have? x_w to x_e, $\phi(x)dx$ equals what is the first term? ϕ_P times Δx right that is the first term. What about the second term?

Student: 0.

0. Third term, what we just got right. This is order Δx^3 term right that is what we got all right, that is kind of sufficient to give us an idea. Now, how do I define a mean value of some quantity? That is nothing but integration of the quantity or the cell divided by?

Student: (Refer Time: 34:03).

Volume of the cell or length of the cell or area of the cell; that means, I would divide everything by Δx. So, essentially divide by Δx, I would get integral x_w to x_e $\frac{\phi(x)dx}{\Delta x}$. So, this is what? This is the definition for mean value right. This is mean value and what did we say? This is equal to, this is equal to now on the right hand side is ϕ_P plus order.

Student: Δx^2.

Δx^2 right ok. So, what did we say? We said ϕ_P right is the value of the cell centroid right. So, in our analysis, what did we say? We represented the mean value using only this quantity right. So, what is the truncation error involved in that?

Student: Second order.

Second order right. Did you say the second order scheme? So, what we have is a second order accurate scheme ok, when you represent the mean value by either a face centroid or by cell centroid right. Now, would the same thing apply for the $\Gamma\nabla\phi$ east also?

Student: Yes.

Yes, because there also you have an integration on the face right and then, you have $\Gamma\nabla\phi$. So, it will apply. So, for both items 1 and 2, we have second order accuracy all right If it is kind of good to write to this coefficient. So, what we have is $\left.\frac{\partial^2\phi}{\partial x^2}\right|_P$ ok. Now, let us say you have a source term, we are talking about represent the integration of the source term using a cell centroid value right, average value.

So, this ϕ is more like a source term or it could be $\Gamma\nabla\phi$ something like that. Now, if you have a source term that is constant ok, what will be the this expression boiled down to? If we have a constant source term, what will happen to this term, $\frac{\partial^2\phi}{\partial x^2}$?

Student: 0 (Refer Time: 36:15).

0 right. You have a constant source and then, what will be the mean value approximation of ϕ_P? That will be exact right. So, essentially, this approximation is exact if S is a constant right. If S is constant, $\frac{\partial^2\phi}{\partial x^2}$; this will be $\frac{\partial^2\phi}{\partial x^2}$ that would be 0. As a result, your mean value for the source term would be only the value at the cell centroid that is exact. Because you do not have any more term surviving right, fine.

We are talking about spatial dependency here. Do not get confused between ϕ_P, do not get confuse with the linearization ok. You are if I used ϕ, but this could be either a source term or it could be $\Gamma\nabla\phi$, some variable right. Do you see that? Ok.

What about if the source is S is linear function of x? If it is a linear function of x, you will have $\frac{dS}{dx}$ would be non zero right. If it is a linear function, this would be non zero; but what about the higher derivatives of the source term?

Student: 0.

0s and but, do you have a linear term coming into play here? No. So, even if you have a linear variation for S in x, this approximation is exact ok; but if you start having a second order representation for your source terms, then this representation is not exact ok. For example, if you would recall I think in one of the assignments you had a source term in the first assignment; source term was given as some $5x^2$; is not it?

So, there if you had used the cell centroid value which is $5x_P^2$ square, you will be making this truncation error right, rather if you have done a integration just like using a analytical expressions. Then, you would actually probably getting an exact value ok, that would be the difference you would have made ok.

But once, you convert your solution, you will not see any difference right. So, that we can cross check by redoing it with a different source term. Fine; everybody understand here ok? So, this is expression is exact for both the constant as well as a linear variation of the source term; but it incurs order Δx^2 truncation error, if you have anything higher variation than that ok.

Student: Sir, (Refer Time: 38:45) you will have first term right.

You will have first term, but the thing is your first term is not is not surviving in the truncation error analysis. Because of the because of the integration here right, it only depends on this mean value approximation only depends on $\frac{\partial^2 \phi}{\partial x^2}$ terms right. The $\frac{\partial \phi}{\partial x}$ terms got cancelled right.

Student: (Refer Time: 39:06) Γ∇ϕ plus x, y is equal to 0. So, that ∇ϕ is equal to S (Refer Time: 39:15) one more derivative means $\frac{\partial^2 \phi}{\partial x^2}$ equal to (Refer Time: 39:17) $\frac{dS}{dx}$ (Refer Time: 39:19).

Student: So, (Refer Time: 39:21) $\frac{dS}{dx}$ is not equal to 0, then $\frac{\partial^2 \phi}{\partial x^2}$ we have (Refer Time: 39:25).

Yeah sure. So, see the question is if I have source term as a function of let us say is a linear function of x right. So, what did we say? We said that we are representing the mean value using only the cell centroid value right. Now, this expressions are one and the same. Either if you have ϕ as a constant or ϕ as a linear function of x because this expression does not contain any first derivatives right. If there are first order derivatives here, then this would be not exact even for a linear function of S right, that is what I mean to say. Other questions? No, clear ok.

Now, one question to you is what will happen if you have non uniform mesh? We consider Δx; will this still be second order accurate, if I have a non-uniform mesh? So, for that we have to go back. So, even if I have a non; so, where is my term getting going out? That is basically the second term right. So, even if I have a non-uniform mesh, what will happen to my limits? Would they be equally on both sides of the centroid?

Student: (Refer Time: 40:38).

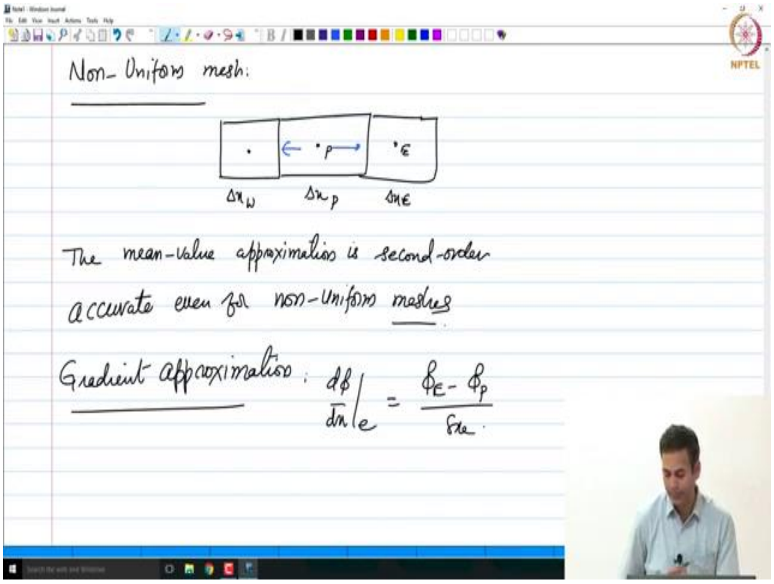
They will be; for the cell they will be. But cell to cell the Δx would be different right. See this is this is integration of limits from x_w to x_e. So, x_w and x_e would be always equidistant from the centroid right. But not the Δx of P would be different from Δx of east, east cell.

So, in that sense, even if I have a non-uniform mesh would I still get a 0 for the second term.

Student: Yes.

Yes, I will right because if I have a non uniform mesh.

(Refer Slide Time: 41:14)



So, if we have a non uniform mesh, then how does the mesh look like? It will look like I have 1 cell here, another cell here and something else here ok. So, this is my P cell, this is my E cell that means, my Δx_P would be different from Δx_E, Δx_W right.

The cell widths of each of the cells are different, but the limits of integration would be $-\Delta x_P/2$ to $\Delta x_P/2$ that is x_w to x_e which will be equidistant within the cell about the centroid right. As a result, your mean value approximation will still have a 0 for the first derivative right, but this.

So, it will be still second order accurate, even if you have a non-uniform mesh, the mean value approximation ok. So, the mean value approximation is second order accurate, even for non uniform meshes ok. Fine? Everybody agrees or questions? Ok all right. Then, let us look at the other term that is the gradient approximation right because this takes care of the source term and the $\Gamma\nabla\phi$ on the faces.

We look at the gradient approximation that is $\frac{d\phi}{dx}\Big|_e = \frac{\phi_E - \phi_P}{\delta x_e}$, that is the only term remaining right in terms of approximation, linear profile assumption that is basically the gradient approximation ok; wherein, we $\frac{\partial\phi}{\partial x}\Big|_e = \frac{\phi_E - \phi_P}{\delta x_e}$ ok.

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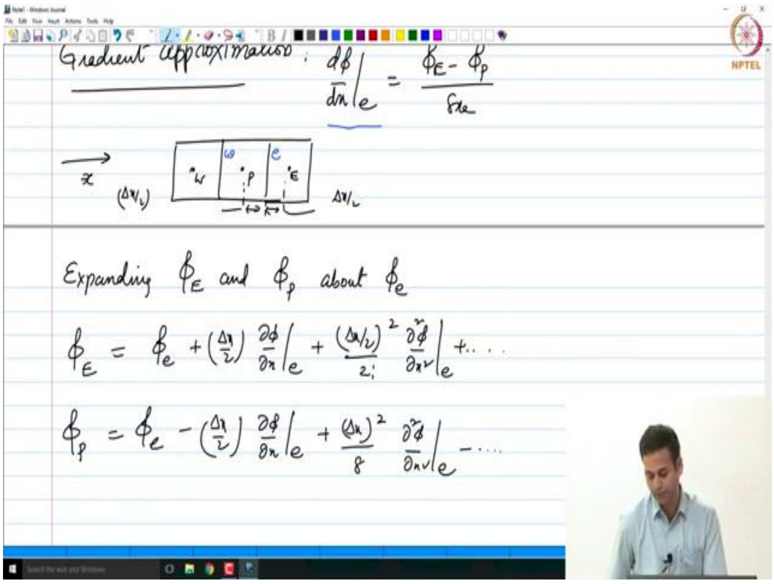

Now, let us consider a uniform grid again ok; its basically same as what we had before, I am considering a uniform grid and this is P, this is capital E, this is west and our faces are east and west ok. Now, let me expand ϕ_P and ϕ_E about the face e ok. So, essentially, we are expanding the values expanding ϕ_E and ϕ_P about ϕ_e ok; about the face value, I am expanding the cell values. I mean essentially, we are interested in what is $\frac{d\phi}{dx}$ right at the face.

So, when would I calculate $\frac{d\phi}{dx}$ at the face, only if I expand about that term right. If I expand about ϕ_P, I will get gradients at P that is not what we want. What we want is $\frac{d\phi}{dx}$ at face value e. So, we are expanding E and P about the face that is little e. So, that we can see if we can get an estimate for this value right because this will show up in the Taylor series that is what we are doing. So, then, what would be the distances now? About ϕ_e, what will be these two distances?

This will be $\Delta x/2$; the other one would be the distance wise $\Delta x/2$, but it will be in the negative x direction ok. So, this is x direction ok. Then, what would be ϕ_E about ϕ_e? ϕ_e

plus x capital E minus x little e that would be $\Delta x/2$ right; $\frac{\partial \phi}{\partial x}\Big|_e$ at e, is that correct? Yes, plus a what else? $(\Delta x/2)^2$ upon 2 factorial, $\frac{\partial^2 \phi}{\partial x^2}\Big|_e$ at east and so on right. What about ϕ_P about ϕ_e?

Student: Phi

ϕ_e minus $\Delta x/2$ $\frac{\partial \phi}{\partial x}\Big|_e$ about east; what next? Plus Δx^2 upon 8 right; $\frac{\partial^2 \phi}{\partial x^2}\Big|_e$ about east minus and so on right. Correct? Ok.

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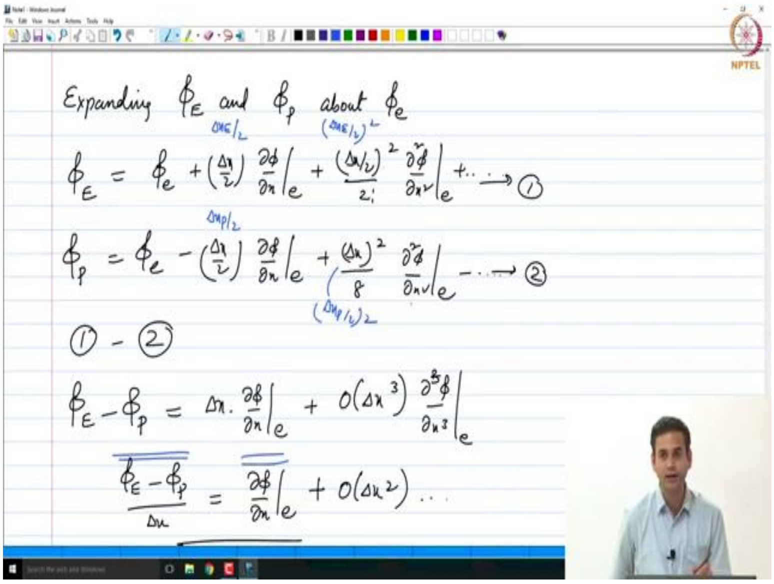

Then, what do we want? We want ϕ_E minus ϕ_P lets subtract 1 subtract 2 from 1; so, essentially we are looking at 1 minus 2. So, on the left hand side, we have ϕ_Et minus ϕ_P equals what will be there on the right hand side? ϕ_e gets cancelled; first term gets cancelled; second term gives rise to?

Student: (Refer Time: 46:42).

Two halves will be 1. So, this will be $\Delta x \frac{\partial \phi}{\partial x}\Big|_e$. What about the third term?

Student: 0.

Gets cancelled. What about the fourth term?

Student: (Refer Time: 46:50).

You will have some value that will be order Δx^3 or 4 or?

Student: Cube.

3 ok, $\Delta x^3 \left.\frac{\partial^3 \phi}{\partial x^3}\right|_e$ right, that is what we will have ok. Then, what is that we want? We want $\phi_E - \phi_P$ by Δx equals $\left.\frac{\partial \phi}{\partial x}\right|_e$ at east plus, what will be this term?

Student: Δx.

Order Δx^2 times something right. So, when we calculate $\frac{\partial \phi}{\partial x}$, if you only approximate it with $\frac{\phi_E - \phi_P}{\Delta x}$, what will be the truncation error that is involved?

Student: Order (Refer Time: 47:36).

Order Δx^2 right. So, this is also second order accurate ok.

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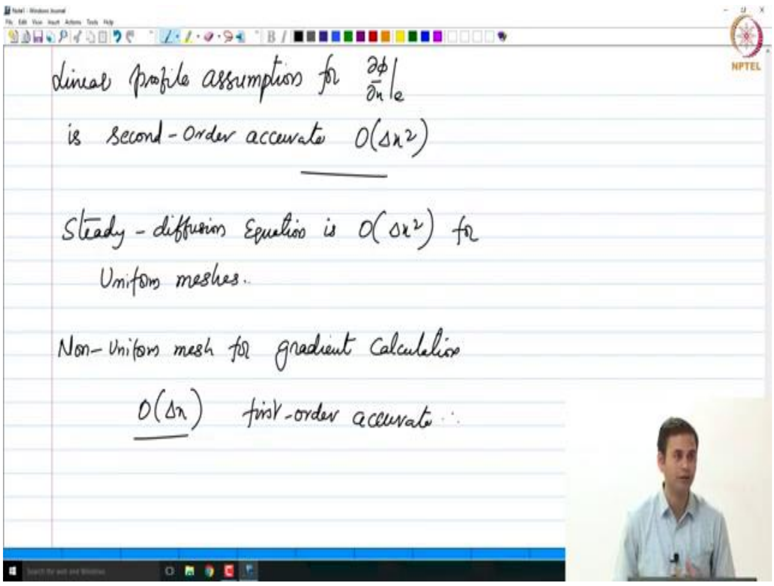

So, essentially, the linear profile assumption for the gradients $\left.\frac{\partial \phi}{\partial x}\right|_e$ is second order accurate right. This is second order, that is order Δx^2. So, with this all the terms that we have or the all the profile assumptions are now done right and all of them are what order of accuracy?

Second order. As a result, the steady diffusion equation is order Δx^2 ok; for I would say uniform meshes right now because that is what we have considered ok, for uniform meshes and for uniform meshes ok. We have also seen that for non-uniform meshes, the mean value approximation is still second order accurate ok. This is order Δx^2 because every term has truncation error of order Δx^2 in this steady diffusion equation.

Now, what about non uniform mesh in the context of gradient approximation? So, this is non uniform mesh for gradient calculation. Is it still second order? Yes? Yes or no?

Student: No.

No; why?

Student: Δx will be different.

Δx will be different right because what will happen to this $\Delta x/2$? This will be $\Delta x_E/2$, what will happen to this guy?

Student: $\Delta x_P/2$ (Refer Time: 49:39).

$\Delta x_P/2$. As a result, these two as a result this would not get subtracted off right. So, sorry yeah, I mean this is the second term right; this term would not get subtracted off. So, it will this will still remain there ok. So, essentially this term would still remain there, this is $(\Delta x_E/2)^2$ and this would be $(\Delta x_P/2)^2$ right. As a result, you would not get order Δx^2 terms. Rather you would start getting terms which are order Δx that means, you would still get it will be only order Δx. This will be only first order accurate, if you have non uniform meshes ok.

As a result, the if you have non uniform meshes, the total order of accuracy will come down to first order because some terms are first order accurate ok. So, we have to now look at the unsteady diffusion equation and in the unsteady diffusion equation, we will look at both spatial and temporal truncation errors. But the spatial errors are pretty much the same as what are there in the steady diffusion equation because those are the same assumption we have made.

But anyway, we will quickly look at that in the next class fine. So, I am going to stop here and see you guys in the next class.

Thank you.