Course Name: Turbulence Modelling

Professor Name: Dr. Vagesh D. Narasimhamurthy

Department Name: Department of Applied Mechanics

Institute Name: Indian Institute of Technology, Madras

Week - 6

Lecture – Lec32

32. Modelling of turbulent kinetic energy (k): production, destruction, and dissipation rate - I

So, let us get started with where we have left off yesterday, which was eddy viscosity models. So, in a two-equation model, one example we discussed is eddy viscosity could be a function of turbulence kinetic energy and its dissipation rate, right? So, in a two-equation eddy viscosity model, one option for ν_t is we can take ν_t as $C_\mu k^2/\varepsilon$ and this we have got it from the dimensional analysis. And if, of course, here the question is what is C_μ, what is k? k is of course turbulence kinetic energy, but we need to see how to get this. This equation when we derived it, it is not closed. There was a turbulence closure problem there also.

And ε is the dissipation rate of turbulence kinetic energy. So now we will see how to go about it. So, the first thing is, of course, to look at it; I will come back to C_μ later. First, we will see the k.

We have an equation, an exact equation for k. So, we will proceed in that direction. So we have an exact equation, or I can write modelling turbulence kinetic energy. So, we have an exact equation for k or your turbulence kinetic energy, which is we have dou k by dou t plus. This is a transport equation, of course, uj dou k by dou xj equal to; we have on the right-hand side the three terms the diffusion rate, production rate and the dissipation rate.

So, dou by dou xj, the diffusion rate of the three terms, which is the pressure diffusion rate p prime ui prime delta ij minus half ui prime ui prime uj prime plus you have the diffusion rate due to viscosity which is dou k by dou xj ok. The exact equation I am writing and then the production rate which is minus ui prime uj prime over bar dou ui over bar by dou xj minus the dissipation rate which is mu dou ui prime by dou xj, dou ui prime by dou xj, the square term. So, this is your dissipation rate. So, I can symbolically write this as this is Dk here, the diffusion rate term for the turbulence kinetic energy, and this is your Pk, this is epsilon. Symbolically, one can write like this in the transport equation three particular terms.

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left\{ -\frac{1}{\rho}\overline{p u_i'}\,\delta_{ij} - \frac{1}{2}\overline{u_i' u_i' u_j'} + \nu \frac{\partial k}{\partial x_j} \right\} - \overline{u_i' u_j'}\frac{\partial \bar{u}_i}{\partial x_j} - \nu\,\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}$$

So, the left-hand side I do not have to do any modeling the mean velocity is available k is what we are computing. But the right-hand side we see that there are far too many unknowns right. So, we have specifically turbulence kinetic energy to compute turbulence kinetic energy I have this Reynolds stresses that is an unknown the dissipation rate and this correlation for the diffusion rate due to turbulence term and the pressure term. So, we have additional unknowns that we have to model. So, this is the exact equation here.

So, now we will proceed towards modeling ok. So, for modeling I will take up first the production rate. How do I model? So, we call it let us say Pk model. The model terms. So, the exact term is available, which is Pk exact, right? which is $\underbrace{- \overline{u_i' u_j'}\frac{\partial \bar{u}_i}{\partial x_j}}_{P_k}$

The mean strain I do not have to model that is available. The Reynolds stresses we need to find a way. So, how do I do this? So, Pk exact is known. So, what should I do for Reynolds stresses? $\overline{u_i' u_j'}$ is an unknown, that is 6 unknowns are there. So, what is the modeling idea here? How do I close this particular term? Boussinesq, right? We already used the Boussinesq hypothesis to close Reynolds stresses in the RANS equation.

So, I am going to use the same Boussinesq here. modeling using Boussinesq hypothesis again. So, I can say the Pk model is now essentially the Pk model term is equal to the minus $u_i' u_j'$, I am going to substitute this as your $2\nu_t S_{ij} - 2/3 k\delta_{ij}$ ok. And, of course, the main strain is your Pk model. So, it was straightforward to use the Boussinesq hypothesis here, but you must understand that we are actually modelling the production rate or the generation rate of turbulence kinetic energy.

in a statistical approach. So, this has a far-reaching consequence by modeling the production rate. You are telling the flow system how much turbulence to be generated using the Boussinesq, and we will see later that this actually leads to errors, ok? So, there is a modelling error associated by using the Boussinesq hypothesis for the production rate here. I will come to that later. What are the issues with modelling like this? But right now, the model is now closed with the 6 unknowns; these 6 unknowns here now have been closed using you know ν_t, which is $C_\mu k^2/\varepsilon$.

And we are going in the direction of closing the k. So, k will be computed I will soon tell you how to get ε and C_μ. So, ν_t is anyway one of the thing that it is not introducing any additional unknown here by closing Pk using Boussinesq nu t is already there existing, and we have figured out how to do that mean strain rate you have access turbulence kinetic energy is what we are modelling right now. So, it has not introduced any additional terms than what we are already discussing about. So, this particular term is now taken care of this is modeled right.

So, now, we look into the diffusion rates. Now, coming to the diffusion rate, there are three different terms here: diffusion due to the pressure, right pressure, diffusion rate, turbulent diffusion rate and the viscous diffusion rate. So, in this, this is clear, there is no change in a sign. It is - $\overline{u_i' u_j'}$is modelled as Pk model. So, Pk is this particular term here.

The Pk is entire part, but what we are modelling is this closing the Reynolds stresses, -$u_i' u_j'$. Follow the Boussinesq; I have not changed any sign here. So, now we have the diffusion rates due to three different terms. So, here we make some modeling assumptions again in the diffusion rates. The first thing is that we look at, we take help from the data here.

So, we have to look at direct numerical simulation data, and then we see that which term is possible to neglect. So, the viscous diffusion rate we cannot neglect because viscous effects are dominant very close to the wall. We have seen even in a turbulent boundary layer, there is a zone very close to the wall called the linear sub-layer where viscous terms are dominant to first-order right turbulence is negligible inside a turbulent boundary layer in the thin zone, so viscous effects are always dominant very close to the wall so viscous diffusion rate we cannot let it go But when we compare all these three terms, viscous diffusion rate, turbulent diffusion rate and pressure diffusion rate, in some canonical flows like a smooth wall boundary layer or, a smooth pipe flow or a smooth plane Couette flow and so on, we see that pressure diffusion rate is much smaller compared to a turbulent and a viscous diffusion rate. So, viscous diffusion rate dominates very close to the wall, turbulent diffusion rate dominates away from the wall and the pressure diffusion rate is minimal and it is also a complicated term to model. So, the modelling argument is we take to model this Dk model.

The first modeling argument is upon comparing, comparing the three diffusion rates. The pressure diffusion rate ok; this is your p prime ui prime bar term, p prime; this particular term pressure diffusion rate is small compared to the pressure diffusion rate is small in some canonical flows. For example, smooth wall turbulent boundary layers pipe flows wherever you have a smooth wall turbulent boundary layer like pipe flow, channel flows, Couette flows or a flat plate boundary layer. In that flow, this is from the DNS data.

We can see this ok. Upon comparing this small, and this is, of course, coming from, information is coming from the DNS data from the budgets that I discussed, once we look at all the terms and see which is big and small in a particular flow, ok. So, this particular term is now small and therefore, for modeling sake we are omitting the term at least in this kind of flows it is omitted. But, this modeling term omitting it is a what to say, if you take any general purpose flow because the models are used for all kinds of flows, we will see that this is not a small term. I can show you the data for example, here. So, this is from our own in-house DNS data of the same turbulent Plane Couette flow that you are used to see the data that I am showing.

The only thing is I have introduced now what is called a roughness. So, I have the same problem as I discussed before. So, this plate was moving in this direction, right? u wall, and this was fixed like before. The only difference is now I have introduced some obstructions here at some separations small distance.

So, I am putting small ribs. this is typically used in heat transfer application. So, I am putting some small rib roughness. This is just to illustrate that all I did is I changed the flow problem only minimally just introduced little bit of roughness close to the wall, and you see from the data here, this is the roughness element here, the rib roughness. I am showing only a single pitch that is the gap between only this part, only this distance I am showing here in this part along the x direction, this is along your x direction, and this is along your wall-normal direction. So, this particular, so as you see this is wall to wall.

This is your turbulent diffusion rate, viscous diffusion rate and the pressure diffusion rate. So, if I come close to the wall let us say the values are going let us say the maximum is like 0.06 for a turbulent diffusion rate and they are pretty large actually compared to the viscous term, viscous term is even much smaller. And I did mention this idea of this inertial sub-layer or the buffer layer linear sub-layer. I think somebody asked this question: is it universal, and it is not because the moment you disturb the inner layer, all this linear sub-layer or the buffer layer vanishes because you are disturbing the flow by introducing a small flow separation or a vortex generator here.

This small roughness acts like a vortex generator, disturbing the near-wall boundary layer. And, so the viscous effects, at least now you see it is an order of magnitude smaller. 10 times smaller than turbulent diffusion rate. Were you able to see these numbers on the screen? So, it is, let us say near to the rib, I have 0.

06, and it is 0.001 in the viscous 10 times smaller, but if you look at the pressure diffusion rate, it is about 0.04, 02 and so on same order of magnitude as turbulent. So, those who are experts in aerodynamics you would know that this bluff element introduces what is called a form drag or pressure drag right. So, the pressure effects are big in this type of regions close to this box here. So, obviously, the pressure diffusion rate transport due to pressure field is now comparable to turbulent diffusion rate and the viscous effect is minimal here.

So, the modeling assumption here will go wrong. So, if you are going to use the model blindly without knowing its assumptions obviously, you will see error and you will not know where you have gone wrong. So, that is why I am showing you the data that the model assumption has been done for simple canonical flows where a smooth wall turbulent boundary layer is considered. Based on that only we are building the entire modeling approach. So, this modeling argument that pressure diffusion rate is small and therefore, this is omitted.

So, modeling argument is upon comparing the three pressure diffusion is small. So, therefore, omitted. But this is not true for general-purpose flow problems where this term can go big. But anyway to proceed further now we have this turbulent and the viscous term, viscous term I do not have to model because k you are computing and viscosity is known ok. There is no need to model this particular term, right? No modelling is required because k and ν are what is being known or computed here.

Only the turbulent diffusion rate needs to be modeled. For that we are going to use what is called a gradient diffusion hypothesis. So, to model turbulent diffusion rate which is basically you have $\overline{u_i' u_i' u_j'}$, half of it. This is your turbulent diffusion rate term here. So the Pk, this particular term, yes, to do this, we use what is called the gradient diffusion hypothesis.

This phenomena is known to some of you. Those who have taken heat transfer courses you would have learned this what is a gradient diffusion hypothesis. For example, if you have high temperature zone and a low temperature zone. So, the flux will go from a high temperature zone to a low temperature zone a gradient diffusion ok. So, similarly we take similar analogy that now the flux of turbulence will move from a high turbulence zone to a low turbulence zone. This is because it is a transport term diffusion is nothing but transport and turbulence itself is transporting.

So, we use this gradient diffusion hypothesis. So, what it does is basically if you have a high or I can write it here if you have a high turbulence kinetic energy zone to a low turbulence kinetic energy zone. So, there will be transport in this particular direction. or the can say the gradient transport from high TKE to a low TKE zone, and this we have seen analogous to your Fourier's law of heat conduction, right? Where you have your flux, let us say qx the x direction, you use the thermal conductivity and the gradient of your temperature, right? So, similar to that we construct the Tk or sorry the model for the turbulence diffusion rate. So, we can say similarly this particular model this is the exact term here, this is the exact.

Similarly, a model for this term would be this entire exact term is replaced by for example, minus nu t by sigma k, we will see what sigma k is here another constant being introduced and of course the gradient dou k by dou xj, this is the model term here.

$$-\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j}$$

So, now together I have this is exactly constructed similar to your Fourier's law of heat conduction. So, nu k is sorry, σ_k is an unknown, σ_k a model constant, ν_t we have access to $C_\mu k^2/\varepsilon$, k you are computing. So, now, if I introduce this into the Dk model, essentially, this is omitted right, so this is neglected or omitted. This no modeling is required and only for this term we have now constructed a model $-\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j}$.

So, that is for the entire term here. So, it is minus of minus it becomes positive. So, essentially it becomes dou by dou xj of nu plus nu t times dou k by dou xj. So, this particular term exact one is only this part. Where the Pk 1 was this entire Pk, this Pk and the Dk is only or the turbulent diffusion rate is only half of this term. So, now I have the Dk model, therefore is constructed as therefore the Dk model term entirely looks like you have dou by dou xj of nu plus nu t by sigma k dou k by dou xj.

the viscous and the turbulent diffusion rate are considered omitting the pressure diffusion rate. So, this is the model for the diffusion. So, I have the modeling ready for the diffusion rate where it is we are falling back on viscosity and eddy viscosity to help us to decide how much turbulence kinetic energy is being transported ok. And, of course, eddy viscosity depends on Boussinesq. It depends on other flow quantities as well as some turbulence quantities, and it depends on the k itself again.

So, you can see the coupling that is involved. You have to, you know, look at this equation carefully again. You will see the coupling. So, the transport of turbulence kinetic energy depends on nu t, and to compute ν_t, you need k, and the transport of k requires ν_t. So, there is a coupling. So, the coupled system of equations is what you are going to solve.

So, you need a robust equation which will be numerically stable. Otherwise, the code will. So, at least diffusion rates are done, production rates are done, only the dissipation rate is now to be modelled epsilon. Epsilon, we know that this is 9 unknowns here.

So, it is a positive term this entire part here. This is your $-\varepsilon$. So, in the eddy viscosity models, epsilon is not modeled based on considering each of these terms. These are far too many unknowns that you can it is a correlation term. nine correlation terms. So, we do not proceed in the direction of coming with an argument for that.

And one can say why not you know solve a transport equation? That is why I told this is a two equation model, two equations implying two transport equations. What is the second transport equation apart from k? Epsilon. So, we can solve a transport equation for epsilon, ok?