Course Name: Turbulence Modelling

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Week - 7

Lecture – Lec37

37. Standard k-ε model, RNG k-ε model, and Prandlt's one equation model - II

So, using equations 2, 3 and 4 in equation 1, I would get the left-hand side is 0, followed by I have the dou by dou y of ν_t, ν_t eddy viscosity is what we found this as $u^*\kappa y$, $u^*\kappa y$ eddy viscosity by sigma epsilon dou by dou y of epsilon itself which is u star cube by kappa y. The diffusion part is done.

$$\frac{\partial}{\partial y}\left(\frac{u_* \kappa y}{\sigma_\varepsilon} \frac{\partial}{\partial y}\left(\frac{u_*^3}{\kappa y}\right)\right)$$

And the production, which is plus C epsilon 1 epsilon by k epsilon, is again u star cube by kappa y epsilon by k. So, it is 1 over k is C_μ raised to minus half u star square and then Pk which is u star cube by kappa y.

$$C_{\varepsilon_1}\left(\frac{u_*^3}{\kappa y}\right)\frac{1}{C_\mu^{-1/2} u_*^2}\left(\frac{u_*^3}{\kappa y}\right)$$

And finally, the destruction rate of epsilon which is minus C epsilon 2 epsilon square by k. which is u star cube by kappa y whole square by k here, by k which is C mu raised to minus half u star square.

$$C_{\varepsilon_2}\frac{\left(\frac{u_*^3}{\kappa y}\right)^2}{C_\mu^{-1/2} u_*^2}$$

$$0 = \frac{\partial}{\partial y}\left(\frac{u_* \kappa y}{\sigma_\varepsilon} \frac{\partial}{\partial y}\left(\frac{u_*^3}{\kappa y}\right)\right) + C_{\varepsilon_1}\left(\frac{u_*^3}{\kappa y}\right)\frac{1}{C_\mu^{-1/2} u_*^2}\left(\frac{u_*^3}{\kappa y}\right) - C_{\varepsilon_2}\frac{\left(\frac{u_*^3}{\kappa y}\right)^2}{C_\mu^{-1/2} u_*^2}$$

So, I substituted all the scales here. So, if I go ahead and differentiate this before that you should know that u^* is a wall friction velocity, it has no y dependence. It is a constant for a given flow problem here, right? Or it will be in a developing boundary layer, it will be a function of x as you walk along x. The u star can change, but it will have no y dependency.

It is defined for all the walls it is coming from wall shear stress. So, u^* has no y dependence, κ is one common constant no y dependence, σ_ε has no y dependence, right? So, you should know here that $u^*\kappa$ and σ_ε are independent of y. So therefore, I can write this as 0 equal to, this thing comes out here, which is $u^*\kappa$ by sigma epsilon u star raise to 4 actually, kappa y will go away, sorry kappa will go away here. Another kappa comes in here, and then I get dou by dou y of y dou by dou y of 1 over y. This is going away.

$$\frac{\cancel{u_*^4}\,\cancel{\kappa}}{\sigma_\varepsilon\,\cancel{\kappa}}\frac{\partial}{\partial y}\left(y\,\frac{\partial}{\partial y}\left(\frac{1}{y}\right)\right)$$

Let me know if there is any mistake. plus, I have this term, which is any term that goes away; u^{*2} is there, kappa y square it becomes. So, I get C epsilon 1, C_μ raise to half, u star square and this goes away giving to u star raise to 4 by kappa y whole square minus, is it, any mistake I have done? No, this is fine. So, C epsilon 2, here I get C_μ raise to half and then mu star again here this will become 4 by kappa y square, ok. So, if I go ahead and differentiate this, so basically, u star 4 goes away here continuously, u star 4, u star 4, u star raise to 4 goes away, and so I have 1 over sigma epsilon differentiating this, this is minus y by y square that is 1 over y plus C epsilon 1 C_μ or the square root of C_μ over kappa square y square minus C epsilon 2 square root of C_μ by kappa square y square, essentially it has come to both.

So, now the difference is only in the $C_{\varepsilon 1}$, $C_{\varepsilon 2}$. If these two values become same, $C_{\varepsilon 1}$ is equal to $C_{\varepsilon 2}$ that means your production rate of epsilon and destruction rate of epsilon is becoming same. So, now you have on the other side So, this goes away here by this minus 1 by y. So, this becomes essentially plus. So, I get 1 by sigma epsilon y square plus $C_{\varepsilon 1}$ by kappa square y square minus $C_{\varepsilon 2}$ by kappa square y square, so y square goes away.

So, this becomes essentially, so I need to find out the value for $C_{\varepsilon 2}$ here, sigma epsilon I know the value from computer optimization I have the value, C mu I know, $C_{\varepsilon 2}$ I know, I know kappa. So, I can substitute all these constants to get the value for C epsilon 1, correct. So $C_{\varepsilon1}$ essentially becomes, so $C_{\varepsilon1}$ square root of C mu by kappa square will be equal to $C_{\varepsilon2}$ square root of C mu by kappa square minus 1 by sigma.

$$
\begin{aligned}
0 &= \frac{\cancel{u_*^4}\,\cancel{\kappa}}{\sigma_\varepsilon \cancel{\kappa}} \frac{\partial}{\partial y}\left(y \frac{\partial}{\partial y}\left(\frac{1}{y}\right)\right) + C_{\varepsilon1} C_\mu^{1/2} \frac{\cancel{u_*^4}}{(\kappa y)^2} - C_{\varepsilon2} C_\mu^{1/2} \frac{\cancel{u_*^4}}{(\kappa y)^2} \\
&= \frac{1}{\sigma_\varepsilon} \frac{\partial}{\partial y}\left(-\frac{\cancel{y}}{y^{\cancel{2}}}\right) + C_{\varepsilon1} \frac{C_\mu^{1/2}}{\kappa^2 y^2} - C_{\varepsilon2} \frac{C_\mu^{1/2}}{\kappa^2 y^2} \\
0 &= \frac{1}{\sigma_\varepsilon \cancel{y^2}} + C_{\varepsilon1} \frac{C_\mu^{1/2}}{\kappa^2 \cancel{y^2}} - C_{\varepsilon2} \frac{C_\mu^{1/2}}{\kappa^2 \cancel{y^2}} \Rightarrow C_{\varepsilon1} \frac{\sqrt{C_\mu}}{\kappa^2} = C_{\varepsilon2} \frac{\sqrt{C_\mu}}{\kappa^2} - \frac{1}{\sigma_\varepsilon}
\end{aligned}
$$

or I can say $C_{\varepsilon1}$ is equal to $C_{\varepsilon2}$ minus 1 by sigma epsilon times kappa square by square root of C mu. So, I have the values for all the constants here.

$$\boxed{C_{\varepsilon1} = C_{\varepsilon2} - \frac{1}{\sigma_\varepsilon} \frac{\kappa^2}{\sqrt{C_\mu}}}$$

$C_{\varepsilon2}$ is known. So, I can substitute this as what was the $C_{\varepsilon2}$, 1.92 we use, 1.92 minus 1 by sigma $C_{\varepsilon1}$.

3. Kappa square 0.41 square by square root of 0.09, right? So, if you do all this, I get C epsilon 1 equal to 1.44. So, the 5 model constants are now given in the equation.

It is C mu equal to 0.09; this is in the standard k epsilon model, and then sigma k equal to 1, σ_ε is 1.3 and then $C_{\varepsilon1}$ is 1.44. and $C_{\varepsilon2}$ is 1.92. So, these are the constants that are used in standard k epsilon model. So, the reference for this standard k epsilon model is if you want to go ahead and look at the original article where they have published the one who have come up with this model Launder and Sharma, 1974. So, this article you can download and look at. This one? Similar reasoning, yes. So, they are using this 1.44 for accommodating the other types of flows. This is accommodating other class of turbulent flows, especially the free shear. Free shear turbulent flows . So, one can go ahead and get these constants for different types of flows. You need to make different assumptions, different considerations right.

We have omitted certain, we looked at log law, we looked at certain types of flows. You can of course, there is no golden rule here. You can go ahead and look at a different flow and see whether this model constants work ok. But I think they have already looked through that by making this 1.92 and 1.44. So, if I use the same value what do you get?

What was the value that was coming out to be here? 1.489. So, now if you see this 1.489 and 1.92, this is C epsilon 1 is much closer to what they have found here.

But if you go back and see the formula for the production rate of epsilon and destruction rate, the main difference that you get here is actually this value here. You see, upon introducing scales, this particular term is now exactly the same. So, what you choose for C epsilon 1, C epsilon 2 decides how much the production rate of epsilon is there. So, you need to see this is a coupled system. So, production rate of epsilon is large means your x more epsilon is being produced more epsilon being produced means turbulence kinetic energy is drained away from the system, right? More epsilon means less k, and that will stabilize, so in some flows, the turbulence kinetic energy can outgrow or become very large, destabilizing your entire calculation, so a higher epsilon leads to a lower k, stabilizing the flow.

But this lower k also means that you have lower turbulence. For example, if you are looking into a turbulent jet and the jet will grow as we see any shear layer or a boundary layer grows based on the turbulence intensity. So, turbulence intensity can be thought of as square root of k, a turbulence velocity. So, that means k is higher means the jet will grow wider. So you can actually manipulate here, right? That's why this entire RANS model is purely statistical.

You can decide the size of the jet purely by micromanaging all these parameters here. You can get a wide jet or a narrow jet just by changing all the model constants or working with a different type of a k-epsilon model. So, you're excessively, it is user dependent. You have a say in how the flow should look like rather than the flow is telling you what it should be. For that, we have to go to eddy-resolved methods where the flow itself decides what it should be.

So, RANS approach is like this. So, I am just explaining to you what it is all about. So, now we can look at another; we can quickly go back and see. Let us say no. I would like to talk about another type of k-epsilon model, which is also very popular.

I will not go ahead and derive this because then the entire course will be just k epsilon models, ok? So, I would like to move on to other types, but there is something called RNG k epsilon model which is also popular. So, I will just quickly give what the main difference between, not in the derivation; derivation is very, very different. But at the end, when you look at standard k epsilon model and RNG k epsilon model what is the main difference that I would like to give you and tell you what is the new thing that they have done. Let us quickly look at this. What is called RNG k epsilon model.

This RNG stands for renormalization group. That is a group. So, there is a reference for this. It is Yakut et al, 1992. So, it is almost like 20 years later this model has come standard k epsilon is 1972, 1974 about 2 decades later this has come.

So, the primary difference in RNG k epsilon is of course, the model constants are different and one model constant is actually calculated. So, the C mu value takes 0.085 here not 0.09. and sigma k takes 1.0, same as before, sigma epsilon takes 0.719, and C epsilon 2 takes 1.68 here. But the interesting thing in this model is $C_{\varepsilon 1}$, that is the production rate of the epsilon that is there is a new influence here because $C_{\varepsilon 1}$ is calculated. So, the most interesting part here is that interesting feature, the feature is $C_{\varepsilon 1}$, is calculated.

So, it is dynamic. This value is not constant. In a given flow this value keeps changing at every point or in time. So, $C_{\varepsilon 1}$ is calculated, and the way they calculate is I call it $C^{*}_{\varepsilon 1}$. If you actually go and look into this model, they may call it a different name a c beta or a c gamma whatever they would they would have call it differently. But for you it is easier to understand if I am using the same terms $C_{\varepsilon 1}$, $C_{\varepsilon 2}$.

So, I am calling it C epsilon 1 star. This formula goes like this: 1.42 minus eta 1 minus eta by eta 0 divided by 1 plus beta eta cube.

$$\boxed{C^{*}_{\varepsilon_1} = 1.42 - \frac{\eta\left(1 - \eta/\eta_0\right)}{1 + \beta\eta^3}}$$

So, this is the formula they are using. So, what is special here is that this η they are looking into so far, we have looked into a turbulent time scale k by epsilon there is also a flow time scale that you can use the strain rate right dou u dou y will give you a flow time scale so they want to use the competition between a flow time scale and a turbulent time scale and introduce that into the flow so that is a very interesting feature here. This does not mean that the RNG Capsulon model is better than the Standard Capsulon in all the flows.

There are flows where Standard K epsilon is good. There are flows where RNG K epsilon is good. You have to figure it out which one is working good for you. So, here, η is the ratio of turbulent timescale over the flow timescale or, I would say, mean strain timescale. So, it is essentially, the formula goes like k by epsilon divided by 1 over S.

So, this S is, or I can rewrite this as s k by epsilon where S is nothing but square root of 2 Sij over bar Sij over bar.

$$\eta = \frac{\text{turbulent time-scale}}{\text{mean-strain time-scale}} = \frac{k/\varepsilon}{1/S} = \frac{Sk}{\varepsilon} \; ; \quad \text{where } S = \sqrt{2\bar{S}_{ij}\bar{S}_{ij}}$$

So, you are making use of this so that means you need to dynamically calculate in your flow and other model constants I will give it to you. So, this is η 0 value is 4.38. and then beta is 0.012. So, η is being computed on the go when simulation is running and they are looking into the computation between these two scales. One can also introduce some other scale. So, since turbulence is a multi-scale problem, if you feel the k by epsilon or any other turbulent scale that you have is not good enough, you can look at that. And many popular models they have already looked into this including this.

But the derivation takes a very different form. I can share the article for you or you can download this yourself and you can see. So, this is one other form of a k epsilon model. There is one more called a realizable k epsilon model. For that, that is actually instead of going into just a realizable k epsilon, I would like to introduce what is realizability. There must be something unrealizable here in the model.

So, I will first discuss what is unrealizable and then the fix which is realizability constraint that you can easily apply to any of your existing model rather than implementing a new model. There is actually a realizable k-epsilon model, but for any of your eddy viscosity model, it is a good idea to just implement the realizability constraint so that it becomes realizable. So, that form I will discuss when we go to realizable constraint, realizability constraints. I can quickly go and look at a one equation model here before proceeding. I told you in the beginning if you do a two equation model, it is very easy to do a one equation model.

So, we will quickly go back and see it is not very popular. There are some models which can be popular for a different reason. So, I will quickly go to what is called a one equation. one equation eddy viscosity model. So, here there are different models Palat, Almaraz and so on.

I will not go into that one. I will discuss one which is called Prandtl's model. So, this model the k model equation is same. no change in that dou k by dou t plus uj bar dou k by dou xj equal to dou by dou xj of nu plus nu t by sigma k dou k by dou xj plus the production rate term that is modeled which is your Boussinesq to nu t Sij minus two-third k delta ij and your strain rate term. minus epsilon.

$$K_{model}\text{ eqn} \Rightarrow \quad \frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left[\nu + \frac{\nu_t}{\sigma_k}\right]\frac{\partial k}{\partial x_j}\right) + \left(2\nu_t \bar{S}_{ij} - \tfrac{2}{3}k\delta_{ij}\right)\frac{\partial \bar{u}_i}{\partial x_j} - \varepsilon_{model}$$

So, up to this it is the same part as any other k epsilon model, but here for epsilon we actually have a model transport equation.

In one equation obviously there is no transport equation for epsilon. Here we simply take an algebraic form using dimensional arguments again. So, no transport equation for epsilon is used here, no transport equation for epsilon is used. Instead, epsilon simply comes out to be the epsilon model term here, this one. Simply dimensional arguments which is C mu k raise to 3 half over 1 turbulent length scale, dimensionally, dimensional arguments here.

So, the question or the main drawback of this model is what is L? You need a turbulent length scale that you have to give and so the question is what is L? So, this is user dependent. You can plug in whatever value and get whatever solution you want which is not a very good So, that is the main drawback here, the main disadvantage since it is user dependent. So, eddy viscosity is of course calculated here, we have eddy viscosity calculated same way as $C_\mu \frac{k^2}{\varepsilon}$, epsilon of course I have given you the formula and the model constants or sigma k is 1 again and what else I have sigma k eddy viscosity that is it epsilon is given and the C_μ value of course, the C_μ value here ranges from 0.07 to 0.09 depending on if you are doing the wall bounded flow to a free shear flow they slightly changes in the value occurs.

and the turbulent length scale as I said is the biggest problem here user defined and that means you can see here if you want if you define a large L here. Let us say you define let us say this is a pipe flow you put turbulent length scale as the length of the diameter of your pipe or something that is not a turbulent length scale, but let us say that is what you are giving a very large L will make epsilon becoming small. So, a small epsilon leads to higher turbulence kinetic energy. So, you can actually again control turbulence kinetic energy in this and turbulence is actually taking away the momentum from the flow. So, that means you can control your flow essentially by controlling the length scale which is not a good idea and that is the main disadvantage here a user defined turbulent length scale.

So, a large L leads to small epsilon which further leads to large k value and vice versa you can just interchange to get whatever you want which is not a good idea. Any questions you have I can take it up. So, one equation, two equations we have mainly one particular equation, k epsilon model.