**Course Name: Turbulence Modelling**

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**Week - 8**

**Lecture – Lec44**

# 44. Damping functions for LRN – I

Let us get started. In the last class, we were looking into the low Reynolds number models, right, LRN models. So, we briefly discussed the near-wall behaviour and the two-component limit. The two-component limit of turbulence and how it can be addressed using LRN models we will see it today. And we did a Taylor series expansion to see the near wall behaviour for the three fluctuations, which gave $u'$ and the higher order terms basically a $y$ dependency and then for the $v'$, it had a $y^2$ dependency in the near wall region and $w'$ has $y$ dependency again this we saw.

So, now we would like to introduce the same concept into the modelled equation and exact equations. So, we will go back and see what is the exact equation that we have. Recall that if you take a k epsilon model, epsilon we did not take the exact equation to model epsilon, right? So, we go back to only the k. That only k equation has an exact equation and its modelled counterparts.

So, we go back to taking the exact and model counterparts for the k ok. So, let us take the k exact equation. k exact equation. So, if you recall, it will be $\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j}$ equal to the right-hand side you have the three terms, which is $\frac{\partial}{\partial x_j}$ of the pressure diffusion, the turbulent diffusion and the viscous diffusion and the production rate term and the dissipation rate term, this is your exact equation k exact, and we also have the k modelled equation. On the left-hand side, it is the same thing, no difference in the modelled and the exact equations.

Left hand remains the same, only on the right-hand side, this particular term was not modelled. This was not modelled. You can recall the arguments that we placed. So, only the this particular term is modelled the turbulent diffusion term. So, we will see the modelled equation here looks like $\frac{\partial}{\partial x_j}$ of basically the $\nu + \nu_t$ or since we are writing it this way, we can write it $\left(\frac{\nu_t}{\sigma_k}+\nu\right)\frac{\partial k}{\partial x_j}$ or you can simply write it two parts here as $\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j}+\nu\frac{\partial k}{\partial x_j}$.

So, only this turbulent diffusion term is the model counterpart that exists between the two equations. The pressure diffusion is never modelled. Viscous diffusion looks the same. So, I do not have to do anything about it; it will have the same behaviour. So, we look at only terms which are modelled here ok.

Then I have this production rate term, which is, of course, we will have the what to say the Boussinesq is used. So, I have plus $\left(2\nu_t\overline{S_{ij}}-2/3k\delta_{ij}\right)\frac{\partial\overline{u_i}}{\partial x_j}$ .So, I also have this particular term now modelled. I am only highlighting using green colour what are the terms that are modelled that we need to focus on; the rest remains the same between the two comparisons. And here I have simply I said it is epsilon where we solve an another equation for it ok.

So, with this, now we go ahead and apply this for a canonical turbulent boundary layer using the same condition, statistically stationary homogeneous in two directions. So, if I apply, applying this to a canonical turbulent boundary layer that is a smooth wall turbulent boundary layer. So, we have statistical stationarity, statistically stationary it is $\frac{\partial}{\partial t}$ of your average terms are 0. and then also statistically homogeneous. Homogeneous in two directions, that is x, z, which implies we have which means $\frac{\partial}{\partial x}$ of all the average quantities and $\frac{\partial}{\partial z}$ of all the average quantities are 0, the same conditions and fully developed also, fully developed giving you $\bar{v}=\bar{w}=0$.

So, if I use that condition and look at the near wall behaviour using this three equations that I have here. So, I would get this equation let us call this 1 here and this equation 2, 1 is the exact and 2 is the modelled equation. So, this reduces to I will not look into the left hand side part or I will only look into the terms that are modelled and it exact counterpart exists. So, the equation would be like the first term the diffusion term here. So, I would write this as exact.

So, I have the diffusion term. So, that is minus half of I have ui prime ui prime. So, if I expand it I get u1 prime u1 prime plus u2 prime u2 prime. This is sum of three terms here plus u3 prime u3 prime multiplied by the uj prime that is there ok. And obviously, I am looking into the y dependency.

$$-\frac{1}{2}\left(u_1'u_1'+u_2'u_2'+u_3'u_3'\right)u_j'$$

So, $\frac{\partial}{\partial y}$ term is what I am looking into. So, this u j, j will take 2, right? Not the other this particular one it is a $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial z}$ of all the statistical terms are 0, statistically stationary, sorry, statistically homogeneous. So, only $\frac{\partial}{\partial y}$ which is $\frac{\partial}{\partial x_2}$ basically, this is $\frac{\partial}{\partial x_1}$ and this is $\frac{\partial}{\partial x_3}$. So, only $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial y}$ or the $\frac{\partial}{\partial x_2}$ term survives here. So, that means j has to take 2 value.

You understand this? This is clear? This we can expand it. Since j is repeated, i is also repeated. So, it eventually becomes a scalar, sum of all the terms. and j I am taking it as 2 not 1 and 3 because when I take j as 1 it becomes $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_1}$ of this statistical quantity this is an average quantity is 0. Similarly, $\frac{\partial}{\partial x_3}$ of this statistical quantity is 0.

Therefore, here only $\frac{\partial}{\partial x_2}$ term survives since $\frac{\partial}{\partial x_2}$ survives, it becomes, j becomes 2 ok. So this becomes only 2, which is $\overline{u_2'}$, statistically homogeneous that is your the other terms $\frac{\partial}{\partial x_1}$ of this is 0 $\frac{\partial}{\partial x_3}$ of this term is 0 here only this particular term is surviving. And if I look at this particular term exact one, what is its y dependency now? I can rewrite this as in terms of your u y for consistency I can write this as u prime u prime v prime plus this is v prime v prime v prime plus w prime w prime v prime. These are the three terms here for consistency I am writing. Now, if I look at this particular let us call this equation star here.

So, what will be its y dependency? The exact term of the turbulent diffusion term u prime is y dependency, another $u'$ *is* y and $v'$ is $y^2$. So, it is $y^4$. So, this will have $y^4$ near wall behaviour. This is the exact term only for this particular term I am looking at here. So, since I am looking into that one let us look at the exact term also for the model term also for the diffusion the turbulent diffusion.

So, now I have the model term here. So, if I take the model term, which is $\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j}$, again, same thing $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_3}$ will be 0, same argument as above here, the same thing. So, only $\frac{\partial}{\partial x_2}$ survives; j has to be 2. So, therefore, this reduces to essentially $\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_2}$. And for consistency, if I write in x, y, z terminology, I get $\nu_t$ as your in a k epsilon model.

You have $C_\mu k^2/\epsilon$. Of course, you have $\sigma_k$ here, and then you have $\frac{\partial k}{\partial y}$ term. So, we look at only the k epsilon and the gradient of the k here. So, $k^2$ we have already figured it out in the last class. So, here if I write it again.

So, the k is your half of u prime u prime plus v prime v prime this we did in the last class w prime w prime. So, I get here order of magnitude of y square because this will be this is y square, this will give $y^4$, and this is $y^2$. So, to first order, this gives me only y square behaviour. Ok? fine. So, now k is done, we also have epsilon.

So, we need to also figure it out what epsilon would do it. So, let us see it here for the epsilon yes. So, for epsilon, let me use another color for epsilon now it is $\nu \overline{\frac{\partial u_i'}{\partial x_j}}^2$ . So, upon expanding this, I would get sum of 9 terms. All 9 terms survive here because this is the average of the 2 correlations, 2 fluctuating quantities.

So, it is a correlation term. So, but we are looking into the y dependency. So, in the sub layer the gradient dou by dou y will be much much larger than dou by dou x. So, this is of course sum of all the terms. So, if I rewrite this if I expand it first I would get let us say nu dou u1 prime by dou xj whole square plus nu dou u2 prime by dou xj whole square plus nu u3 prime by dou xj whole square average. So, in the sublayer or the in the near wall near wall region.

What is so special in this part is that you will get typically your dou by dou y gradients are much much larger than your dou by dou x and dou by dou z or dou by dou x2. If I write it in terms of that, it is basically dou by dou x2 much, much larger than dou by dou x1 or dou by dou x3 gradients in the near wall zone. This y gradients are much larger in that, and therefore, this particular term will take up. So, epsilon will be essentially the j will take 2 dou by dou x2 of these 3. So, which is nu for consistency, I am writing it in terms of dou u prime u prime by dou y square average plus nu dou v prime by dou y square average plus nu dou w prime by dou y square average.

So, what is the near-wall behaviour here? So, dou u prime by dou y if I go back here? u prime is y dependency dou u prime by dou y will give me will give me no y dependency, right? So, dou u prime by dou y gives me no y dependency, and v prime will give me that. So, this will give me no y dependency y raise to 0. but dou v prime by dou y that is y square. So, it will give me y behaviour, but to first order it is only y 0. So, this becomes the higher order term here which is y and this is again y 0 that makes it essentially.

So, this will be the higher order term here. So what does this mean? Epsilon is having a y-0 behaviour. Epsilon has no y, it is not showing any y dependence in the near wall zone. So, epsilon is available now for the model k is available, and the dou k by dou y will figure it out. So, I get essentially the near wall behaviour $k^2$.

So, k is y square here. So, I get order of magnitude of y square y raise to 4 by I have y 0 for the epsilon and dou k by dou y will give me order of magnitude of just y. Because k is y square. So, this is giving me. So, the near wall behaviour of the modelled term here, which is here if I take this, it is giving me y raise to phi behaviour and if I take the exact one. This is the exact one that is giving me to raise to 4 behaviour in the near wall zone.

Something is not correct, right? So, we must do something about it. So, it looks like dimensionally, the model is consistent, The k epsilon model, the standard k epsilon model, but as you approach the near wall very close to the wall in the linear sub-layer buffer layer where viscous effects are dominant, it is not accounting for as I said the two-component limit as well as