Mechanical Measurement Systems

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Lecture - 19
Second Order System- Step Response (1)

Hello, I welcome you all in this course on Mechanical Measurement Systems. Today we will discuss the second order system and mainly we will focus on step response of second order systems.

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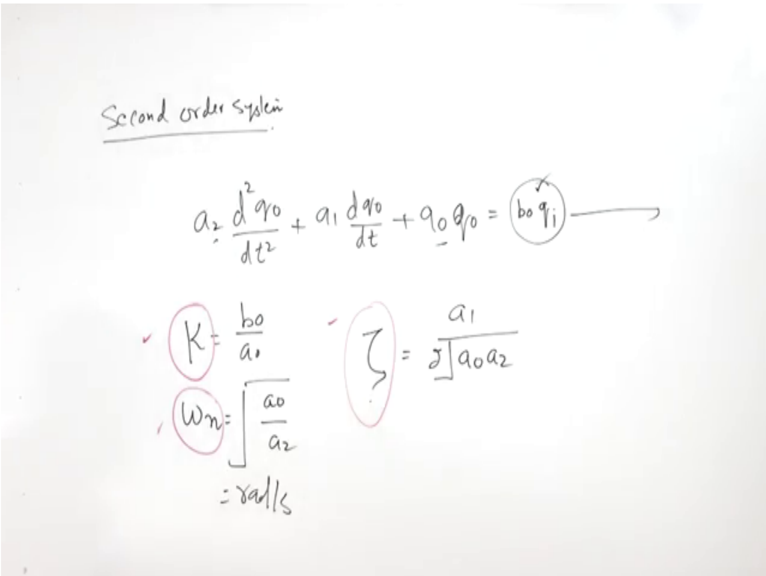

Now, in the second order system first we will deal with the basics of the second order system. If you remember the generalized governing equation for dynamic response of measuring systems in that equation if the degree of differential in output that is of second order that is a 2 D square sorry, D square qo by dt square plus a 1 D qo by dt plus ao qo is equal to bo qi. So, this type of system; the system the response of the system governed by this equation are known as second order systems.

Now, on this side we can go for the higher orders not necessarily we have to restrict for bo qi only, but for all engineering uses common entering uses this up to this it is good enough to represent many of the engineering systems especially mechanical engineering systems right.

Now, here there is a term sensitivity of the instrument and sensitivity of the instrument we get by dividing bo by ao in dynamic response of the system the system does vibrate also. I mean in first order system the response of the system like this the when the moment you give the input, the output will slowly rise and will become close to the input.

But in dynamic response of the instrument it is possible that there is overshoot in the output and after certain oscillations the system gets settled. So, where regions are also involved in this type of system, that is why undamped natural frequency of the system is also taken into the account and undamped natural frequency is equal to ao by a 2, this ao divided by a 2 this is undamped natural frequency.

Now, in the simple suppose a system is vibrating in air, air will produce some damping effect right and not only air even a suppose a if shaft is rotating with a certain speed all of a sudden the power is stopped the shaft will also stop up rotating after certain period of type due to damping by the lubricating oil.

So, a damping ratio it is normally represented by zeta it is known as damping ratio. So, the unit of this is radians per second. Now, damping ratio has unit of Newton meter per meter per second. So, the damping ratio is defined as a 1 divide by 2 under root ao and a 2. Now, why I am doing this I will be I will be explaining you later on, because these 3 terms because this equation because a 2 a 1 a 2 ao and bo for any instrument they are known not known to us for any measuring instrument.

Let us say for example, voltmeter I do not know what is the value of a 2 a 1 ao bo for voltmeter, but definitely for a voltmeter I know the value of K sensitivity, w n also I can find out natural vibration frequency and damping ratio is also given for a for dampers the damping ratio is also given. So, these are normally known quantities.

So, now, we will replace will be putting these quantities here and the expression we are going to get. Now, omega n is equal to under root ao by a 2.

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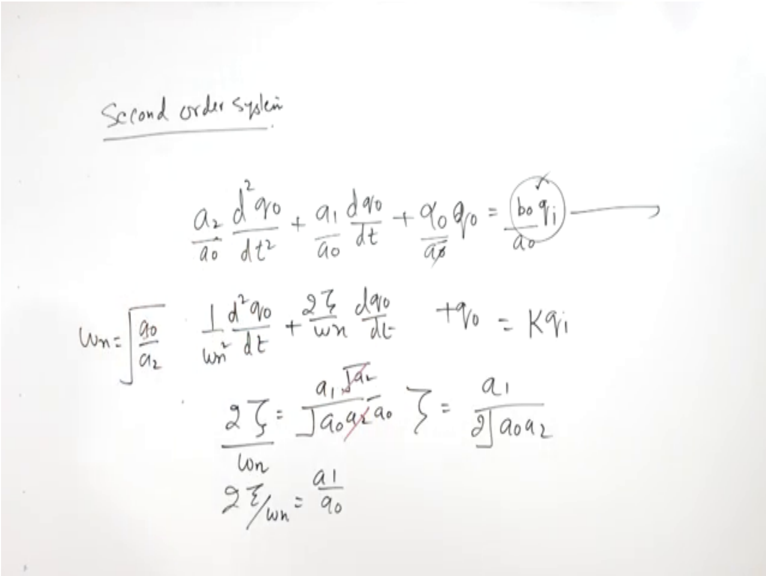

So, if we divide entire equation by ao. So, here we will be getting ao, here will be getting ao, here will be getting ao and here will be getting ao right. Now, this is omega n. So, we can write D square qo by dt square 1 by omega n square. This will take later on. Here bo by ao we can always say K q i we have done this for first order system also. Now, this ao by ao they will be cancelled out q. Now, a 1 by ao is remaining a 1 by ao.

Now, a 1 by ao and zeta we have defined as a 1 by 2 under root ao a 2 this is zeta. So, if we take 2 zeta then 2 zeta is going to be equal to a 1 by under root ao a 2. And if we divide this by omega n if we divide this by omega n, when we divide this by omega n omega n then it becomes ao and under root a 2. Now, these two will be cancelled out and then we will get the value of a 1 by ao. So, to 2 zeta by omega n is equal to a 1 by ao. So, here we will put plus 2 zeta by omega n D qo by dt. So, this equation has been modified by this equation.

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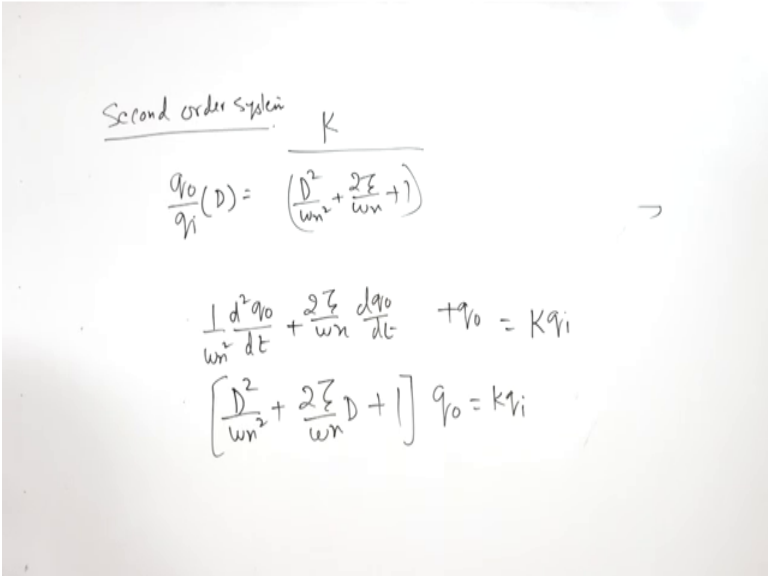

Now, we can write that D square by omega n square plus 2 zeta by omega n D plus 1 qo is equal to K q i right. And this becomes the transfer function because the transfer function is for transient response qo by qi D is equal to K by D square by omega n square plus 2 zeta by omega n plus 1. This is transfer function.

For example, some there is a spring balance I will I will give you an example of the spring weighing machine there is a weighing machine let us take one example. There is a weighing machine and it is a spring loaded weighing machine. So, there is a vertical scale right and there is a shaft, on this shaft there the platform and shaft is mounted over a spring. Normally, it is assumed that it is a weightless a spring if it is spring has weight comparable to the applied load then one third weight of the or mass of the spring is added right.

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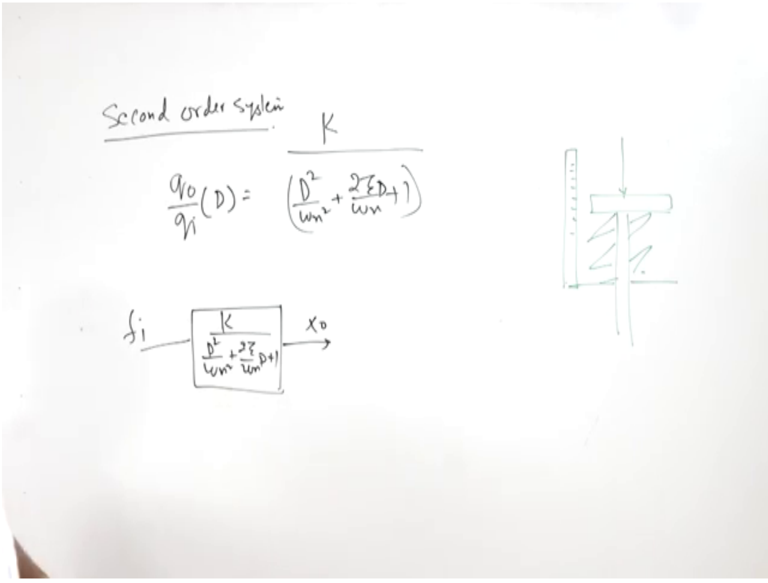

Now, this shaft has divisions though this vertical scale has divisions. So, when the force is applied the spring get compressed right and we get the output. So, input is, so input if it is a second order system the input is a fi input force this is transfer function K by D square by omega n square plus 2 zeta by omega s K and D plus 1 omega n D plus 1 [FL] right, and then it is xo is the output. So, now, if you know this transfer function we can immediately get the value of output for a given input.

So, this transfer function will be directly transfer. So, we will get the output for a given input. Now, will be start with the force balance here to for the second order analysis we will start with the force balance and f; this force has frequency well below or well above or sorry well below the natural frequency not well above well below the natural frequency. So, that there is no interference. So, there is no this f force has frequency well below well below the natural frequency of the instrument. So, frequency of f is well below the natural frequency of the instrument and spring is weightless and it is linear right.

Now, in this case if we do the force balance then force is equal to sigma force is equal to mass into acceleration.

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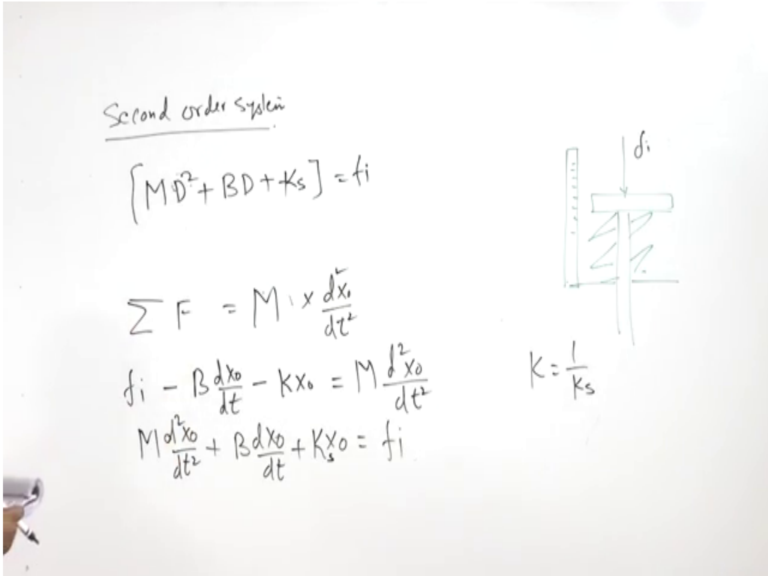

Now, sigma force sigma force is input force minus this here we have assumed that there is a air film otherwise these machines are well lubricated. So, damping of lubricating oil also becomes effective here. So, when the damping of lubricating oil becomes effective then it is B dx over dt right this is this will resist this damping of the lubricating oil will resist fi. So, it is opposite direction minus K xo spring force spring this is the sum of the forces is equal to this let us take capital M. So, total M M into D square xo by dt square. Now, we will rearrange these terms and we will get M D square xo by dt square plus B d xo by dt plus K xo is equal to fi input force.

Now, simply we will divide this by K stiffness of the spring will represented by s its stiffness of this spring because stiffness of the spring is. So, sensitive the instrument is 1 by Ks this is the sensitive of the instrument. So, further we can modify this equation as M D square plus BD plus 1 this is Ks. Now, we have not divided it by Ks it has to be like this. So, plus Ks is equal to fi. Now, this is the differential equation for a second order system for this type of system and. Now, we want to have the value of xo multiplied by this fi.

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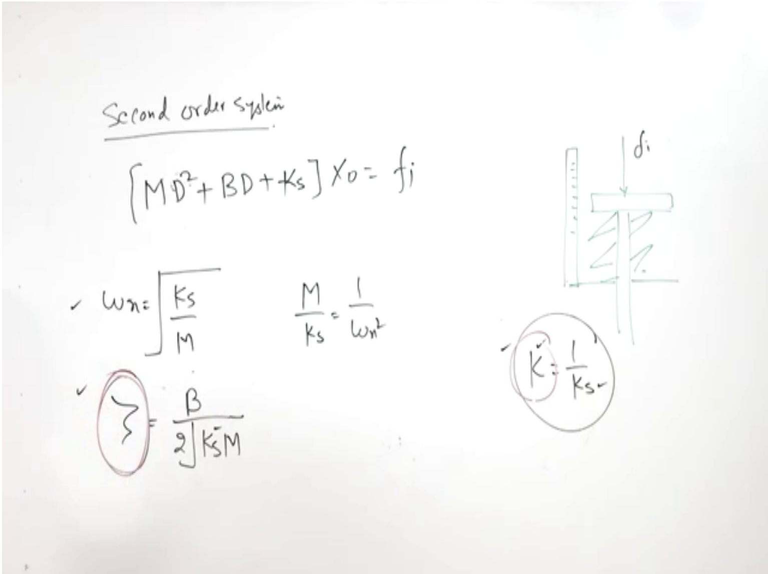

Now, from this equation if you compare with the original equation we can find that K static sensitivity K is 1 by Ks right and omega n natural frequency is under root Ks by M, Ks by M because where the entire equation is divided by Ks. So, this will become M by Ks M by Ks is equal to one by omega n square. So, from here we can get omega n is equal to Ks by M and similarly we can find the value of zeta s b by 2 under root Ks multiplied by M. So, we have got all these 3 and the static sensitivity this is the sensitivity of the instrument.

Now, if we have high spring stiffness, if we have high spring stiffness sensitivity will be low. Now, here there is a very good observation that if this is high the K will be low sensitivity will be low and here this is high damping ratio will also be low. If this is high this is low this is high this is also low it means if we increase the stiffness of the spring sensitivity of the instrument will reduce damping ratio will also reduce that reduction in damping ratio means the response will become quicker.

So, system will have quicker response if we have lower value of Ks, but at the same time we will have to sacrifice the sensitivity if you going to have very high order sensitivity then damping with increase and the response of the system will be sluggish. So, a trade off has to be struck between these two right. This is a particular case we have taken for a spring balance.

Now, we will start with a step response of second order system we generalized solution for the step response of second order system.

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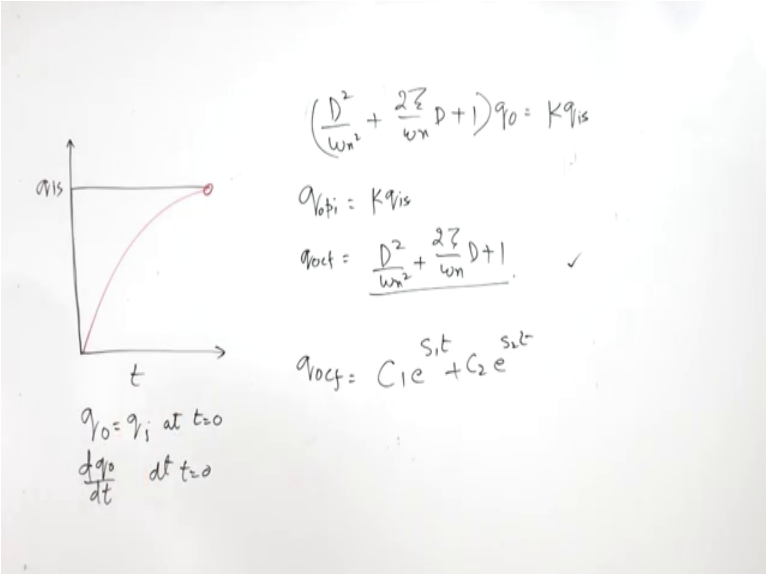

As you know that in a step input there is a change in the input from 0 to certain value within no time 0 duration of time and input is q i s and this is time dimension this is step input. And if you remember for the first order system we have drawn the characteristic curve for the response of the curve like this, and after certain time interval maybe 4 or 5 or 6 time constants of nearly 5 time constants this become almost equal to this right. So, then dynamic error is reduced to almost 0 right.

But here in this case the situation is different because here vibrations we have considered the natural frequency also and where there is a frequency of the damped vibration also the all those things will be discussed here. First we will be discussing here. First of all let us start with the basic governing equation for second order system.

So, the basic is D square by omega n square plus 2 zeta by omega n D plus 1 qo is equal to Kq is right boundary condition boundary condition is same qo is equal to q i at t is equal to 0 and sorry qo is equal to q i. And q i is equal to sorry and D qo upon dt rate of change in output is also 0 at t is equal to 0. So, at t is equal to 0 output is 0 and rate of change of output is also 0 these are two boundary conditions.

Now, here for finding out the solution first is particular integral and second is complementary functions, [FL] two things we have to drive. So, particular integral is yes particular integral is qo p is, qo p is, Kq is and qo complementary function. So, now, we will have to solve this differential equation. It means D square by omega n square plus 2 zeta by omega n D plus 1 you have to find roots of this.

Once we find the roots of this then we will take the solution qo complementary function is equal to, because D there are 3 possibilities or 4 possibilities. First is roots are unequal and real will start with the first one roots are not equal and they are real for that the solution is C 1 e raised to power S 1 t plus C 2 e raised to power S 2 t, right.

So, first of all we have to calculate the root for this equation and the root for this equation is, if you compare this with ax square plus bx plus c is equal to 0 in this equation the roots are roots are minus b by 2 a plus minus under root b square minus 4 a c divided by 2 a. So, if 1 root is plus and another root minus.

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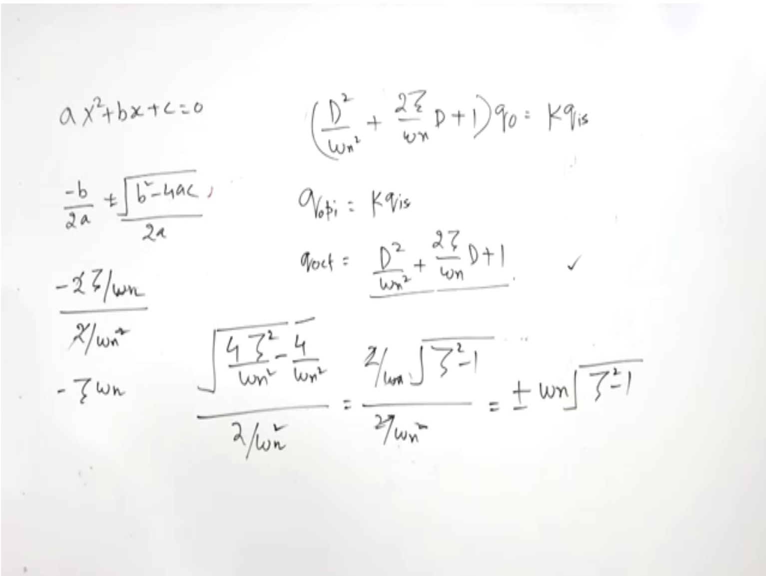

Now, here D is minus 2 zeta by omega n, minus 2 zeta by omega n minus 2 a, 2 a is 2 by omega n square. So, this omega n omega n will be cancelled out 2 will also be cancelled. So, it will be minus zeta omega n. So, this is minus b by minus b by 2 a. Now, the second part is b square minus 4 ac. So, it is going to be 4 zeta square by omega n square that is b minus 4 times ac that is again omega n square divided by 2 by omega n; this is 2 by omega n 2 a 2 by omega n square.

Now, here 2 by omega n square it is going to be further simplified as 2 by omega n square 2 by omega n divided by 2 by omega n square multiplied by zeta square minus 1. So, here 2, this 2 will be cancelled we will be cancelling with this one omega n with omega n 1 and then we will be getting plus minus omega n under root. So, the solution for this equation solution for this equation we are going to have, there are 2 roots, there are 2 roots.

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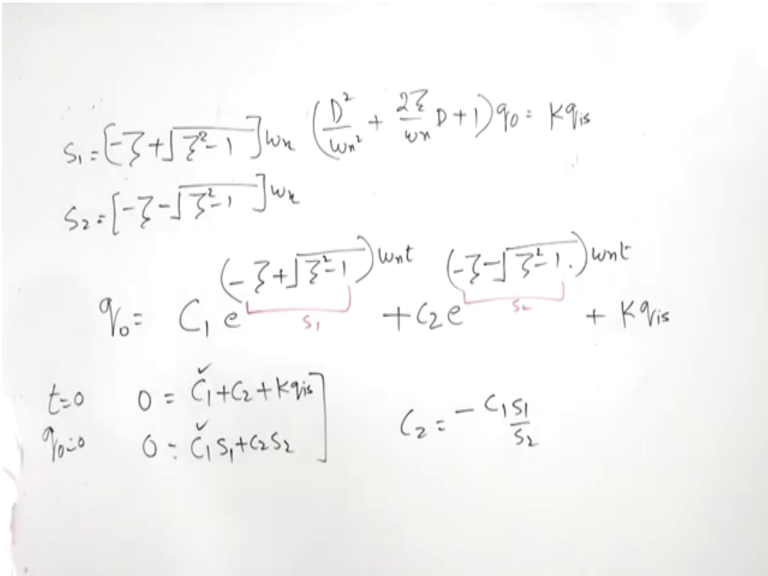

S 1 is equal to minus zeta plus under root zeta square minus 1 multiplied by omega n. The second route is S 2 is minus zeta minus under root zeta square minus 1 omega n, right. Now, we have two roots, solutions are not equal they are real zeta is greater than 1 this is the case when zeta is greater than one and then we get solution as q is equal to S 1 sorry not S 1, C 1 some constant C 1 e to power S t, S 1 t. So, it is going to be minus zeta plus under root zeta square minus 1 multiplied by omega n t and then we get again another one C 2 e raise to power minus zeta minus minus under root zeta square minus 1 multiplied by omega n t plus particular integral plus Kq is. So, this is the generalized.

Now, we will be putting the boundary condition. The boundary condition there are two boundary conditions. Now, the first boundary condition is at x is equal to 0 qo is equal to 0 this is qo at x is equal to 0 sorry, at t is equal to 0, at t is equal to 0 this is the boundary condition qo is equal to 0. So, we will be putting qo here 0 is equal to t is equal to 0 it means C 1 plus C 2 plus Kq is. Now, another condition is derivative of these if this is if you replace this by S 1 if we replace this by S 1 for the sake of convenience and this is S 2 right. So, if we differentiate this then we get 0 is equal to C 1 S 1 plus C 2 S 2, this is S 1 so when we differentiate this C 1 e raised to power S 1 we will get S 1 C 1 e raised to power S 1 and when we are putting 0 then this term will become 1. So, we are getting C 1 S 1 plus C 2 S 2.

Now, we have two simultaneous equations and if we solve these two simultaneous equations we will be getting the value of C 1 and C 2 right. So, here we can say that C 2 is equal to minus C 1 S 1 by S 2. So, this C 2 is equal to minus C 1, S by 2. So, we will be putting the value C 2 here and when you put the value of C 2 here, when we put the way to C 2 here, we get 0 is equal to C 1 minus C 1 S 1 by S 2 plus Kq is right or C 1 S 1 by S 2 minus C minus 1 is equal to Kq is right.

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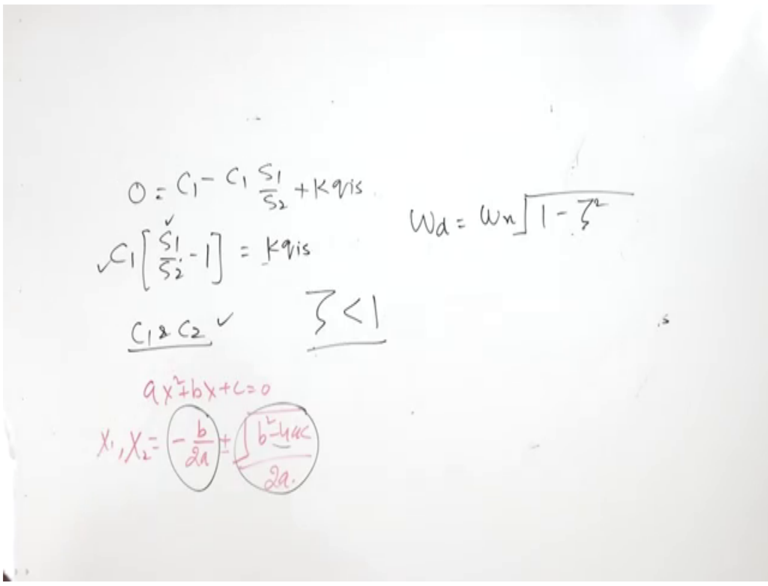

S 1 and S 2 are known to us, and then by putting the value of S 1 and S 2 we will get the value of C 1 and similarly we can get the value of C 2. So, this is how we can get the value of C 1 and C 2 by putting the boundary conditions. When in the original equation the boundary conditions are put we get solution like this we say that a system is over damped when the roots are real and unrepeated and in over damped system this is going to be the governing equation.

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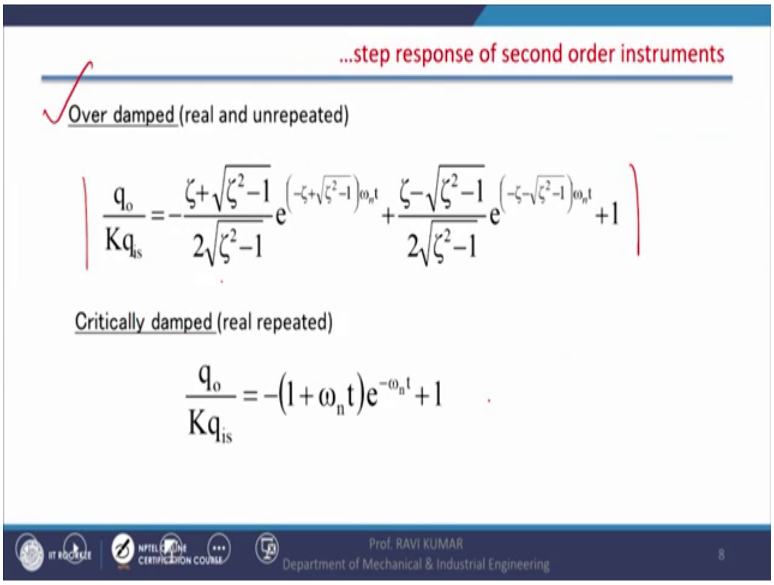

Another type of solution can be the roots are real and equal. Say for a equation ax square plus bx plus c is equal to 0. If I say the roots are real and equal and roots generalized solution for the roots is x 1 and x 2 is minus b by 2 a plus under root b square minus 4 ac by 2 a. So, here in this case in such type of solution if I want to have roots equal roots this has to be 0. So, we are going to have minus b by 2 n.

So, the critically damped system the value of zeta is 1. So, that is a critical damp system and the value of input and output ratio K qo by Kq is can be expressed by this equation.

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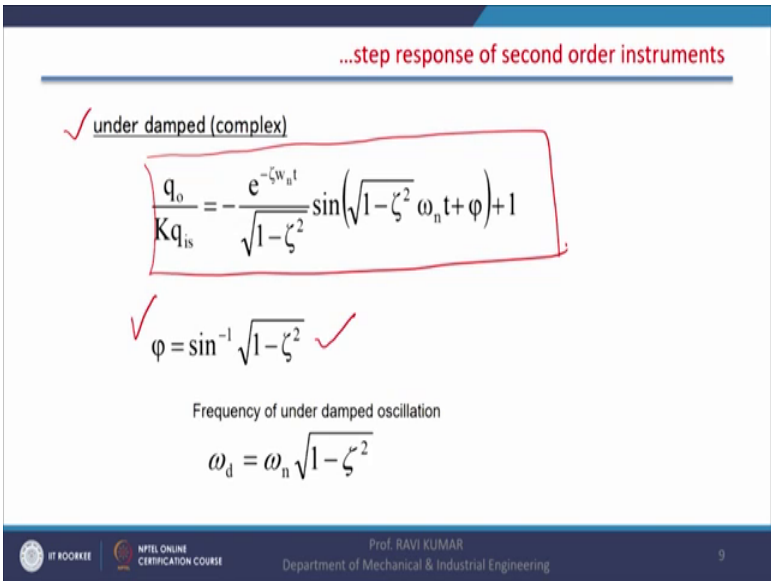

Now, the third one is under damped system when the roots are complex, I mean zeta is less than 1, this zeta is less than 1. When zeta is less than 1 then we are doing to get complex expression in the, so in that case under damped system the final expression will be like this K before K by q is and for the value of phi it is sine inverse 1 minus sorry. So, phi is sine inverse under root 1 minus zeta square or it is cos inverse zeta both are same.

And frequency of under damped vibrations, because when the system is under damped the system will vibrate for certain time period and this frequency of damping frequency of the second order system which is under damped is natural frequency , but multiplied by 1 minus zeta square right. So, now we have discussed all 3 systems I mean over damped system, under damped system and critical damped system.

The further discussions on these system will have in the next lecture, that is all for today.

Thank you very much.