**MECHANICS**

**Prof. Anjani Kumar Tiwari**

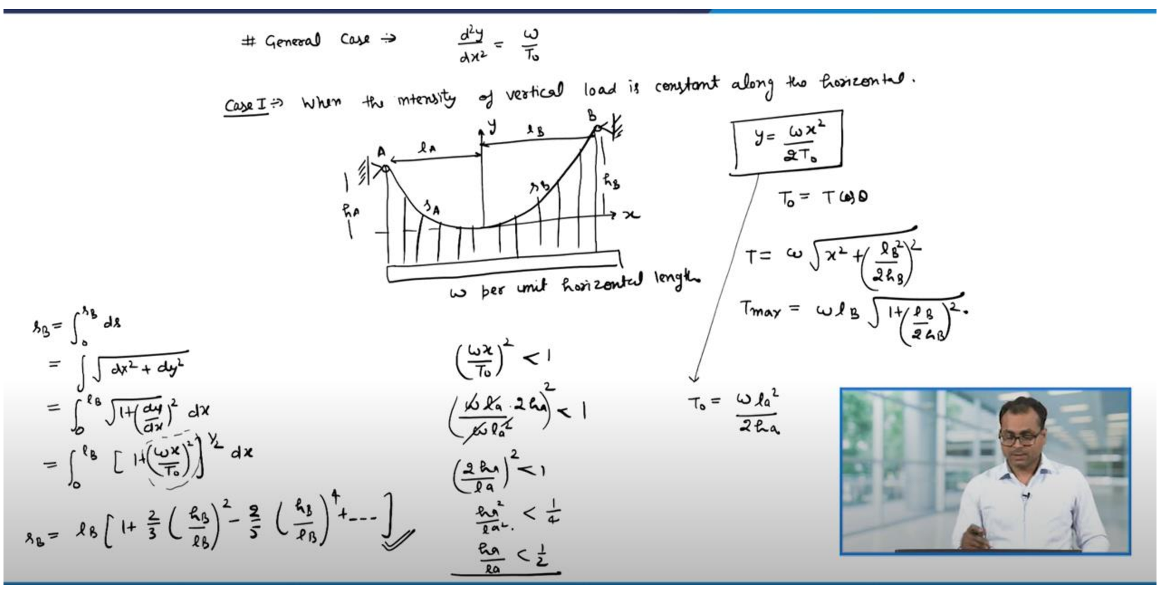
**Department of Physics**

**Indian Institute of Technology, Roorkee**

**Lecture 12**

**Flexible Cable, Catenary curve**

Hello everyone, welcome to the lecture again. In the last class, we look at cable, particularly flexible cable and we saw that when the load is uniformly distributed along the horizontal line, then the equation of the cable comes out to be the equation of the parabola.




So, let me summarize those results first and then we will look at the case wherein the weight is uniformly distributed along the cable and in that case you will see that the equation of the cable comes out to be catenary. So, first let me summarize the result of the last lecture. We discussed the general case and there we saw that the equation is $\frac{d^2y}{dx^2} = w/T_0$ . We also saw that when the intensity of vertical load is constant along the horizontal so, for example, you have a cable and with this cable, a uniform weight is suspended. So, this was the case and it has a weight w per unit horizontal length. So, in this case, the equation of the cable becomes $y = wx^2/2T_0$, where this $T_0$ was the horizontal component of the tension in the cable and it was $Tcos\theta$.

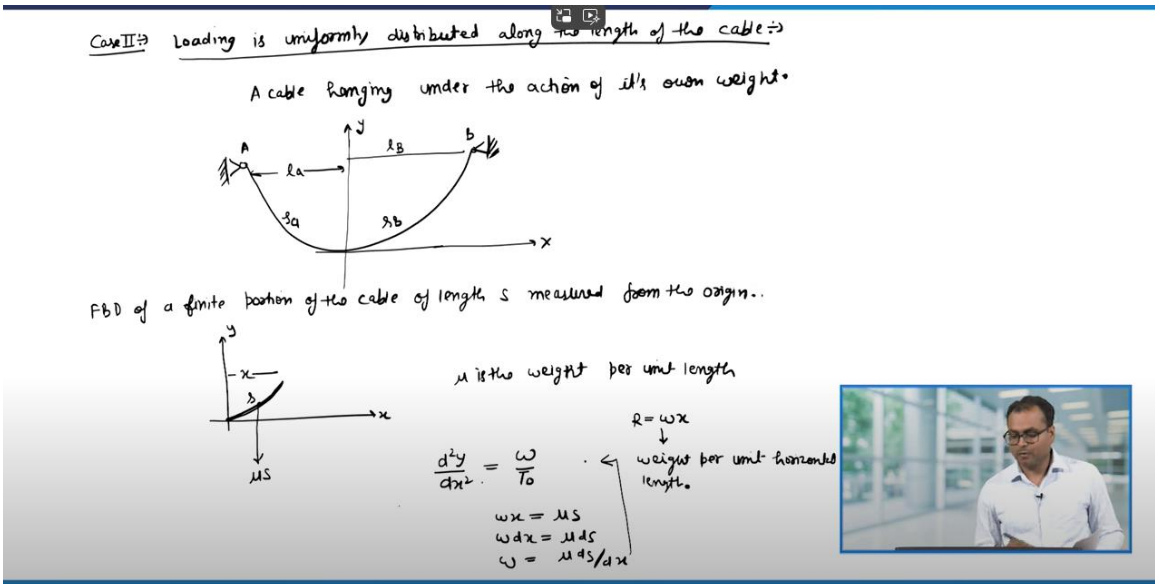
Note that here we fix our axis at the bottom of this cable. So, this was our x axis and this one was the y axis. We also saw that the equation of T becomes $w\sqrt{x^2 + \left(\frac{l_B^2}{2h_B}\right)^2}$ wherein this point was B, this point was A, this length was $l_A$, this length was $l_B$, this height was $h_B$ and this height was $h_A$.

Now, this was $s_B$ and this was $s_A$. Now, to maximize the tension, so we find out where the tension is maxima. So, it will be maxima when x is equal to $l_B$ that is the maximum x that is possible. So, we saw that $T_{max}$ was $wl_B\sqrt{1 + \left(\frac{l_B}{2h_B}\right)^2}$.

Now, we have also calculated the length of the cable and for that we find out $s_B$ and $s_B$ was 0 to $s_B$ $ds$ and this was $\sqrt{dx^2 + dy^2}$ which can be rewritten as 0 to $l_B$, $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ and then we put the value of $dy/dx$. We got 0 to $l_B$ ,$\left[1 + \left(\frac{wx}{T_0}\right)^2\right]^2 dx$ and then to find out this integral, we use the Taylor expansion and when we did the Taylor expansion, then we got $s_B$ equal to $l_B[1 + \frac{2}{3}\left(\frac{h_B}{l_B}\right)^2 - \frac{2}{5}\left(\frac{h_B}{l_B}\right)^4 + \cdots]$ Note that we can use the Taylor expansion only if this quantity is less than 1.

That is my $\left(\frac{wx}{T_0}\right)^2$ whole square should be less than 1. Now, the maximum value of x can be $l_A$. So, therefore, it will be $wl_A$. Now, $T_0$ I can find out from here at the boundary. So, my $T_0$ will be $wl_A^2/2h_A$. So, I put $y = h_A$ and $x = l_A$. So, let us put this $T_0$ over here. So, we get $wl_A^2\ 2h_A$, this should be less than 1 and of course, there is a square.

So, $w$ will get cancelled, $l_A$ will get cancelled with the square and we get $\left(\frac{2h_A}{l_A}\right)^2$ should be less than 1, ok. This implies that $h_A^2/l_A^2$ should be less than then $1/4$ or $h_A/l_A$ should be less than $1/2$. So, only if this condition is satisfied, then we can use this formula to find out the length of the cable.

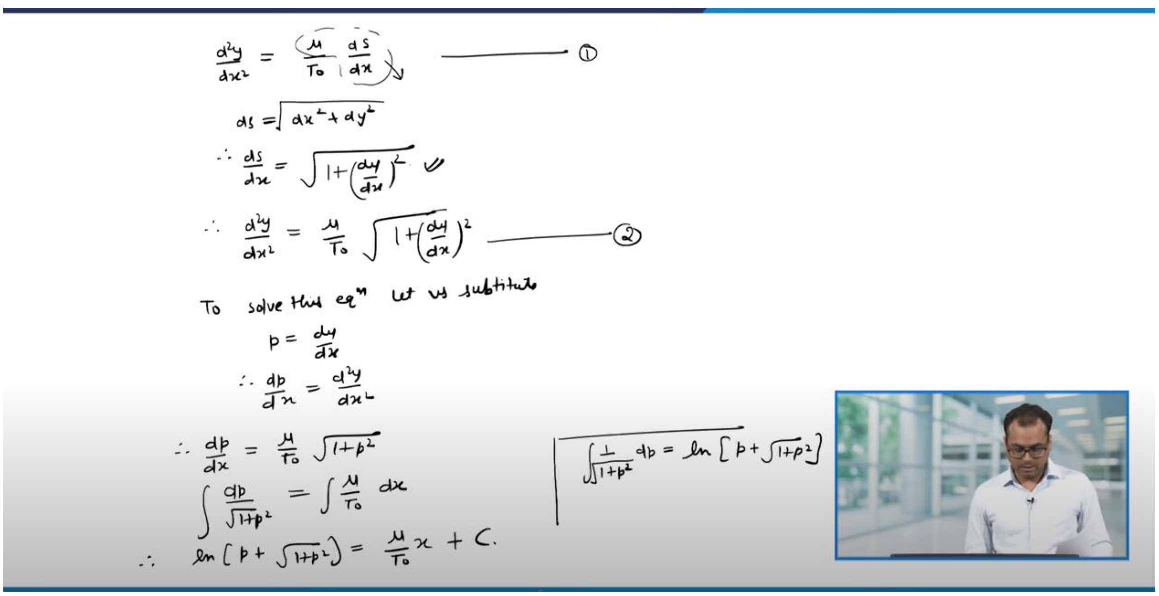



Now, let us discuss the second case where the loading is uniformly distributed along the length of the cable. So, this is case second. Here, the loading is uniformly distributed along the length of the cable. For example, you can think of a cable which is hanging under the action of its own weight. So, let us say I have a cable and this cable as I said is hanging under its own weight.

So, let us say this point is A, this point is B and again let us put the coordinate. At the lowest point of the cable. So, this is my x-axis, this is the y-axis, let us say this length is $s_B$, this length is $s_A$, this length is $l_B$ and this length is $l_A$. To analyze this cable, let us look at the free body diagram of a finite portion of the cable measured from the origin.

So, we are going to look at the free body diagram of a finite portion of the cable of length s measured from the origin. So, we have the x-axis, the y-axis and a small portion of the cable of length s. Now, if mu is the weight per unit length of the cable, then its weight will be $\mu s$ and it will act at the center. Now, to start this, recall the general equation of the cable and it was $\frac{d^2y}{dx^2} = w/T_0$.

This was our starting point. Now, in the previous case, we said well, let us say the length in the horizontal direction is x and In that case, the total load was $R = wx$ where this $w$ was weight or the load per unit horizontal axis. So, this was the weight per unit horizontal length. So, the only difference will be earlier it was $w_x$, now it is $\mu s$ and since $\mu$ and $w$ are constant, we can write down $wdx = \mu ds$ or $w = \mu ds/dx$.

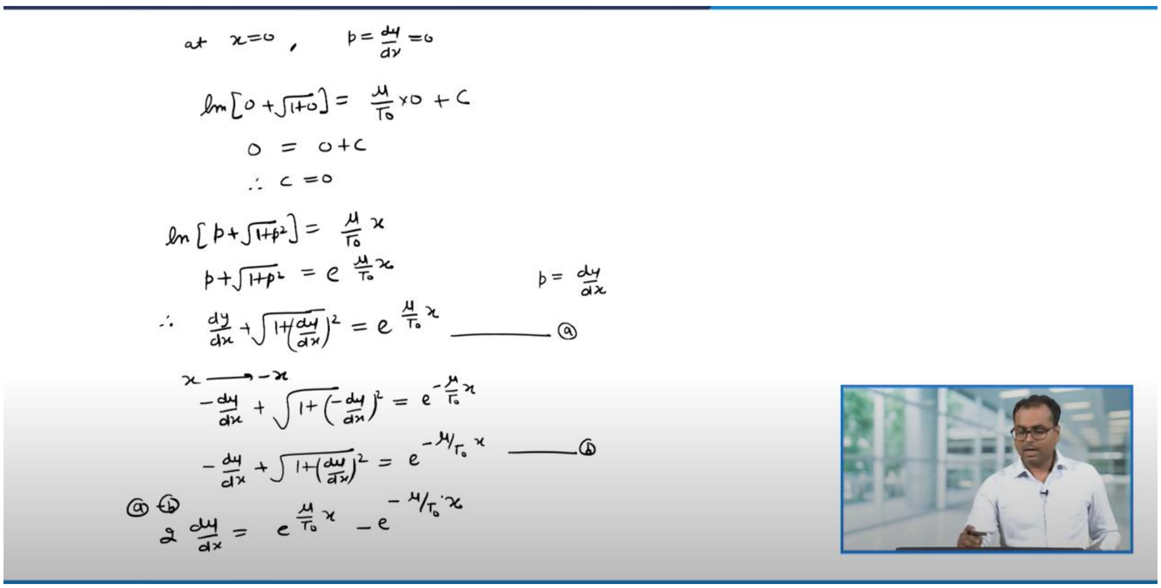



Let us put it in the above equation and analyze the equation. So, we have $\frac{d^2y}{dx^2} = \frac{\mu}{T_0}(\frac{ds}{dx})$ , wherein we have put $w = \mu ds/T_0$. Now, this d s is of course, d x square plus d y square square root.

Therefore, d s over d x will be $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$. Therefore, I can write down $\frac{d^2y}{dx^2}$ equal to $\mu/T_0$. So, I just put the value of $ds/dx$ from here. So, it will be $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$.

Let us call this equation as equation number 2 and this equation as equation number 1. Now, this is a differential equation in x and y and we can solve this equation if we substitute $dy/dx$ equal to some $p$. So, to solve this equation, let us substitute $p = dy/dx$. Therefore, $dp/dx$ will be $d^2y/dx^2$.

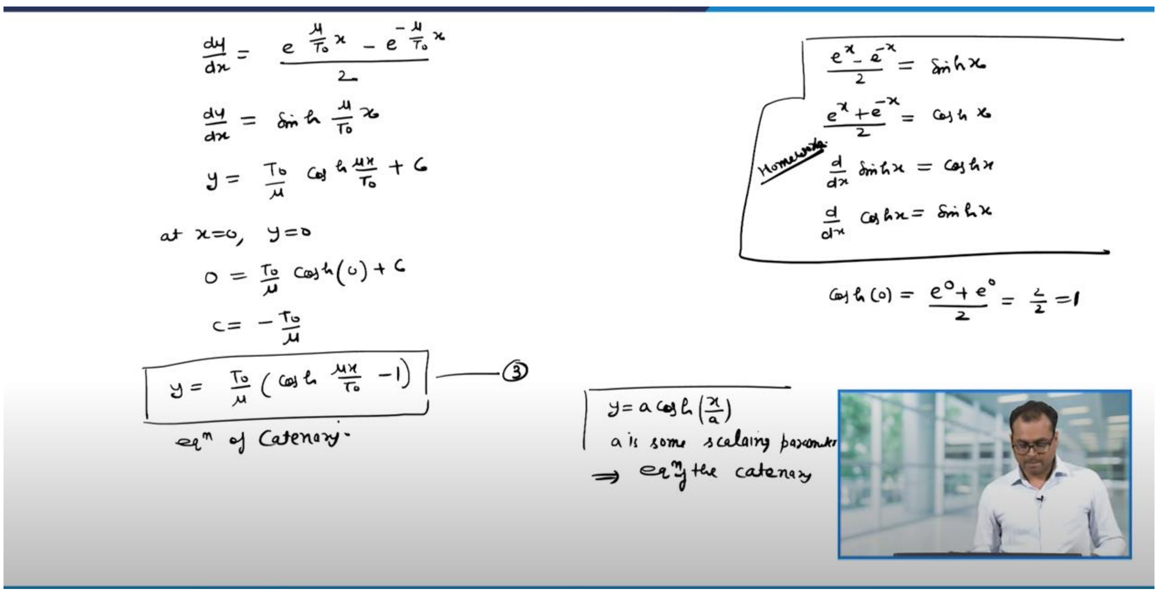
And let us put it above. So, we have $\frac{dp}{dx} = \frac{\mu}{T_0}\sqrt{1+p^2}$. Now, we can solve this equation using variable separable method. So, we have $\frac{dp}{\sqrt{1+p^2}} = \frac{\mu}{T_0}dx$. Now, this is a standard integral, integral $\frac{1}{1+p^2}dp = \ln[p + \sqrt{1+p^2}$. Therefore, this can be written as $lnp + \sqrt{1+p^2} = \frac{\mu}{T_0}x + C$.



Now, to determine C, let us look at the boundary condition. So, you can see here at $x = 0$, my $dy/dx$ or the slope is also 0 and $\frac{dy}{dx} = p$. Therefore, $p$ will be 0. Let us use that to find out $C$. So, at $x = 0$, we have $p$ which is nothing but $dy/dx$ that will be 0.

Now, we have this equation $lnp + \sqrt{1+p^2}$. So, we have $lnp$ is $0 + \sqrt{1+0}$ equal to we have $\mu/T_0$ and then $x$, $x$ is $0 + C$. So, $ln1$ is nothing but 0 equal to $0 + C$. Therefore, $C$ will be 0. So, therefore, the equation becomes $lnp + \sqrt{1+p^2} = \frac{\mu}{T_0}x$. Now, this equation I can rewrite as $p + \sqrt{1+p^2} = e^{\frac{\mu}{T_0}x}$. Now, remember my $p$ was $dy/dx$. Therefore, Let us change the variable again.

So, we have $\frac{dy}{dx} + a^2\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = e^{\frac{\mu}{T_0}x}$. Let us call this equation number a. Now, in this equation, let us replace $x$ by $-x$. So, suppose I replace $x$ with $-x$, then I have $-\frac{dy}{dx} + \sqrt{1 + \left(-\frac{dy}{dx}\right)^2} = e^{-\frac{\mu}{T_0}x}$ or I can rewrite this as $-\frac{dy}{dx}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = e^{-\frac{\mu}{T_0}x}$. Let us call it equation number b. Now, to find out $dy/dx$, let us do $a - b$. So, we have $\frac{2dy}{dx} = e^{\frac{\mu}{T_0}x} - e^{-\frac{\mu}{T_0}x}$.

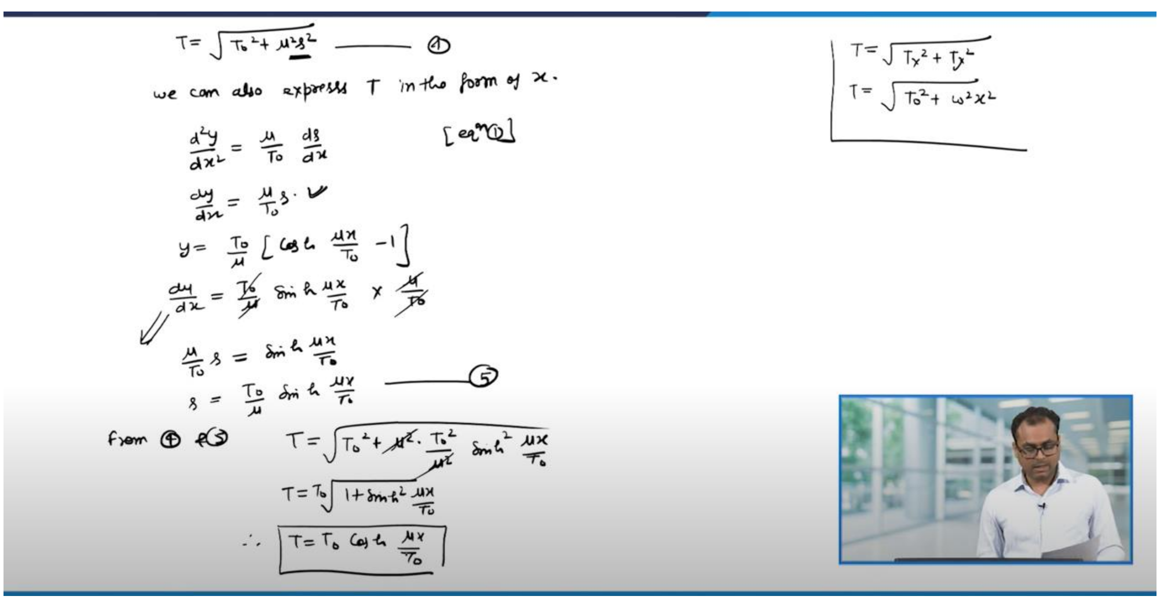



Or i can write down $\frac{dy}{dx}$ as $\frac{e^{\frac{\mu}{T_0}x} - e^{-\frac{\mu}{T_0}x}}{2}$ Now, this is in the form of $\frac{e^{x} - e^{-x}}{2}$ and this function is called the $sinh$ function.

So, this is nothing but $sinhx$. Similarly, we can have the function e to the power $\frac{e^{x} - e^{-x}}{2}$ and this is the definition of $coshx$. Now, you can check by yourself that if you do $\frac{d}{dx}$ of $sinhx$, then you get $coshx$. This you can take as a homework and if you do $dy/dx$, $coshx$ then you get $sinhx$. So, I can write down this equation as $\frac{dy}{dx} = \sinh(\frac{\mu}{T_0})x$. Now, to find out y, because we are interested in finding out the equation of the cable, I can again integrate this. So, we have $y = \frac{T_0}{\mu}\cosh\left(\frac{\mu x}{T_0}\right) + C$, where again $C$ is the constant of the integral. Now, to find out $C$, again let us look at the boundary condition.

So, at $a\ x = 0$, my y was also 0 because we have placed the coordinate at the minimum point of the cable. So, we have $y = 0 = \frac{T_0}{\mu}(coshx)$ is $0 + C$. Now, you can see here that $\cosh(0)$ is 1 because $\frac{e^0+e^0}{2}$ will give you 2 divided by 2 equal to 1. So, we get $C = -T_0/\mu$.

Let us put it above. So, we got the equation of the cable as$\frac{T_0}{\mu}\cosh(\frac{\mu x}{T_0} - 1)$. Let us call this equation number 3. Now, the equation in the following form $y = acosh(\frac{x}{a})$ where $a$ is some scaling factor or scaling parameter is the equation of catenary. So, here you can see that this equation is in the form of the catenary and therefore, a cable which is hanging under its own weight will have the equation of the catenary.



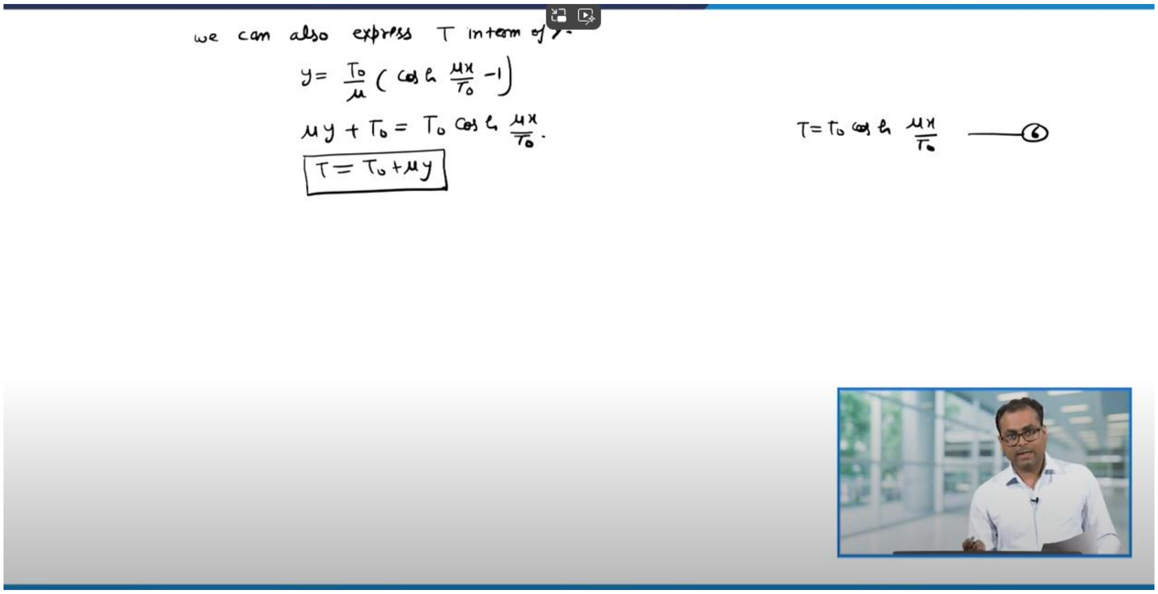

Now, let us find out the tension in the cable. When we calculate the tension, we said $T = T_x^2 + T_y^2$ and in the previous case, it was $T_x$ was constant because it was along the horizontal direction. So, it was $T_0^2$ and $T_y$ was $w^2x^2$. So, same as earlier, in this case, $T$ will be $\sqrt{T_0^2 + \mu^2 s^2}$.

Here, $\mu$ is the load per unit length along the cable. So, let us call it equation number 4. Now, here the tension is in the form of x. We can express this tension also in the form of x. So, let us express the tension T in the form of x. For that, first let us use equation number 1. So, equation number 1 was this.

This was our starting point. So $\frac{d^2y}{dx^2} = \frac{\mu}{T_0}(\frac{ds}{dx})$ . So, let us use $\frac{d^2y}{dx^2} = \frac{\mu}{T_0}\left(\frac{ds}{dx}\right)$. This was our equation number 1.

From here, I can see that $\frac{dy}{dx}$ is equal to $\frac{\mu s}{T_0}$. Now, this $\frac{dy}{dx}$, we can find out using the equation of the catenary. So, note that my $y$ was $\frac{T_0}{\mu}[\cosh\left(\frac{\mu x}{T_0}\right) - 1]$. And from here, I can find out what is $\frac{dy}{dx}$. It will be $T_0/\mu$.

$Cosh(\frac{\mu x}{T_0})$, its differentiation will be $\sinh(\frac{\mu x}{T_0})$ and multiply by the $\mu/T_0$. So, $T_0$ and $\mu$ will get cancelled. And $dy/dx$, just now we have find out it is $\frac{\mu}{T_0}s = \sinh\left(\frac{\mu x}{T_0}\right)$. And from here, I get $s = \frac{T_0}{\mu}\sinh(\frac{\mu x}{T_0})$.Let us call it equation number 5. So, from equation number 4 and 5, we can write down $T = \sqrt{T^2 + \mu^2}$ and $s^2$ we can put from 5, it will be $\frac{T_0^2}{\mu^2}\sinh^2\left(\frac{\mu x}{T_0}\right)$ . Now, this $\mu$ will get cancelled and we will have $T = T_0\sqrt{1 + \sinh^2\left(\frac{\mu x}{T_0}\right)}$ or $T = T_0\cosh(\frac{\mu x}{T_0})$. So, equation number 4, which was earlier in T and $s$, now we are able to write this equation in terms of T and x. We can also express this tension T in terms of y.




So, let us express T in terms of y. For that, I have to use equation number 3. We had $y = \frac{T_0}{\mu}(\cosh\left(\frac{\mu x}{T_0}\right) - 1)$. So, let us write this down $y = \frac{T_0}{\mu}(\cosh\left(\frac{\mu x}{T_0}\right) - 1)$. This I can write down as $\mu y + T_0 = T_0 \cosh(\frac{\mu x}{T_0})$. and look at equation number 6. So, we had $T = T_0 \cosh(\frac{\mu x}{T_0})$. This was our equation number 6. So, therefore, I can write down $T = T_0 + \mu y$. So, we are able to write down the tension in terms of x, in terms of y and also in terms of s. So, this was all about the cable which were hanging under its own weight. Thank you.