MECHANICS

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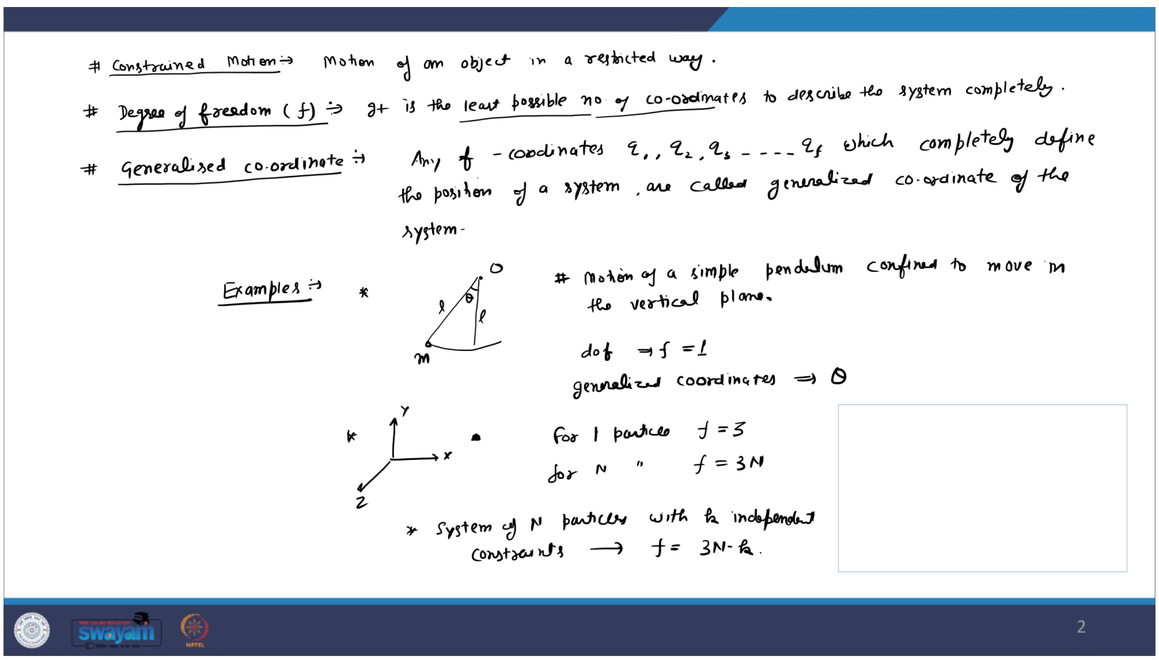
Lecture 19

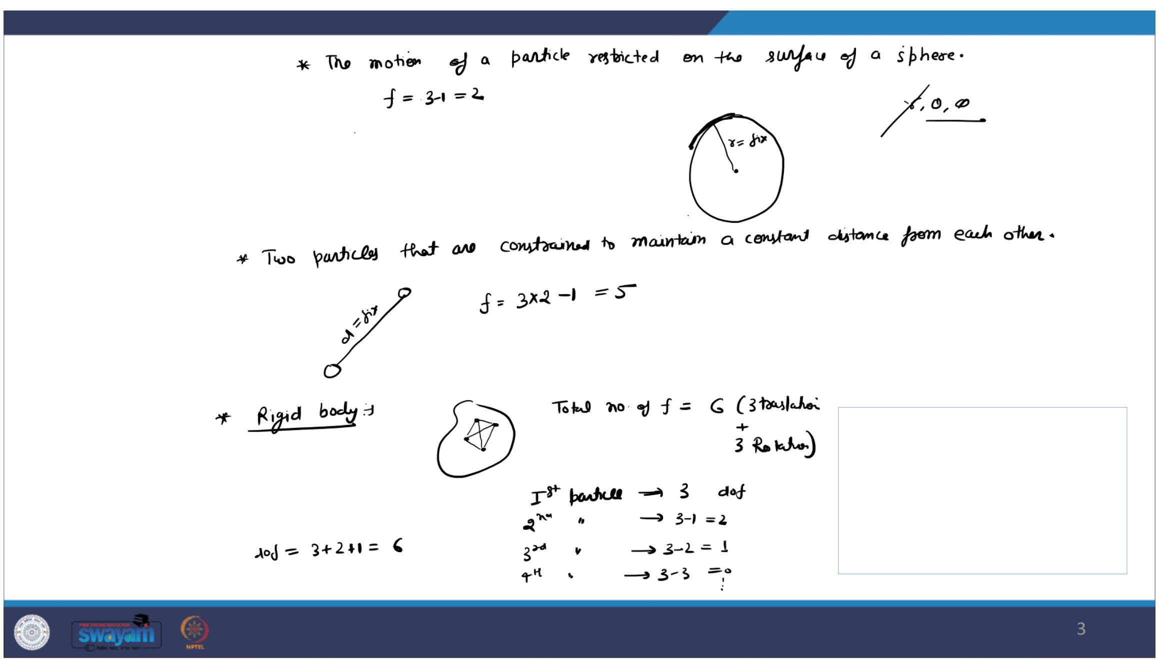
Principle of virtual work

Hello everyone, welcome to the lecture again. Today, we are going to discuss the principle of virtual work. Principle of virtual work is required to study the equilibrium of a particle or of a rigid body or the connecting rigid bodies. Now, to develop the principle of virtual work, we need some basic concept like what is the constrained motion, what is degree of freedom, generalized coordinate, virtual work, virtual displacement and so on.



So, let us first discuss what is a constrained motion. So, suppose I have an object and this object is restricted to move in certain direction. So, these are the example of the constrained motion. So, by definition, this is the motion of an object in a restricted way. We will look at this example, but as of now take this definition. Now, another concept that is required is the degree of freedom. So, generally this degree of freedom is denoted by f and it is the least possible number of coordinate to describe the system completely.

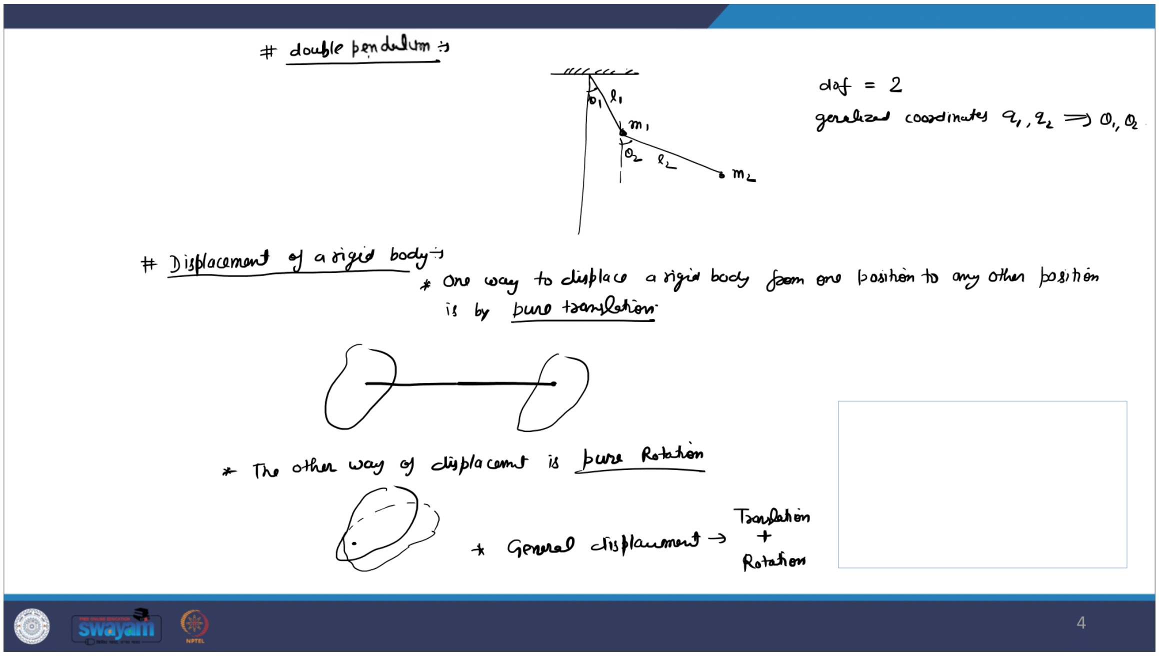
So, for example, if you have a particle and let's say that particle is moving in one dimension, in that case, the degree of freedom of the particle is 1. We will again look at, you know, the examples of, you know, various degrees of freedom. Now, let us look at generalized coordinate. So, any f coordinate. Let's say $q_1, q_2, q_3 \ldots\ldots q_f$ which completely define the position of a system okay and here note that f is the degree of freedom, are called generalized coordinate of the system. Now, with this definition, let us look at the example. So, let's say I have a simple pendulum and let's say this pendulum is suspended from point O and the length of the string is l. So, this length will also be l and let's say the mass of the particle that is attached to it is m. In that case, so this is just the example of the motion of a simple pendulum which is confined to move in the vertical plane. So, here you can see that just by one coordinate which is θ, I can define where the particle is because l is fixed. So, therefore, its degree of freedom or DOF which we also denote by $f = 1$. Again, why? Because just by one coordinate θ, I can define the position of the particle. And the generalized coordinate is θ. Note that it is not x and y because then you require two coordinate system x and y, but here I am saying just by one θ you can define the position of the particle. So, again it is the least possible number of coordinates. So, θ is 1. So, you know this is the least number of possible coordinate. So, therefore, degree of freedom is 1 and the generalized coordinate is θ. Now, let's say I have a particle which is free to move in three dimension. So, let's say this is x-axis, y-axis and z-axis and I have a particle here. So, for one particle, the degree of freedom will be 3. And the generalized coordinate can be x, y or z or in cylindrical coordinate, they can be r, ϕ, z or in spherical coordinate, they can be r, θ, ϕ. But the degree of freedom will be 3 because it can move in 3 dimensions.



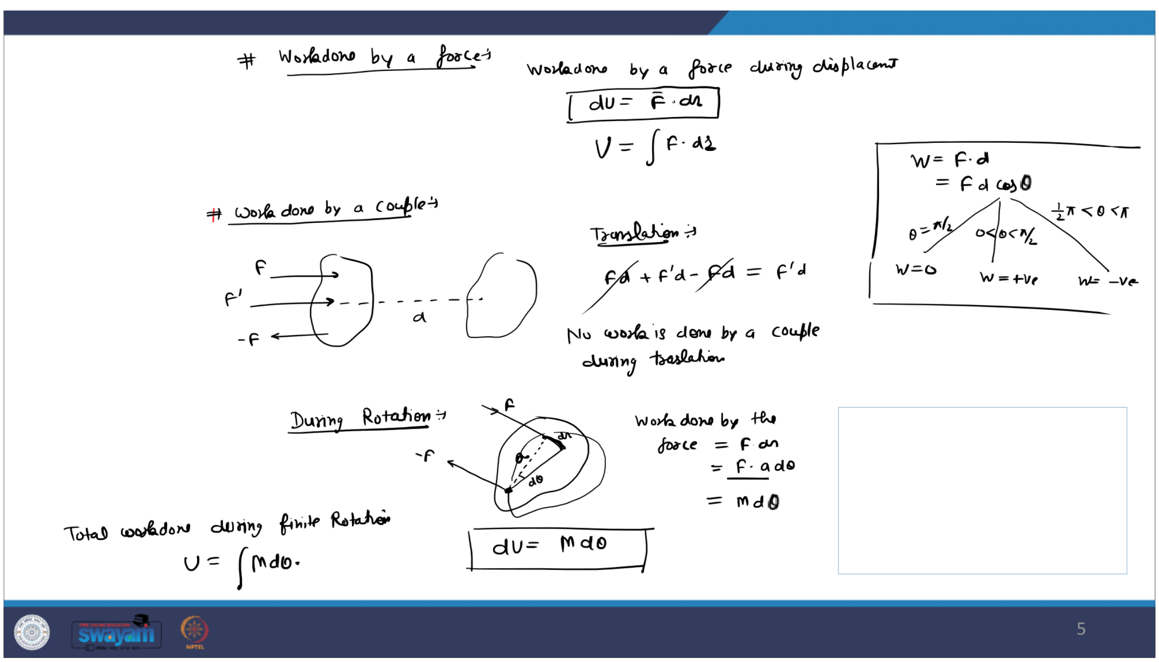
Similarly, if there are N particles, in that case, the degree of freedom will be $3N$. Now, if the system of N particles has k independent constraints, then the degree of freedom will be $3N - k$ because in k ways it is restricted.

So, for example, let us look at the motion of a particle which is restricted on the surface of a sphere. So, for example, let's say this is the sphere and a particle can only move on the surface of the sphere. So, in this case, its distance from the centre of the sphere r, this has to be fixed. So, this is one constraint. And for a particle which is free to move, degree of freedom is 3 and there is one constraint. Therefore, in this case, the degree of freedom will be $3 - 1 = 2$. And what are the generalized coordinates? Well, we can take because of the spherical symmetry, we can take r, θ and ϕ. But since r is fixed, so therefore, θ and ϕ will be the generalized coordinates. Now, let us look at the example of two particles that are constrained to maintain a constant distance from each other. So, for example, this is one particle, this is another particle and in between the distance is fixed. So, in this case, the degree of freedom $f = 3 \times 2 - 1 = 5$ constraint. So, it will be 5. Now, let us look at a rigid body. Now for the rigid body, a rigid body can move in the x, y and z direction and it can also rotate along x, y and z direction. Therefore, total number of degree of freedom will be 6. So, there are 3 translations plus 3 rotations. You can also think of it like following. So, let's say in the rigid body, you have n particle. So, let us consider the first particle. So, for the first particle, which is independent to move, we have 3 degree of freedom. Now, for the second particle, so let us take another particle, for this also, we have 3 degree of freedom, but then it has to maintain a constant distance between them. So, therefore, for this, the degree of freedom will be $3 - 1 = 2$. Now, let us look at the third particle. In this case, the degree of freedom will be 3 minus. So, it has to maintain a constant distance from this and a constant distance from this also. So, $3 - 2 = 1$. Now, you take the fourth particle. So, in this case, we have three coordinates and then it has to maintain a constant distance from all three particles. So, therefore, it will be 0 and subsequently for all the particles, it will be 0. So, total number of degree of freedom for the rigid body will be $3 + 2 + 1 = 6$ again.

Now, let us look at another example, which is let's say double pendulum. So, let's say this double pendulum. So, one arm of this has a length l_1 and mass let's say m_1 is attached to it and then you have another arm of length l_2 and mass m_2 is attached to it. Now, to define this system or the position of mass m_1 and m_2, I need θ_1 and I need θ_2. Therefore, its degree of freedom will be 2 and the generalized coordinate will be θ_1 and θ_2. So, my q_1 and q_2 which are generalized coordinate in this case will be θ_1 and θ_2.



Now, let us look at the displacement of the rigid body. So, suppose I have a rigid body and how can I displace it? So, one way to displace a rigid body, let's say from one position to any other position is by pure translation. So, what do I mean by that? Suppose I have a rigid body and you want to displace it. So, we can just do pure translation. Rotation is not involved. The other way of displacement is by pure rotation. So, again you have a rigid body and let's say you fix some axis on it and then you rotate the rigid body about that



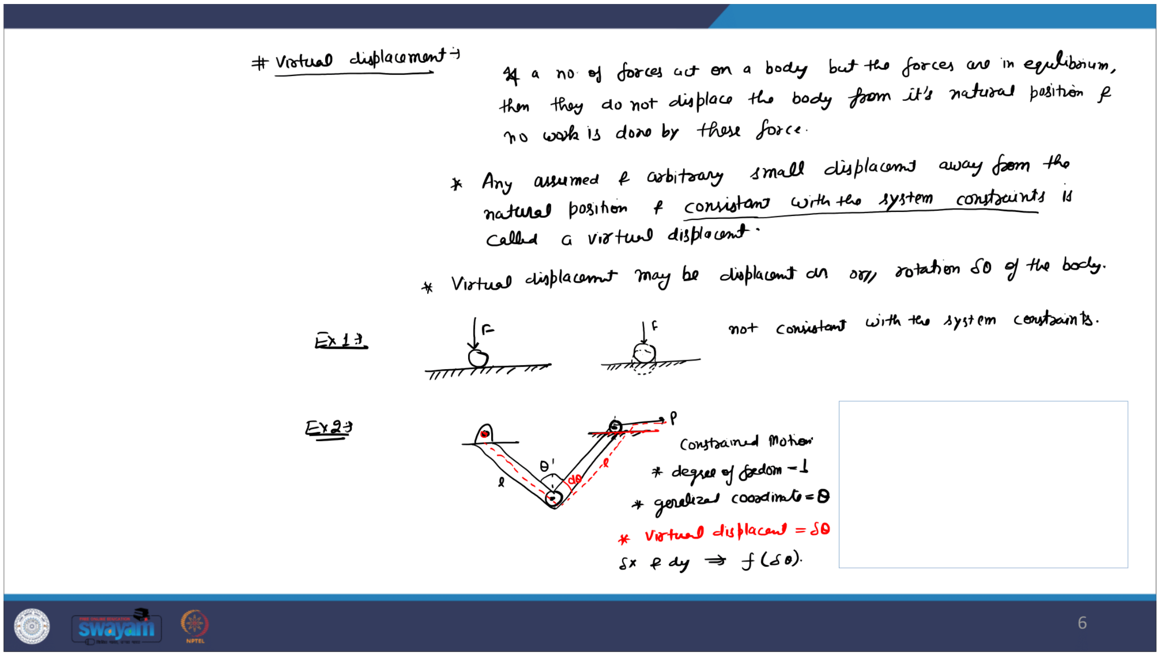
fixed axis. But in general, the general displacement is both translation and rotation. So, this is translation plus rotation.

Now let us look at the work done by a force. So, all these concepts are required to us. So let us look at the work done by a force. So, we know that the work done by a force during displacement is $dU = F.dr$. And if you want to calculate the total work, so, $U = \int F.dr$ and you will get the total work. Now, we also know from the high school that $W = F.d$. This can be written as $Fdcos\theta$. And now, if your θ is let's say 90°, so that means you are applying a force which is perpendicular to the direction of the motion. In that case, the work done will be 0. If your is between $0 < \theta < \pi/2$ in that case work done is positive and if your $\theta > \pi/2$ and let's say $\theta < \pi$ in that case your work done will be negative.

Now, let us look at the work done by a couple. So, let's say I have a rigid body and on this rigid body, a couple is acting. So, let's say F and $-F$ and let's say there is one more force. So, let us call it F' and because of this, let's say you have the translation. So, we are discussing the translation. So, now let us look at the total work done by these forces. So, total work done will be $Fd + F'd - Fd$ and this comes out to be $F'd$. So, you can see that no work is done by a couple during translation because the effect of Fd and $-Fd$ got cancelled.

Now, let us look at the work done by the couple during rotation. So, we have a rigid body and on this rigid body, let's say a couple is acting. So, you have $-F$ and you have $+F$ and let's say the distance between them is a and because of this, the body is going to rotate. So, let us fix our axis at $-F$ and this body rotates. Okay. So, let's say it rotates by a angle $d\theta$. So, therefore, this will be dr and in this case, the work done by the force will be $F.dr$ because you have this force and the body is displaced by dr. This force is not going to work because I have passed the axis through this. So, therefore, for this point dr will be 0. So, it will be $F.dr$ and this will be $F.dr = F.ad\theta$. Now, you also know that force into the distance between them. So, one of the force and the distance between them is the moment. So, therefore, this becomes $Md\theta$. Therefore, during rotation, the work done by the couple will be $dU = Md\theta$. Now, if you have finite rotation, so total work done during finite rotation will be $U = \int Md\theta$.

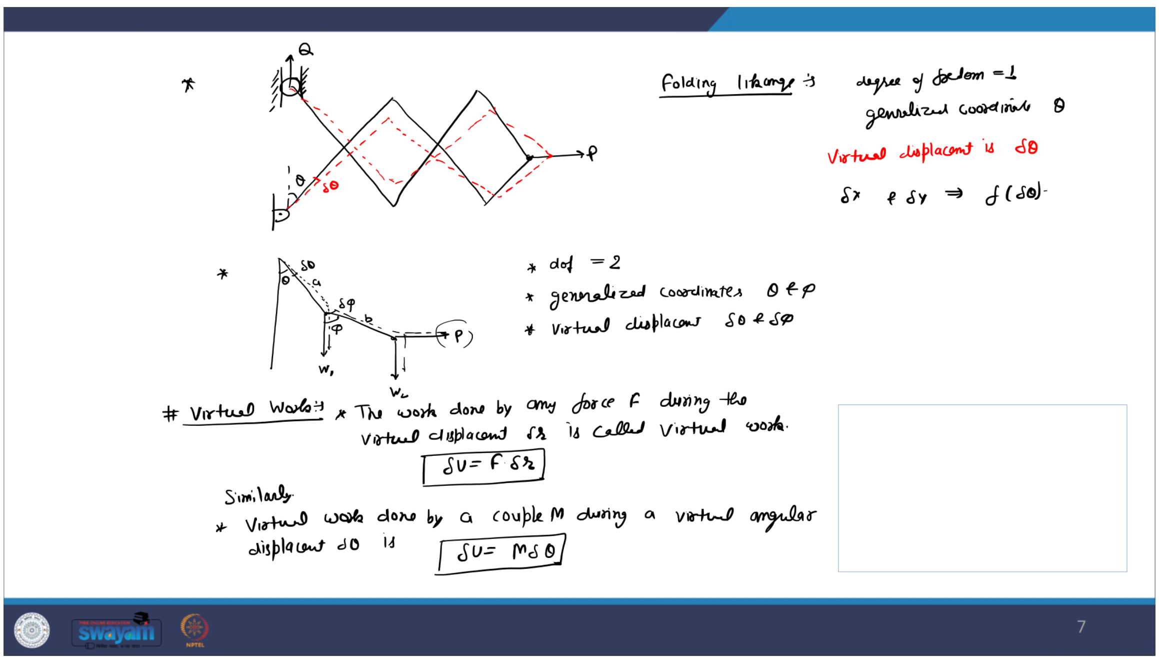
Now, with this general definition, now let us look at what is virtual displacement. So, first of all, we know that if a number of forces act on a body but the forces are in equilibrium, then the forces do not displace the body from its natural position and no work is done by these forces. So, this is the definition of the equilibrium. Now, what is virtual displacement? Any assumed and arbitrary small displacement, let's say away from



the natural position and it should be consistent with the system constraints is called a virtual displacement. The important point is the displacement that you are assuming, it should be consistent with the constraint that is imposed on the system. Let us look at the example.

So, first of all, let me tell you that this virtual displacement it can be just displacement, let's say dr or it can be rotation $d\theta$ of the body. Now, let us look at the virtual displacement by example. So, let us look at example of a body that is kept on the surface and let's say you apply the force like this. So, you have to note that what are the constraints? The constraint is that this body has to be on the surface. But suppose you say well that a small displacement I am going to assume like this. under the action of this force, I am going to assume that this is the displacement. Note that this is not virtual displacement and the reason is it is not consistent with the constraint that we imposed.

So, this is not consistent with the system constraints. So, therefore, this is not the virtual displacement. Now, let us look at the second example. let's say you have the following assembly. So, you have a pin joint. So, let's say I have two bars which are hinged at this point and it is supported by a pin joint and a roller support. So, this roller can move only in the horizontal direction. This point is fixed and the length of the bar will also be fixed. So, therefore, this is the example of a constrained motion. Let me ask what is the degree of freedom? Well, by single coordinate θ, I can define the position of this roller. So, therefore, the degree of freedom is 1 and the generalized coordinate is θ because with this θ, I can define the position of this roller. So, the generalized coordinate is is θ. Now, what is the virtual displacement? Again, we have to respect the constraint that is applied on the system. So, when you apply this force, under the action of this force, you can assume that this body or this connecting body, it moves like this. So, note that this point is fixed. The roller is moving in the horizontal direction and the length of the bar is also fixed. So, therefore, this qualify to be called the virtual displacement. So, we have virtual displacement and virtual displacement is $d\theta$. Okay. So, note that you can also claim that you know I can write down this in terms of x and y coordinate and therefore, the virtual displacement will be dx and dy. But since our generalized coordinate is θ therefore, you have to convert this x and y in terms of θ. So, the point is that this δx and δy if you write down in that form then it will

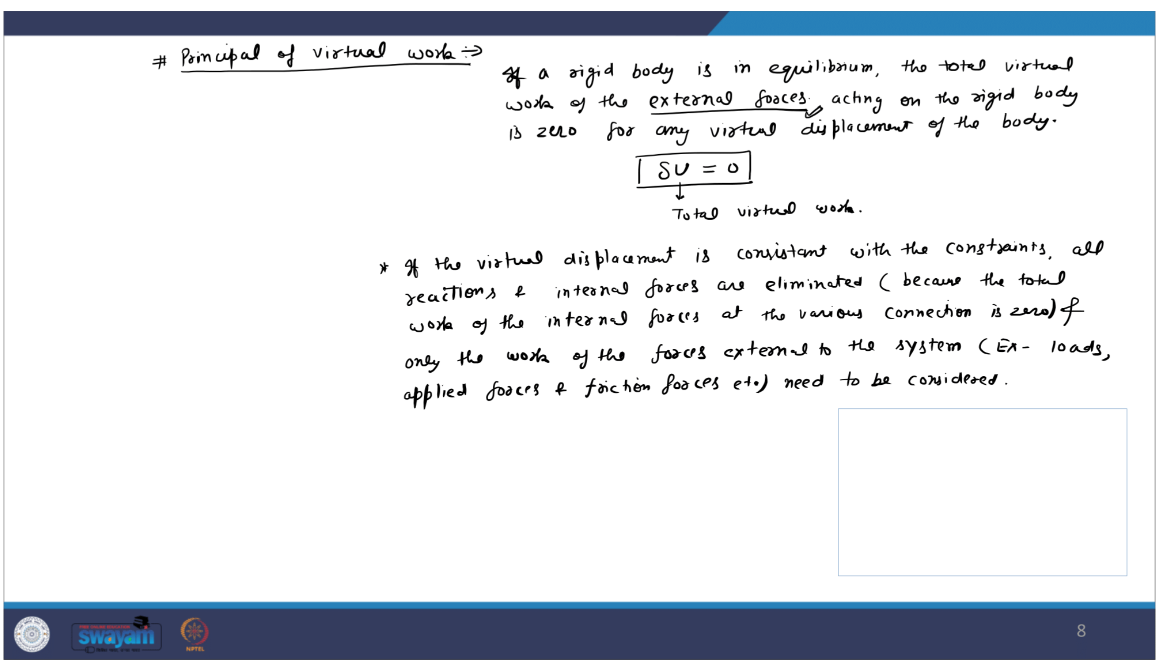


be a function of $\delta\theta$.

Now, let us look at another example. So, this is the example of a folding linkage. So, the concept is following. You have a roller and you have a hinged pin and with this, a folding linkage is connected. Okay. So, let's say you have a force in that direction and you apply a force in this direction. Now, this is nothing but a folding linkage. So, again with a single coordinate θ, I can define where, you know, this link is. So, therefore, its degree of freedom is 1, generalized coordinate is θ. Now, let us look at the virtual displacement. Well, under the action of this force P, I can assume that this link is going to be like that. So, let's say this angle is $\delta\theta$. So, note that I have respected whatever constraint were applied. Okay. So, therefore, the virtual displacement is $\delta\theta$. And again, if you want to write down this in terms of x and y, then you have to note that since my generalized coordinate are θ, therefore, this x and y, I have to write down in terms of θ.

Now, let us look at another example. So, this is a, let's say, a double pendulum. So, you have this length as a and you have weight w_1. Then you have this length as b and weight is w_2. And to keep this in equilibrium, you apply a force P. So, let's say this angle is θ, this angle is ϕ. So, therefore, the degree of freedom for this system will be 2. The generalized coordinate will be θ and ϕ and the virtual displacement will be $\delta\theta$ and $\delta\phi$. So, under the action of this force, you can assume that this assembly is going to move like that and this angle is $\delta\theta$ and this angle is $\delta\phi$. Now, with the concept of virtual displacement, now let us discuss what is virtual work.

So, the definition is the work done by any force F during the virtual displacement, let's say δr is called virtual work. So, for example, if you have translation, then δU will be $F.dr$. Similarly, if you have, so let me write virtual work done by a couple M during a virtual angular displacement $\delta\theta$ will be $\delta U = M\delta\theta$. After understanding the concept of virtual displacement and virtual work, now let us look at the principle of virtual work.



So, the statement is following. If a rigid body is in equilibrium, then the total virtual work of the external forces, so we know what is the external forces now. They are acting on the rigid body is 0 for any virtual displacement of the body. In equation form, if you calculate δU which is a virtual work, it should be 0, $\delta U = 0$. So, note that here δU is total virtual work and you have to calculate for the external forces. For example, the forces that you are applying or the mass of the object and so on. Now, if the virtual displacement is consistent with the constraint then all reaction and internal forces are eliminated. That means you do not need to consider them and the reason is because the total work of the internal forces at the various connections or reaction points is 0 and only the work of the forces external to the system. For example, loads, applied forces and friction forces, etc., they need to be considered. So, this is the same point that I mentioned here that you need to consider only the external forces. In the next class, we will look at the virtual displacement, virtual work, and the principle of virtual work by various examples. With this, let me stop here. Thank you.