**MECHANICS**

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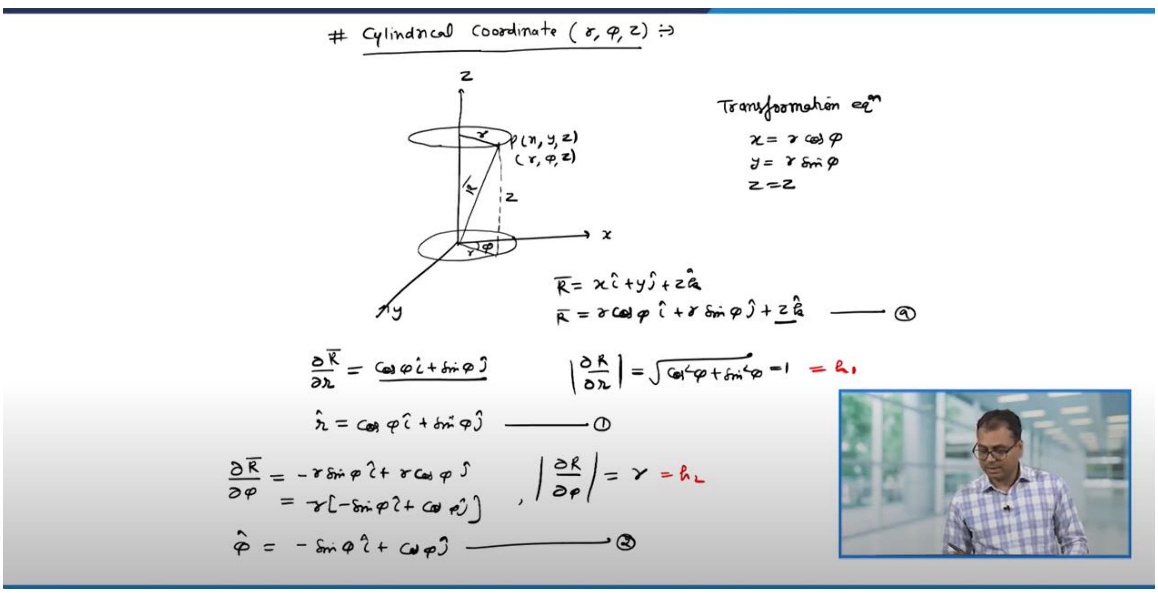
**Lecture: 32**

**Coordinate systems: cylindrical coordinates**

Hello everyone, welcome to the lecture again. In the last two lectures, we learned about the Cartesian coordinate, planar polar coordinate, spherical coordinates.




Today, we are going to study the cylindrical coordinate system. So, cylindrical coordinates, here the axes are r, $\phi$ and z and_1et me define them. So, suppose you have again the x-axis, the y-axis and the z-axis and there is a point P which has a coordinate of x, y and z and its position vector is let us say R. Then in cylindrical coordinate, you draw a cylinder of radius small $r$ with a height $z$.

So, let me draw a cylinder of radius small r and its height is z and the angle that this radius vector r makes from the x-axis is your $\phi$. So, this angle is $\phi$. Therefore, the coordinate of P in the cylindrical coordinate becomes $r, \phi$ and $z$. Let me repeat again, r is the radius of the cylinder, $\phi$ is the angle that this radius vector makes from the x-axis and

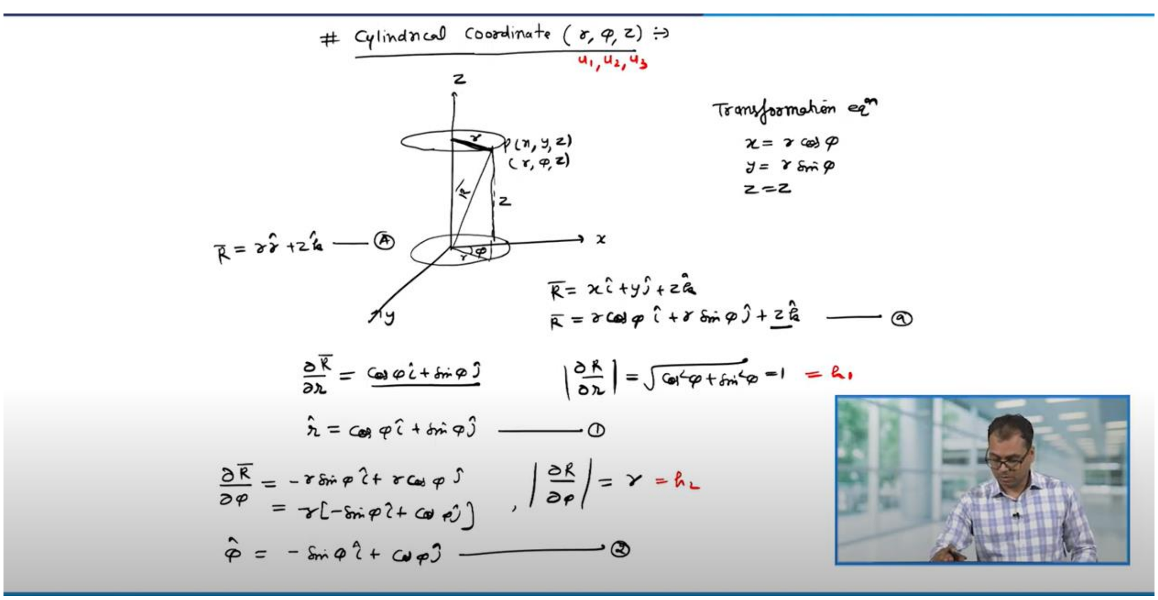
$z$ is the height of the cylinder. Now, from the geometry, you can see the transformation equation.

The transformation equations are $x$ is equal to the value of $x$ will be the projection of $r$ along the x-axis. So, that is $rcos\phi$ the value of y is $rsin\phi$ and z is z. Therefore, the vector r which is $x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ I can write down in the cylindrical coordinate system as x is equal to $rcos\phi\hat{\imath} + rsin\phi\hat{\jmath} + z\hat{k}$. Now, as we have done previously, let us look at the unit vector along the r, ϕ and z direction. For that, I have to differentiate this equation with respect to small r, small ϕ and small z. So, let us call it some equation number a and differentiate $\delta R/\delta r$. So, you can see it is $cos\phi\hat{\imath} + sin\phi\hat{\jmath}$.

And what is the value of $|\delta R/\delta r|$ is $\sqrt{cos^2\phi + \sin^2\phi}$ which is equal to 1. Therefore, this becomes our $h_1$. Therefore, the unit vector $\hat{r}$ becomes $cos\phi i + sin\phi j$. Let us call it equation number 1. Now, let us calculate $\delta r/\delta\phi$. So, from equation number a, you can see that $\delta r/\delta\phi$ is $-rsin\phi i + rcos\phi j$, which can be written as, so I can take r as a common factor.

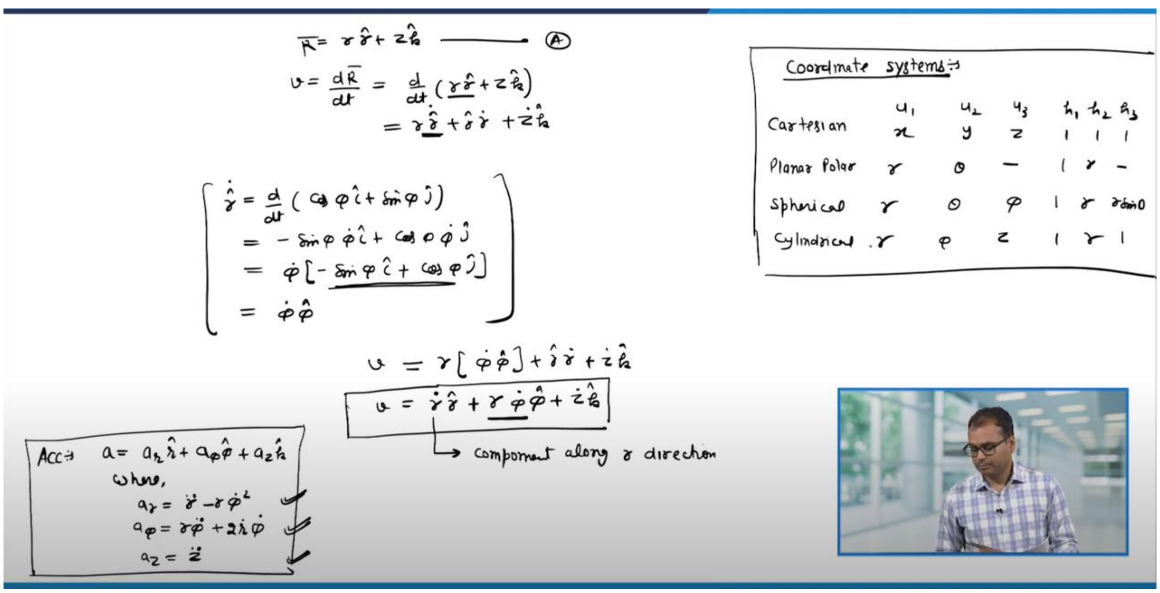
This is $-sin\phi i + cos\phi j$. Now, let us see what is $\delta r/\delta\phi$. This is indeed r because the $\sqrt{cos^2\phi + \sin^2\phi} = 1$. So, therefore, this becomes your $h_2$ and whatever is there in the bracket is your $\widehat{\phi}$. So, this is $-sin\phi i + cos\phi j$. Let us call it equation number 2.

Now, let us calculate $\delta r/\delta z$ and from equation number a, it is clear that $\delta r/\delta z$ will be just $\hat{k}$. So, $\delta r/\delta z$ will be $\hat{k}$. Therefore, $\delta r/\delta z$ modulus will be 1 and this becomes our $h_3$ and $\hat{z}$ becomes $\hat{k}$.



Now, let us look at the differential length, differential area, and differential volume in the cylindrical coordinate system. So, we know the general formula $d_1$ is $h_1 du_1 \widehat{u_1} + h_2 du_2 \widehat{u_2} + h_3 du_3 \widehat{u_3}$, and our $u_1 u_2 u_3$ are $r\phi z$ and $h_1, h_2, h_3$ is 1, r and 1. So, let us put them in this equation. So, we get $d_1, h_1$ is 1, $u_1$ is r, so $dr\hat{r} + h_2$ is r, $u_2$ is $\phi$, so $d\phi\hat{\phi} + h_3$ is 1, and $u_3$ is $z$, so $dz\hat{z}$. Therefore, $d_1$ can be written as $dr\hat{r} + rd\phi\hat{\phi} + dz\hat{z}$. This is the differential length or you can call it the displacement $dr$. Now, let us look at differential area. Differential area, let us say we are looking in the z direction. So, it will be $h_1, du_1, h_2, du_2$ and then $\widehat{u_3}$. Okay. So, $h_1$ is 1, $u_1$ is $dr$, $h_2$ is $r$ and $d\phi$. So, therefore, it becomes $rdrd\phi$ in the $z$ direction.

Now, the differential volume in cylindrical coordinate system is in $h_1, h_2, h_3$ . $du_1, du_2, du_3$ And the value of $h_1, h_2, h_3$, let us put $1, r,$ 1 and $dr, d\phi, dz$. So, the volume $dv$ becomes $r, dr, d\phi, dz$. Okay. Now, let us find out the velocity and acceleration in this coordinate system. So, you can see from the figure that vector r can be written as $r\hat{r} + z\hat{k}$ because this $r$ will be this $r$ plus that r from the vector. So, let us call it equation number let us $\widehat{a_y}$ a and we can find out the velocity by differentiating this with respect to t.




So, let us do that. So, as we said vector r is $r, \hat{r} + z\hat{k}$. This is equation number a. So, velocity $v$ is nothing but $dr/dt$ and it will be $d/dt$ $\vec{r}$ is r, $\hat{r} + z\hat{k}$.

So, let us differentiate this. So, we have $r, \dot{\hat{r}} + \hat{r}\dot{r}$ and k is constant. So, therefore, it will be just $\dot{z}\hat{k}$. Now, as earlier, we have to find out what is the derivative of $\hat{r}$. So, let us look at the derivative of $\hat{r}\dot{\hat{r}}$, this is nothing but $d/dt$ of $\hat{r}$ and $\hat{r}$. We have already find out from equation number a, it is $cos\phi i + sin\phi j$. So, let us look at the derivative, it will be $-sin\phi\dot{\phi}i + cos\phi\dot{\phi}j$ and I can take $\dot{\phi}$ outside. So, it becomes $-sin\phi i + cos\phi j$ and this is the value of $\hat{\phi}$. Let us look at equation number 2. So, this becomes $\hat{\phi}\dot{\phi}$.

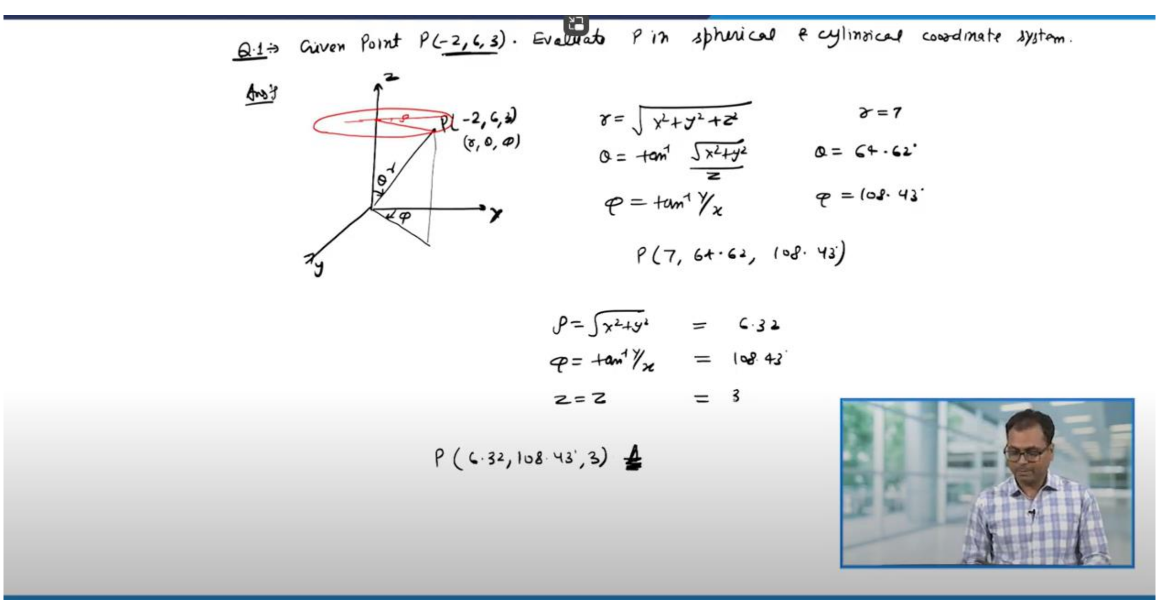
Let us put this in the above equation. So, we have $v = r\dot{\phi}\hat{\phi} + \hat{r}\dot{r} + \dot{z}\hat{k}$ or we can write down $v = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} + \dot{z}\hat{k}$ . This is the expression for the velocity in cylindrical coordinate system.

So, $\dot{r}$ is the component along r direction and similarly $r\dot{\phi}$ is the velocity component along $\hat{\phi}$ direction and $\dot{z}$ is of course the velocity along the k direction. Now, this is your homework problem to find out the acceleration I am giving you the final answer. The acceleration $a$, let us say it is written in this form$a_r\hat{r} + a_\phi\hat{\phi} + a_z\hat{z}$. Then the value of $a_r$ will be r double dot minus r ϕ dot square. $a_\phi$ will be $r\ddot{\phi} + 2\dot{r}\dot{\phi}$ and $a_z = \ddot{z}$.

So, these are the component along the i direction, this is the component of the acceleration along the $\phi$ direction and of course, this is the component of the acceleration along the z direction. Now, let me summarize the result that we have obtained so far. So, we studied various coordinate systems. We studied the Cartesian coordinate. Planar polar coordinate, spherical coordinate and today we studied the cylindrical coordinate.

We used $u_1, u_2, u_3$. $h_1, h_2$ And $h_3$, where $u_1, u_2, u_3$ are the coordinates and $h_1, h_2$ and $h_3$are the scale factors. So, everything can be written down in this term. For Cartesian coordinate, $u_1$ was x, $u_2$ was y, $u_3$ was z, and $h_1, h_2, h_3$ were 1, 1, 1. The planar polar coordinate system was defined on two dimensions.

Therefore, there was no $u_3$. We have r and $\theta$ and $h_1$ and $h_2$were 1 and r. For a spherical coordinate system, we have r, $\theta$ and $\phi$ and $h_1, h_2, h_3$ were 1, r, r sin theta. In the cylindrical coordinate system, it was r, $\phi$ and z and $h_1, h_2$ and $h_3$ were 1, r and 1.

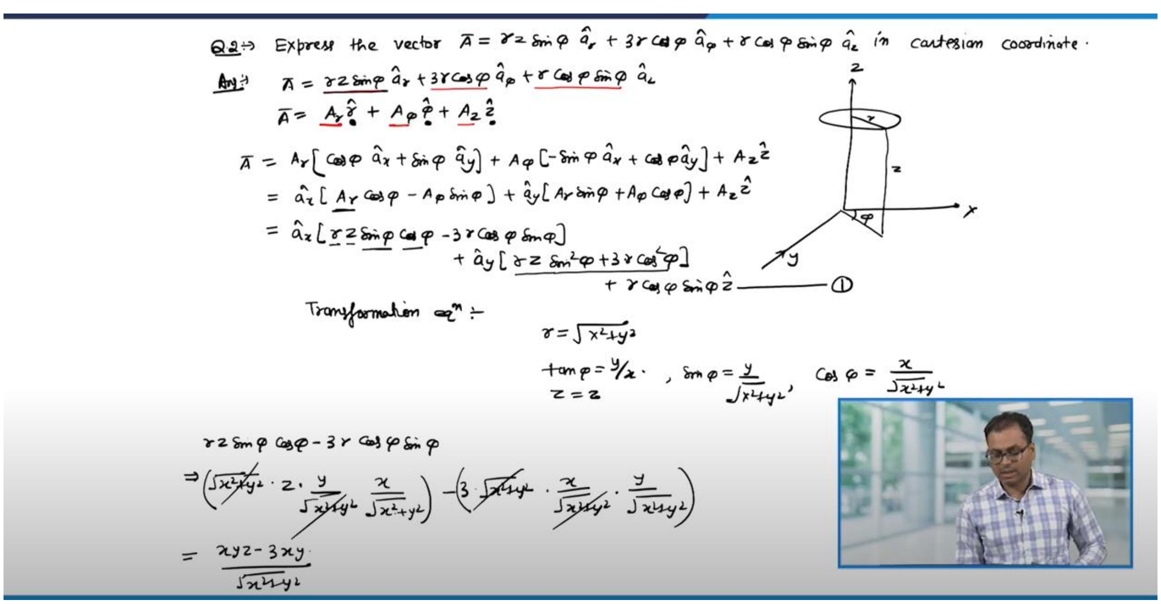


Now, let us look at a very simple example and the problem statement is following. Given point $P$ equal to$-2, 6, 3$, evaluate P in spherical and cylindrical coordinate system. So, the point is given in Cartesian coordinate. So, let me make the Cartesian coordinate x, y and z and_1et us say the point P is here $-2, 6, 3$. Then, in a spherical coordinate system, you have the coordinates $r$, $\theta$ and $\phi$. So, let us say this is r and the angle that this r is making from the z axis is $\theta$ and its projection that makes the angle from the x axis is $\phi$.

So, from the geometry, you can see that $r$ is $\sqrt{x^2 + y^2 + z^2}$. $\theta$ Is $\tan^{-1}\sqrt{x^2 + y^2 + z^2}$, and ϕ is tan inverse y over x. Now, since the value of x, y, and z is given here, so you can put it in these equations. You will get $r = 7$.

Theta is equal to $64.5^0 \ \ 62^0$ and $\phi = 108.43^0$. Therefore, in a spherical coordinate system, the coordinate of point P is $7^0$, $64.62^0$ and $108.43^0$. Now, let us represent this point P in cylindrical coordinate system. You know in cylindrical coordinate system, the coordinates are r, ϕ and z where r is the radius of the cylinder and $\phi$ is the angle that this radius vector makes from the x-axis and z is the height of the cylinder. So, you can see from the figure that $\rho$ is $\sqrt{x^2 + y^2}$. $\phi$ Is $\tan^{-1}(\frac{y}{x})$ and z is z. Now, x, y and z is given. They are minus 2, 6 and 3.

Therefore, rho you can calculate. It comes out to be $6.32$. $\phi$ Comes out to be $108.43^0$ and $z$ is of course $3^0$^. Therefore, in cylindrical coordinate system, the point P becomes $6.32^0, 108.43^0$and $3^0$.



Now, let us look at one more problem statement. Question number 2. Express the vector which is given as $rzsin\phi\widehat{a_r} + 3rcos\phi\widehat{a_\phi} + rcos\phi sin\phi\widehat{a_z}$ in Cartesian coordinate. So, note that this vector is given in terms of $A_r, A_\phi$ and $A_z$. Therefore, this is written in the cylindrical coordinate system. So, we have x-axis, y-axis and z-axis and in cylindrical coordinate, you have r which is the radius of the cylinder, its height z and then $\phi$ is the angle that this r makes from the x-axis.

So, this is your $\phi$. So, vector A is given as$rzsin\phi\widehat{a_r} + 3rcos\phi\widehat{a_\phi} + rcos\phi sin\phi\widehat{a_z}$. Now, in general, a cylindrical coordinate, a vector A is written as $A_r\hat{r} + A_\phi\hat{\phi} + A_z\hat{z}$, where $A_r$ is $rzsin\phi$, $A_\phi$ here is $3rcos\phi$ and $A_z$ is $rcos\phi sin\phi$.

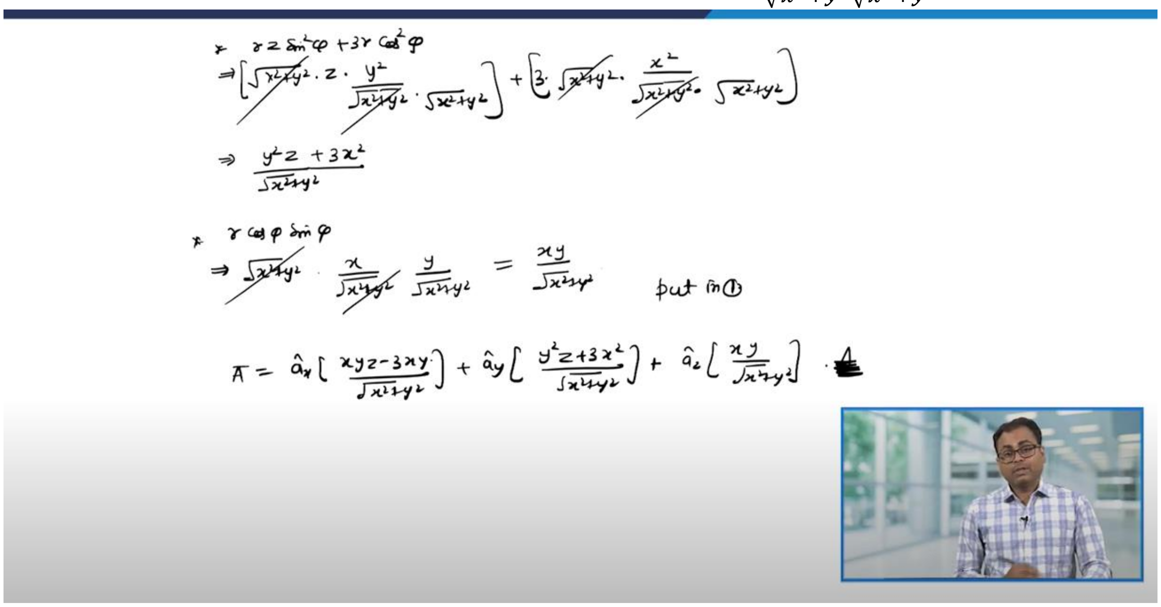
So, we know the value of $\hat{r}\hat{\phi}$ and $\hat{z}$ in Cartesian coordinate. So, therefore, this A I can write down as $A_r\hat{r}$ is $cos\phi\widehat{a_x} + sin\phi\widehat{a_y}$ plus the value of $\hat{\phi}$ is$-sin\phi\widehat{a_x} + cos\phi\widehat{a_y} + A_z\hat{z}$. I can collect the coefficient of $\widehat{a_x}$, $\widehat{a_y}$, and $\widehat{A_z}$. So, this I can write down as $\widehat{a_x}[A_r cos\phi - A_\phi sin\phi] + \widehat{a_y}[A_r sin\phi + A_\phi cos\phi] + A_z\hat{z}$. Now, the value of $A_r$ is already known. It is $rzsin\phi$. Therefore, I can write it as $\widehat{a_x}$. $rzsin\phi$ And $cos\phi - A_\phi$ is also known.

It is $3rcos\phi$ . So, this will be $-3rcos\phi sin\phi + \widehat{a_y}rzsin^2\phi + 3rcos^2\phi$ plus $A_z$ is $rcos\phi sin\phi\hat{z}$. Let us call this equation number 1. Now, what I have to do is I have to write down this r, z and ϕ in terms of the, you know, Cartesian coordinate.

So, for that we have the transformation equation. The transformation equation from cylindrical to Cartesian coordinate is $r = \sqrt{x^2 + y^2}$. This you can see from the geometry and $\phi$ or $tan\phi$ is $y/x$ and $z$ remains $z$. So, if $tan\phi$ is $y/x$ and we need what is the value of $sin\phi$ and $cos\phi$, so that we can find out. So, $sin\phi$ will be $y/\sqrt{x^2 + y^2}$ and $cos\phi$ therefore, will be $x/\sqrt{x^2 + y^2}$.

Now, all these transformation I can put in equation number 1. So, let us look at the first term, the coefficient of $\widehat{a_x}$. The coefficient of $\widehat{a_x}$ is $rzsin\phi cos\phi - 3rcos\phi sin\phi$. So, let us put the values r is $\sqrt{x^2+y^2}$, z is $z$ and $sin\phi$ is $y/\sqrt{x^2+y^2}$ and $cos\phi$ is $x/\sqrt{x^2+y^2}$. This is r $zsin\phi cos\phi$ minus 3, $r$ is $\sqrt{x^2+y^2}$ into $cos\phi$ is $x/\sqrt{x^2+y^2}$ and $sin\phi$ is $y/\sqrt{x^2+y^2}$. So, this square root will get cancelled with this and this will get cancelled with that. So, I can write down this as $xyz - 3xy/\sqrt{x^2+y^2}$. This is the first term. The coefficient of $\widehat{a_x}$. Similarly, we can find out the coefficient of $\widehat{a_y}$. So, which is r, $zsin^2\phi + 3rcos^2\phi$. So, let us find out that. So, we have r, $zsin^2\phi + 3rcos^2\phi$. So, again r is $\sqrt{x^2+y^2}$, z is as it is. And $\sin^2\phi$is$y^2/\sqrt{x^2+y^2}\sqrt{x^2+y^2}$ . Okay.

So, this is your $r$, $zsin^2\phi$ plus $3r$ is $\sqrt{x^2+y^2}$ and $cos\phi$ is $\frac{x^2}{\sqrt{x^2+y^2}\sqrt{x^2+y^2}}$. So, again this




square root will cancel with this and this will get cancelled with that. So, therefore, we are left with $y^2z + 3x^2/\sqrt{x^2+y^2}$.

Now, let us look at the coefficient of $\hat{z}$, which is $rcos\phi sin\phi$. So, now, we are looking for $rcos\phi sin$. So, r is $\sqrt{x^2+y^2}$ $cos\phi$ is $x/\sqrt{x^2+y^2}$ and sin ϕ is $y/\sqrt{x^2+y^2}$. This will get cancelled with that. So, we are left with $xy/\sqrt{x^2+y^2}$.

So, now we have the coefficient of $\widehat{a_x}, \widehat{a_y}$ and $\widehat{a_z}$. Therefore, we can put that in equation number 1. So, we get $\widehat{a_x}$ and then the coefficient $\widehat{a_x}\left[\frac{xyz-3xy}{\sqrt{x^2+y^2}}\right] + \widehat{a_y}\left[\frac{y^2z+3x^2}{\sqrt{x^2+y^2}}\right] + \widehat{a_z}[\frac{xy}{\sqrt{x^2+y^2}}]$. So with this, let me stop here. See you in the next class.

Thank you.