MECHANICS

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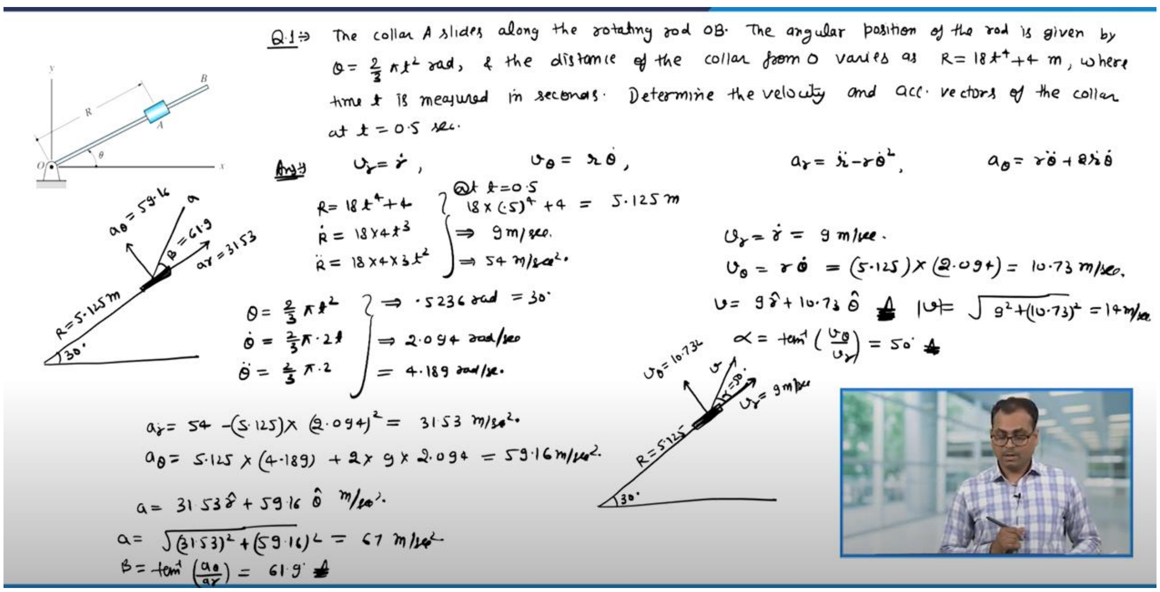
Lecture: 34

Spherical and cylindrical coordinates: examples

Hello everyone, welcome to the lecture again. In the last class, we look at various examples of the velocity and acceleration in Cartesian coordinates and planar polar coordinate system.



Today, we are going to solve some examples on the spherical and cylindrical coordinate system. But first, let me look at one more example in planar coordinate system. So, here the problem statement is following. The coller A slides along the rotating rod OB. The angular position of the rod is given by $\theta = \frac{2}{3}\pi t^2$ radian and the distance of the collar from O varies as $R = 18t^4 + 4$ where time t is measured in seconds.

Determine the velocity and acceleration vector of the collar at $t = 0.5\, sec$. Now, it is obvious in this question that the collar is making planar motion. Therefore, we will use the planar polar coordinate and there the velocity in r direction v_r is $\dot{r}$ and v_θ is $r\dot{\theta}$.

The acceleration along r is $\ddot{r} - r\dot{\theta}^2$ and the acceleration along θ direction is $r\ddot{\theta} + 2\dot{r}\dot{\theta}$. Now, it is given that R is equal to$18t^4 + 4$ therefore, $\dot{R}$ will be $18 \times 4t^3 + 4$ and $\ddot{R}$ will be $18 \times 4 \times 3t^2$. Now, all the quantities in this question, we have to calculate at $t = 0.5\ s$. So, let us put at $t = 0.5\ s$.

So, we get $R = 18 \times 0.5^4 + 4$ which comes out to be $5.125\ meter$. Similarly, $\dot{R}$ comes out to be $9\ m/s$ and $\ddot{R}$ comes out to be $54\ m/s^2$. Now, let us find out $\dot{\theta}$ and $\ddot{\theta}$ because that is required to find out the velocity and acceleration. So, it is given that θ is equal to $\frac{2}{3}\pi t^2$.

Therefore, $\dot{\theta}$ will be $\frac{2}{3}\pi \times 2t$ and $\ddot{\theta}$ will be $\frac{2}{3}\pi \times 2$ and again at $t = 0.5\ s$, this becomes $0.5236\ radian$ which you can convert into degree. So, it comes out to be 30^0 and $\dot{\theta}$ is $2.094\ radian$. And this is $4.189\ radian/sec$. So, now we know $\dot{r}$, $\ddot{r}$, $\dot{\theta}$ and $\ddot{\theta}$. Therefore, we can write down v_r, v_θ, a_r and a_θ.

So, now let us look at v_r. v_r Was $\dot{r}$ and $\dot{r}$ we have calculated. It is $9\ m/s$, v_θ is $r\dot{\theta}$, r is $5.125\dot{\theta}$, $\dot{\theta}$is 2.094 and this comes out to be $10.73\ m/s$. Therefore, v can be written as $9\hat{r} + 10.73\hat{\theta}$.

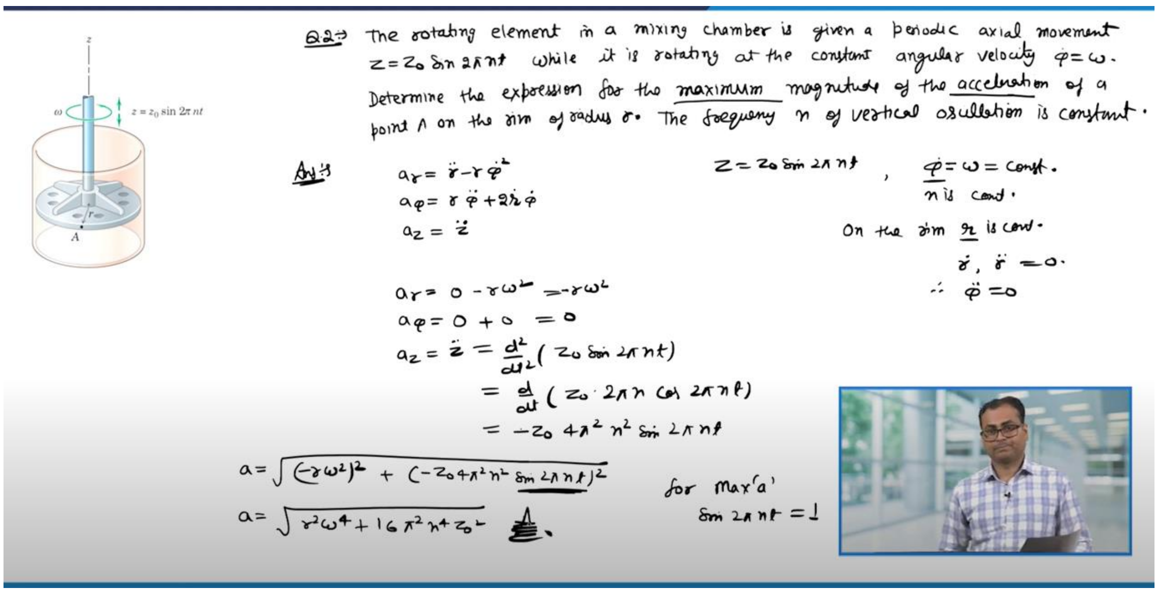
And its magnitude will be $\sqrt{9^2 + 10.73^2}$ which comes out to be $14\ m/s$. Let us look at the angle between v_θ and v_r so that you can find out $\alpha = \tan^{-1}(\frac{v_\theta}{v_r})$. So, that comes out to be Now, let us look at the acceleration, acceleration a_r was $\ddot{r}$ which is $54 - r$, r is $5.125\dot{\theta}^2$ and $\dot{\theta}$ is 2.094^2. And this comes out to be $31.53\ m/s^2$.

Now, let us look at a_θ, a_θ is $r\ddot{\theta}$ and r is 5.125 into $\ddot{\theta}$ which is $4.189 + 2\dot{r}$ which is $9\dot{\theta}$which is 2.094 and this comes out to be $59.16\ m/s^2$. Therefore, acceleration a, i can write down as $31.53\hat{r} + 59.16\hat{\theta}\ m/s^2$. Its magnitude a will be $\sqrt{31.53^2 + 59.16^2}$ and this comes out to be $67\ m/s^2$ and let us say the angle between them is β.

So, this is $\tan^{-1}(\frac{a_\theta}{a_r})$ which comes out to be 61.9^0. Now, let us look the velocity by the diagram. So, we have this collar. This angle is given as 30^0. The collar has a velocity along r. So, r is given $5.125\ m$.

v_r Will be along this direction and it is $9\ m/s$ at $t = 0.5\ s$ and v_θ is perpendicular to it which is 10.732 and its resultant makes an angle α that we have calculated comes out to be 50^0. So, this is your v and this angle is 50^0. Now, let us look at the acceleration by the diagram. So, at $t = 0.5\ s$, this angle was 30^0 and r was $5.125\ m$.

The radial component of the acceleration is ar which we have calculated. It was 31.53 and the θ component a_θ was 59.16 and the resultant makes an angle β and β we have find out 61.9^0 and this was our a. So, in this question, we have find out the velocity and acceleration when $t = 0.5\ s$.



Now, let us look at another problem. Question number two and the problem statement is following. The rotating element in a mixing chamber is given a periodic axial movement this is given by $z = z_0 sin 2\pi nt$ while it is rotating at the constant angular velocity $\dot{\phi}$. Which is nothing but ω. And we have asked to determine the expression for the maximum magnitude of the acceleration of a point A on the rim of radius r and it is given that the frequency n of vertical oscillation is constant.

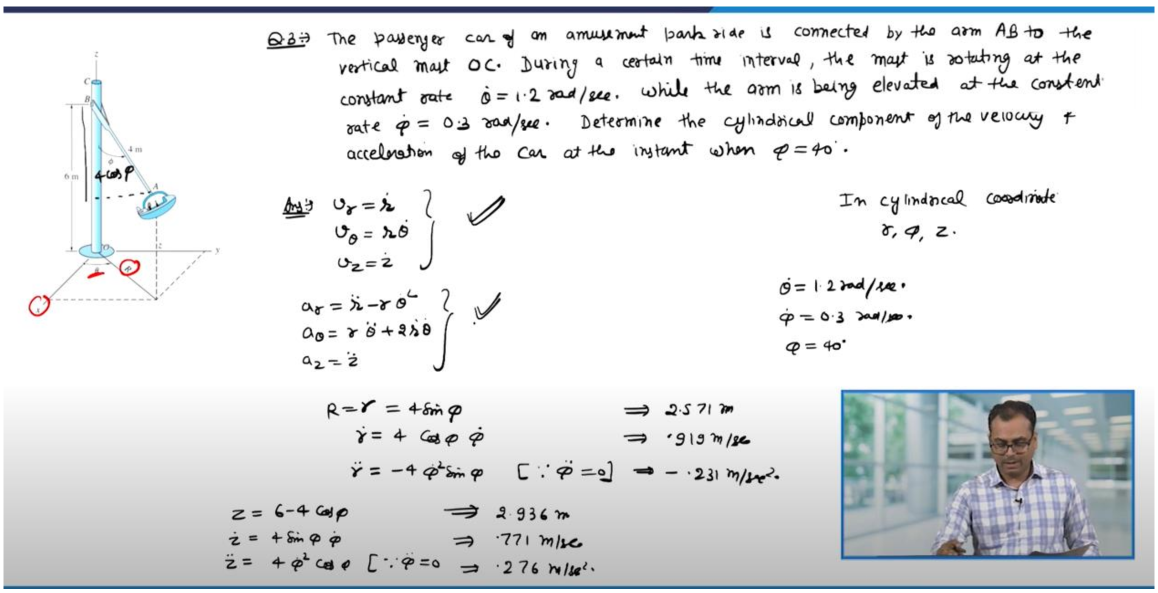
In this question, you can see that because of the cylindrical symmetry, the preferred coordinate choice is the cylindrical coordinate. And in cylindrical coordinate, the acceleration along the r, ϕ and z direction are given as $a_r = \ddot{r} - r\dot{\phi}^2$, $a_\phi = r\ddot{\phi} + 2\dot{r}\dot{\phi}$ and $a_z = \ddot{z}$. Now, we have been given a relation between z and t. It is given that z is equal to $z_0 sin 2\pi nt$ and $\dot{\phi}$ which is ω is equal to constant. It is also given that n is also constant.

And note that because we have been asked to find out the maximum acceleration of point A on the rim. So, on the rim r is also constant. r is not changing. Therefore, let us see what is a_r. Because r is constant, therefore $\dot{r}$ and $\ddot{r}$ will be 0 and because $\dot{\phi}$ is constant, therefore $\ddot{\phi}$ will also be 0.

Therefore, a_r will be $0 - r\omega^2$ which is nothing but $r\omega^2$, a_ϕ is $r\ddot{\phi}$ which is $0 + \dot{r}$ is 0. Therefore, a_ϕ is 0. Now, let us look at a_z. a_z was $\ddot{z}$, which is $\frac{d^2z}{dt^2}$, which is $z_0 sin2\pi nt$. So, this becomes $\frac{d}{dt} z_0 2\pi n cos2\pi nt$ and this becomes $-z_0 4\pi^2 n^2 sin2\pi nt$. Therefore, I can write down the magnitude of a. It will be $\sqrt{a_r^2 + a_\phi^2 + a_z^2}$.

So, this is also minus. So $(-r\omega^2)^2 + (-rz_0 4\pi^2 n^2 \sin^2 \pi nt)^2$. Now, note that in the question we have asked to find out the expression for maximum acceleration.

Now, the maximum acceleration will happen when $sin\theta = 1$ because that is the maximum value of a sin function. So, for maximum a, $sin2\pi nt$ has to be 1. Let us put that. So, we get $a = \sqrt{r^2\omega^4 + 16\pi^2 n^4 z_0^2}$.



Now, let us look at another problem statement. The passenger car of an amusement park ride is connected by an arm AB to the vertical mast OC. During a certain time interval, the mast is rotating at constant rate. And the rate is $\dot{\theta} = 1.2\, radian/sec$. While the arm is being elevated at the constant rate $\dot{\phi} = 0.3\, radian/sec$. And in this question, we have asked to determine the cylindrical component of the velocity and acceleration of the car at the instant when ϕ is equal to 40^0, okay. Now, in this question, we have been asked to find out the cylindrical component of the velocity and acceleration. Therefore, we have to use the cylindrical coordinate.

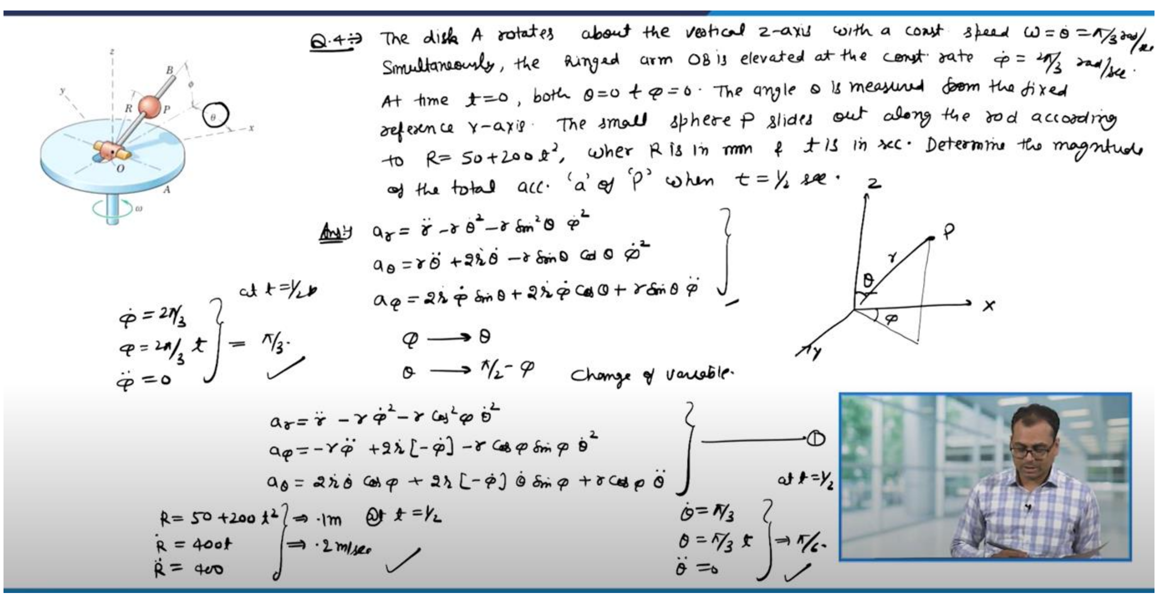
Now, note that in cylindrical coordinates, our axes were $r, \phi,$ and z. Where this ϕ was the angle that r makes from the x-axis. However, in this question, it is given that the angle that

r makes from the x-axis is θ. Therefore, in the formula that we have derived, we have to replace ϕ by θ. So, let me write down the expression for the velocity and accelerations v_r is $\dot{r}$. v_ϕ Becomes v_θ and $r\dot{\phi}$ becomes $r\dot{\theta}$. Similarly, v_z is $\dot{z}$. a_r Is $\ddot{r} - r\dot{\theta}^2$. a_θ Becomes $r\ddot{\theta} + 2\dot{r}\dot{\theta}$ and a_z is $\ddot{z}$.

Now, it is given that $\dot{\theta}$ is 1.2 $radian/sec$ and $\dot{\phi}$ is 0.3 $radian/sec$. And we have to find the acceleration and velocity when $\phi = 40^0$. Now, from the diagram, you can see that R, which is also r, is $400sin\phi$ therefore $\dot{r}$ is $4cos\phi\dot{\phi}$ and $\ddot{r}$ will becomes $-4\dot{\phi}^2 sin\phi$ because $\ddot{\phi}$ is 0.

Now let us put the value of $\phi = 40^0$ so we get $r = 2.571\ m$, $\dot{r} = 0.919\ m/s$ and $\ddot{r} = -0.231\ m/s^2$. Now z also you can find from the figure and you can see that $z = 6 - 4cos\phi$ because this distance will be $4cos\phi$. Therefore, z becomes $6 - 4cos\phi$ and $\dot{z}$ will be $4sin\phi\dot{\phi}$ and $\ddot{z}$ will be $4\dot{\phi}^2 cos\phi$ because $\ddot{\phi} = 0$.

Now, let us look at its value when $\phi = 40^0$. So, z becomes $2.936\ m$ and $\dot{z}$ becomes $0.771\ m/s$ and $\ddot{z} = 0.276\ m/s^2$. So, now we have the value of $\dot{z}, \ddot{z}, \dot{r}, \ddot{r}$ and $\dot{\theta}, \ddot{\theta}$. Those values we can put here to find out the velocity and acceleration.



So, let me first write down v_r. v_r Was $\dot{r}$. It was $0.919\ m/s$. v_θ Was $r\dot{\theta}$. So, this was 2.571×1.2. So, that was $3.085\ m/s$. v_z Was $\dot{z}$. This was $0.771\ m/s$. And a_r was $\ddot{r} - r\dot{\theta}^2$. So, this was $(-0.231 - 2.571 \times 1.2)^2$ comes out to be $-3.9331\ m/s^2$ and

a_θ was $r\ddot{\theta} + 2\dot{r}\dot{\theta}$, but not that $\dot{\theta}$ was constant. Therefore, $\ddot{\theta}$ will be 0. So, it will be $2\dot{r}$ was $0.919\dot{\theta}$ was 1.2. So, this becomes $2.206\ m/s^2$ and a_z was $\ddot{z} = 0.276\ m/s^2$.

So, this is the answer. Now, let us look at one more problem. Question number 4 and the statement is following the disc a rotates about the vertical z-axis with a constant speed $\omega = \dot{\theta} = \frac{\pi}{3} radian/sec$. Simultaneously, the hinged arm OB is elevated at the constant rate $\dot{\phi} = \frac{2\pi}{3} radian/sec$ at time $t = 0$, both $\theta = 0$ and $\phi = 0$.

The angle θ is measured from the fixed reference x-axis, the small sphere P slides along the rod according to $r = 50 + 200t^2$ where r is in mm and t is in second. The magnitude of the total acceleration a of P when $t = \frac{1}{2}\ sec$. So, the obvious choice to solve this question is the spherical polar coordinate and in spherical polar coordinate, you have the $x, y,$ and z. You have, let us say, some vector r.

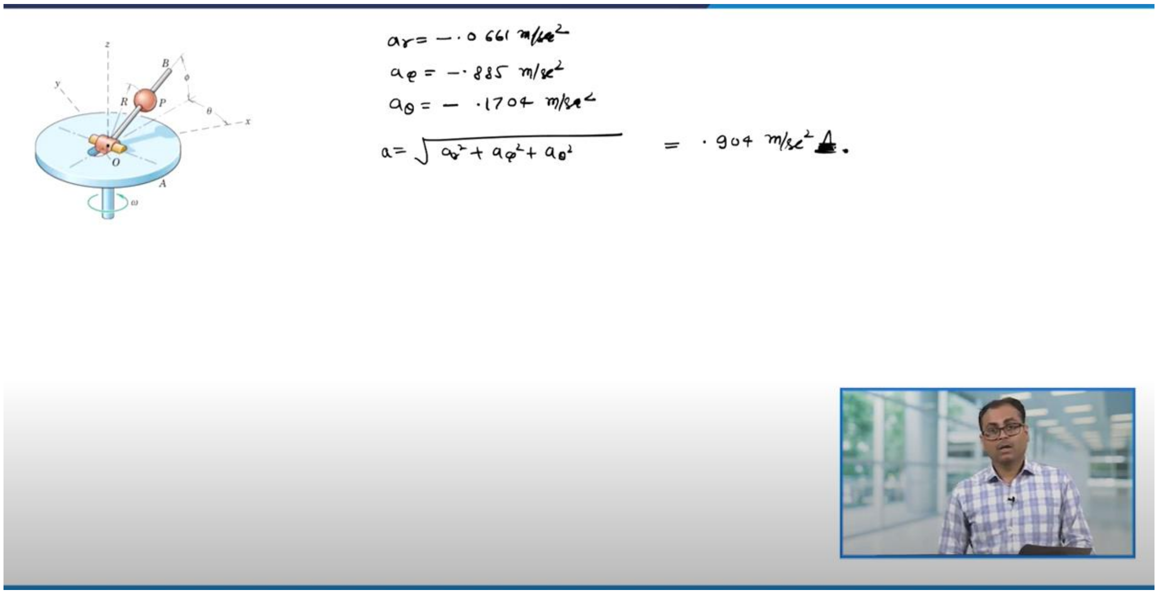
It makes an angle θ from the z axis and its projection on the xy plane makes an angle ϕ from the x axis. So, this is some point P. Now, for this case, we have derived the expression for the acceleration. And our a_r was $\ddot{r} - r\dot{\theta}^2 - rsin^2\theta\dot{\phi}^2$. Our a_θ was $r\ddot{\theta} + 2\dot{r}\dot{\theta} - rsin\theta cos\theta\dot{\phi}^2$. And a_ϕ was$2\dot{r}\dot{\phi}\sin^2\theta + 2\dot{r}\dot{\phi}cos\theta + rsin\theta\ddot{\phi}$. Note that when we derived this expression, we said that the angle ϕ is the angle that the projection of r makes with the x-axis.

However, in this question, that angle is given as θ. Therefore, we have to make the transformation. In our formalism, we have to replace this ϕ by θ and in this formalism the angle θ was the angle that r makes from the z-axis which will be replaced by $\frac{\pi}{2} - \phi$ because that angle will be $\frac{\pi}{2} - \phi$. So, let us make these transformations or we can call it the change of variable. In the above equation. So, we will have $a_r = \ddot{r} - r\dot{\theta}$ will becomes $\dot{\phi}$. So, it will be$-r\dot{\phi}^2 - rcos^2\phi\dot{\theta}^2$.

a_ϕ Will be $-r\dot{\phi} + 2\dot{r} - \dot{\phi} - rcos\phi sin\phi\dot{\theta}^2$ and a_θ will be$2\dot{r}\dot{\theta}cos\phi + 2r - \dot{\phi}\dot{\theta}sin\phi + rcos\phi\ddot{\theta}$. This is the acceleration along r, ϕ and θ direction after having the change of variable. Now, in the problem statement, R is given, R is $50 + 200t^2$. And we have been asked to find out the acceleration, etc.

At $t = \frac{1}{2} second$. So, therefore, at $t = \frac{1}{2} second$, this R comes out to be $0.1\ m^2$. And $\dot{R}$ will be $400t$. Therefore, it will be $0.2\ m/s$. And $\ddot{R}$ will be 400. Now, let us look at $\dot{\theta}$.

It is given that $\dot{\theta}$ is $\pi/3$. Therefore, θ will be $\frac{\pi}{3}t$ and $\ddot{\theta}$ will be 0. Now, at $t = 1/2$, θ becomes $\pi/6$ and $\dot{\phi}$ is also given. $\dot{\phi}$ Is $2\pi/3$. Therefore, ϕ will be $\frac{2\pi}{3}t$ and $\ddot{\phi}$ will be 0.



At $t = \frac{1}{2} second$. Φ Will be $\pi/3$. So, now, we have $r, \dot{r}, \ddot{r}, \theta, \dot{\theta}$, $\ddot{\theta}$ and $\phi, \dot{\phi}, \ddot{\phi}$. This we can put in equation number 1 to find out what is a_r, a_ϕ and a_θ and when you do that, then a_r comes out to be$-0.0661\ m/s^2$, a_ϕ comes out to be $-0.855\ m/s^2$ and a_θ comes out to be $-0.1704\ m/s^2$.

Therefore, the magnitude of a will be \ $\sqrt{a_r^2 + a_\phi^2 + a_\theta^2}$. And this comes out to be $0.904\ m/s^2$. With this, let me stop here. See you in the next class. Thank you.