MECHANICS

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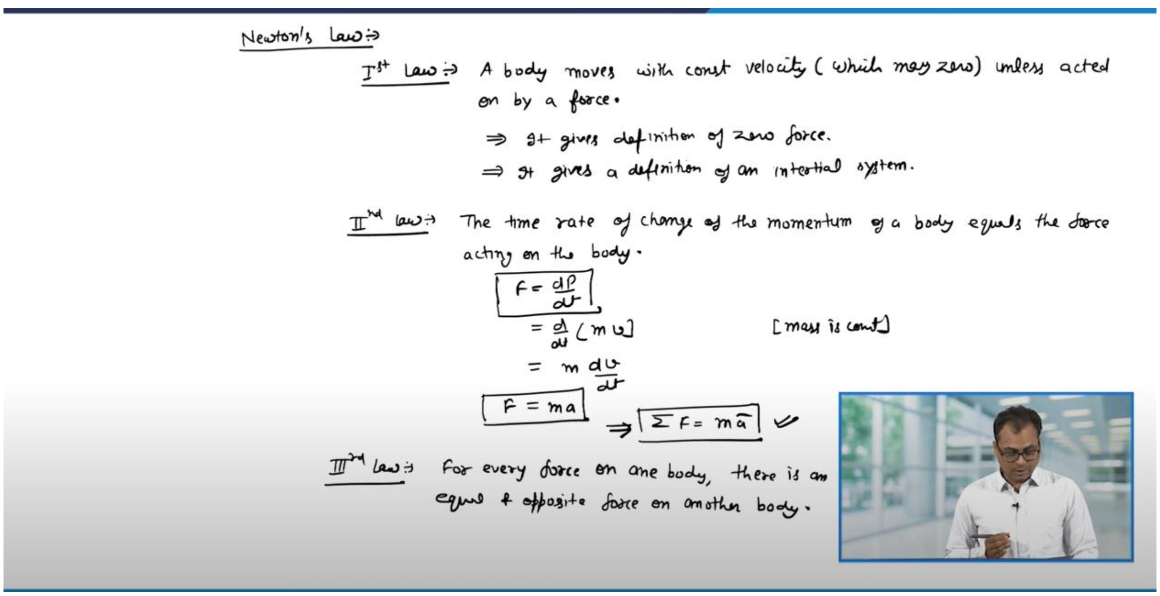
Lecture: 35

Equation of motion in different coordinate systems

Hello everyone, welcome to the lecture again. In the last few lectures, we learned about how to find out the velocity and acceleration in different coordinate system.



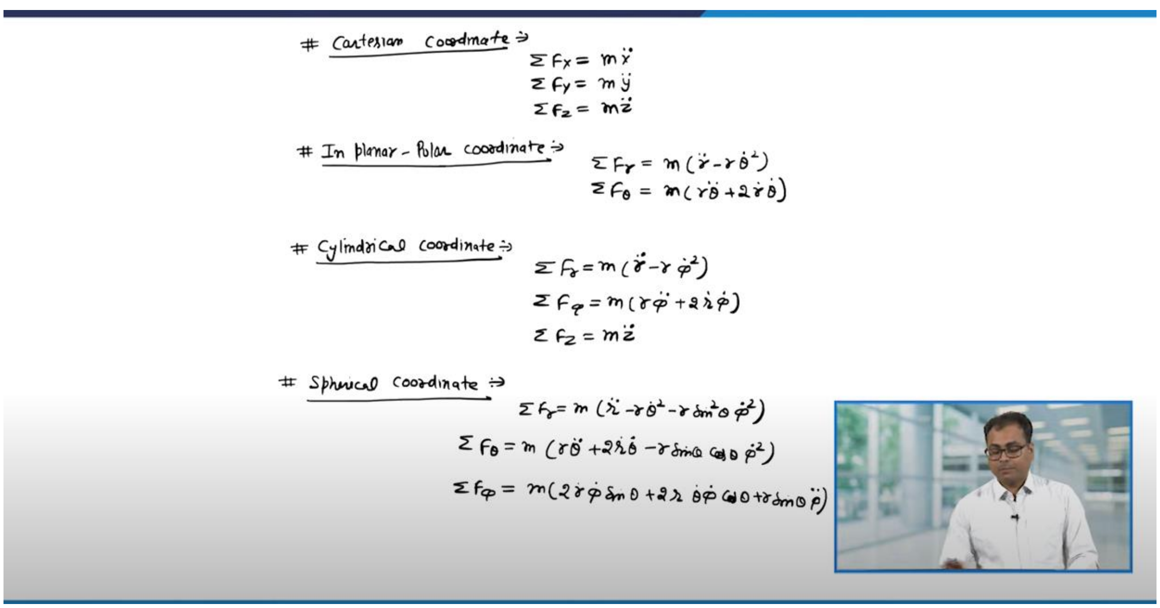
Today, we are going to study the equation of motion in different coordinate system. By equation of motion, I mean the Newton's second law. So, let me very briefly discuss what Newton’s law is because we are already aware of that.

So, Newton's law, the statement of the first law is following. A body moves with constant velocity which may be zero also. So, that means the body is at rest unless acted on by a force.

The implication of this law is this law defines what is zero force. So, it gives you the definition of zero force and very importantly, it also gives you the definition of the inertial system. It gives a definition of an inertial system. Now, inertial system is that system in which the Newton's second law is applicable. So, let us look at the second law.

It states that the time rate of change of the momentum of a body equals to the force acting on the body. So, F is nothing but dP/dt and we know that momentum is mv. So, this is $\frac{d}{dt}mv$ and let us say the mass is constant. In that case, we can write it down as $\frac{mdv}{dt}$ and dv/dt is the acceleration. So, therefore, it can be written as ma.

So, either $F = dP/dt$ or you can say if the mass is constant, then $F = ma$. Now, I am not elaborating more on this because we are already familiar with that. However, we are interested in writing this equation $F = ma$ in different coordinate system. So, let me briefly state the third law also. So, this is something that we have extensively used when we studied the equilibrium for every force on one body, there is an equal and opposite force on another body ok. And Here, I wrote $F = ma$ when there are one force acting on a particle, but if there are many forces, then this equation can be generalized as $\sum F = ma$ and the acceleration because of all the forces.

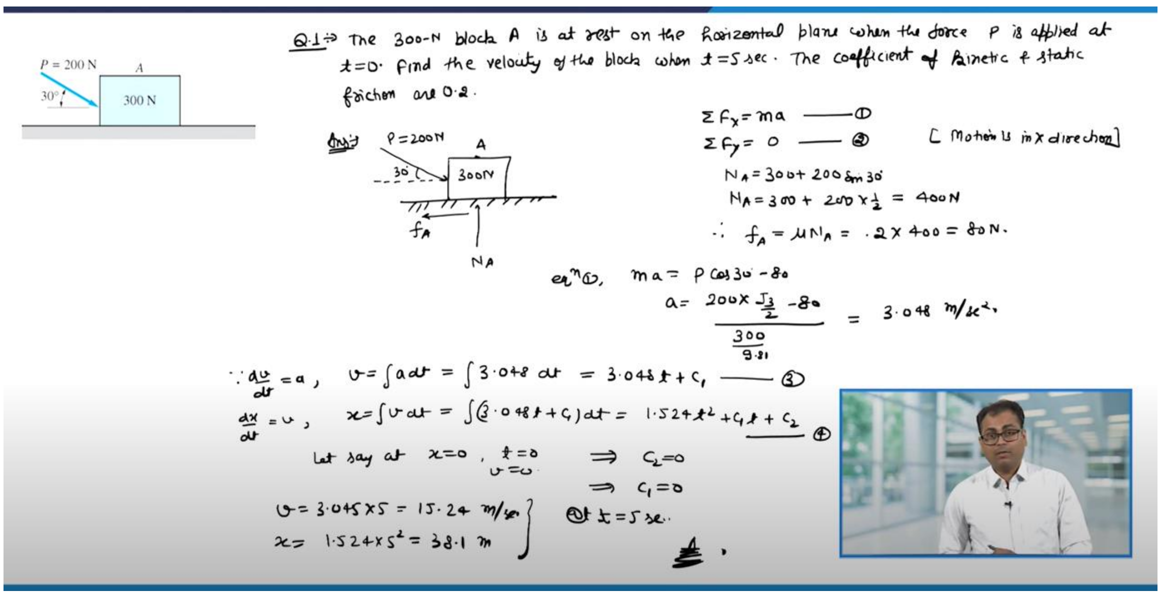


Now, let us write this down $F = ma$ equation for different coordinate system. What are the coordinate system that we have studied so far are the Cartesian coordinate, planar polar coordinate, cylindrical coordinate and spherical coordinate. So, let me write down this equation in those system. In Cartesian coordinate, we can decompose that equation $\sum F_x =$

$m\ddot{x}$ where $\ddot{x}$ is the acceleration along the x direction and similarly, $\sum F_y = m\ddot{y}$ and $\sum F_z = m\ddot{z}$. In planar polar coordinate, we have the force along the r direction and θ direction.

$\sum F_r$ Equal to m and then the acceleration along the r direction is something that we have derived $\ddot{r} - r\dot{\theta}^2$ and $\sum F_\theta$ equal to m acceleration along the θ direction that is $r\ddot{\theta} + 2\dot{r}\dot{\theta}$. In cylindrical coordinates, we have the $r, \phi,$ and z a xis. Therefore, $\sum F_r = m(\ddot{r} - r\dot{\phi}^2)$ and $\sum F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})$ and $\sum F_z = m\ddot{z}$. In a spherical coordinate system,

We have $r, \theta,$ and ϕ axis. Therefore, $\sum F_r = m(\ddot{r} - \dot{\theta}^2 - rsin^2\theta\dot{\phi}^2)$ and $\sum F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta} - rsin\theta cos\theta - \dot{\phi}^2)$ and $\sum F_\phi = m(2\dot{r}\dot{\phi}sin\theta + 2r\dot{\theta}\dot{\phi}cos\theta + rsin\theta\ddot{\phi})$. These are the Newton's law in different coordinate systems in different direction. Now, with this very basic introduction, now let us look at the examples.



The first problem statement is following. The 300 N block A is at rest on the horizontal plane when the force P is applied at $t = 0\ s$. Find the velocity of the block when $t = 5\ s$ and it is given that the coefficient of static and kinetic friction are 0.2.

Here, it is obvious that the problem is easy to solve in Cartesian coordinate. Let me first make the free body diagram of this. So, we have a block, its weight is 300 N and there is an external force that is acting at 30^0 with the horizontal. Its value is $P = 200\ N$ and because of that, this block is going to move. First of all, there will be a normal force, let us say N_A because this is A and the frictional force will oppose the motion.

So, therefore, they will develop in the opposite direction of the motion. So, let us say this is the frictional force. This completes the free body diagram. Now, let us write down the equation of motion in the x and y direction. So, in the x direction, we have $\sum F_x = ma$. Let me call this equation number 1 and in the y direction, there is no acceleration because the block is moving along the horizontal. Therefore, in the y direction, the forces has to be balanced. So, $\sum F_y = 0$ and the reason as I said because the motion is in x direction. Okay.

So, from equation number 2, we can find out what is the value of N_A and therefore, I can find out what is the value of f_A because the frictional coefficients are given. So, let me use equation number 2 to find out what is N_A. So, we can balance the forces along the y direction. So, we have $N_A = 300 + 200sin30^0$ Therefore, N_A is $300 + 200 \times \frac{1}{2}$ which gives you $400\ N$.

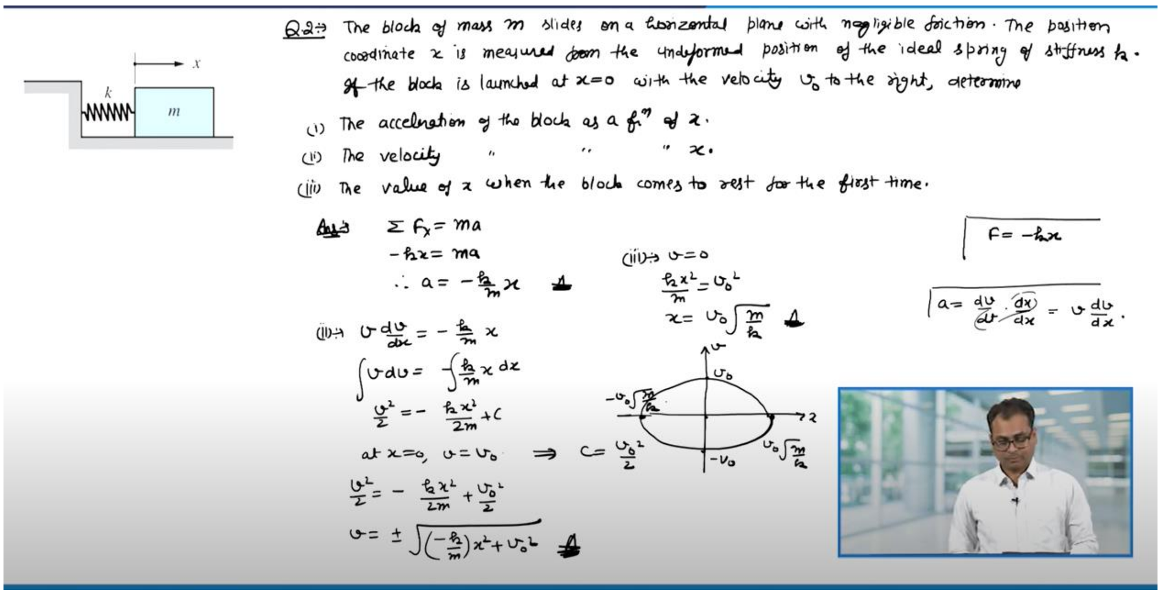
Therefore, the frictional force f_A will be μN_A and that will be $0.2 \times 400 = 80\ N$. Now, I can use equation number 1 because now the f_A is also known, and I know all the forces that are acting along the x direction. So, from equation number 1, we have N_A equal to total force along the x direction is $Pcos30^0 - 80\ Newton$. Therefore, A becomes P is $200cos30$ is $\sqrt{3/2} - 80$ divided by the mass of the object. So, weight is $300\ N$.

So, we have to divide it by 9.81 and this gives you $3.048\ m/s^2$. So, this is the acceleration. Now, we have to find out the velocity of the block. So, velocity we can find out $v = adt$. So, we integrate the acceleration with respect to time because dv/dt gives you a.

So, since $\frac{dv}{dt} = a$, therefore v is $\int adt$ and it will be integral a, we already know $3.048\ dt$ and this gives you $3.048t + c_1$. Let me call this equation number 3 because I have to find out what is c_1 and x again I can find out by integrating vdt. Because v is nothing but dx/dt. So, v is dx/dt. Therefore, x is integral vdt.

So, it becomes $3.048t + c_1 dt$. So, it will be $1.524t^2$ because the integral of t is $\frac{t^2}{2} + c_1 t + c_2$. Let me call it equation number 4. Now, the value of c_1 and c_2, I can find out using the initial condition and let us say at $x =$, everything is start.

So, therefore, t is 0. Now, if I put $x = 0$ and $t = 0$ in equation number, let us say 4, then I get $c_2 = 0$. And since at this instant, the initial velocity v is also 0, therefore, I get $c_1 = 0$. Now, in the question statement, we have been asked to find out the velocity when $t = 5\ second$. So, let us put $t = 5$ in equation number 3 and 4. So, we have $v = 3.045 \times 5 = 15.24\ m/s$ and x will be $1.5\ 24 \times 5^2$, which is $38.1\ m$. Note that we have put $t = 5\ second$ in equation number 3 and 4.



Now, let us look at another problem based on spring. The question statement is following. The block of mass m slides on a horizontal plane with negligible friction. The position coordinate x is measured from the undeformed position of the ideal spring of stiffness K. If the block is launched at $x = 0$ with the velocity v_0 to the right, determine (1) the acceleration of the block as a function of x, (2) the velocity of the block as a function of x and (3) the value of x when the block comes to rest for the first time. Okay. So, again, this is a one-dimensional problem and can be easily solved in Cartesian coordinates. Therefore, let us use $\sum F_x = m\ddot{x}$, which is acceleration.

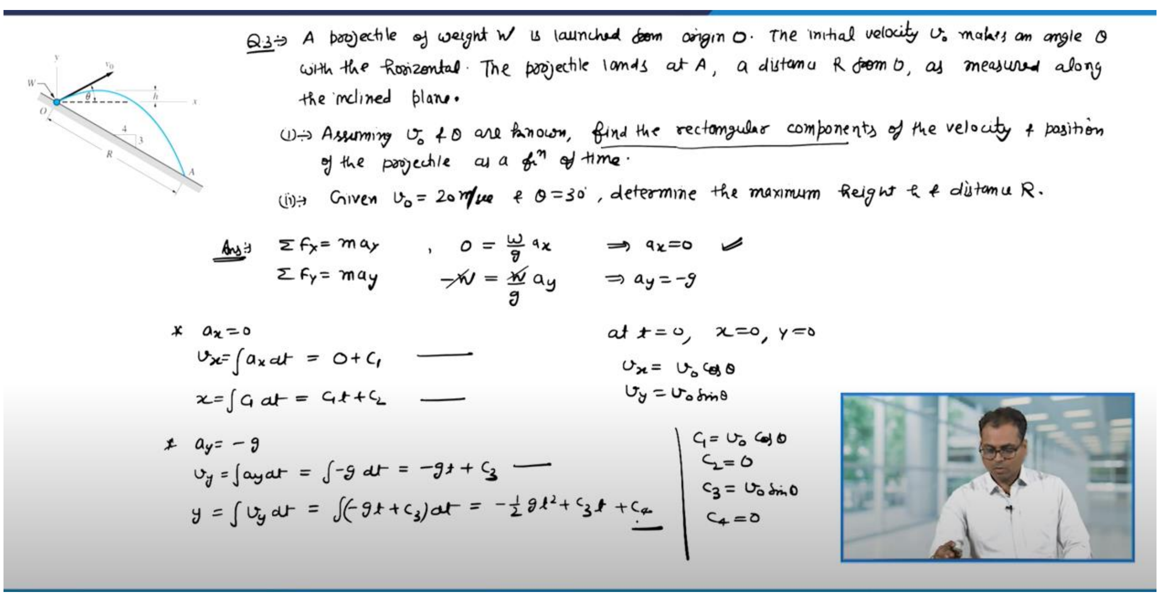
So, let us say m a. And note that for a spring, the force is equal to $-kx$, where x is the extension from the equilibrium and minus because it always oppose the extension. So, therefore, in place of F, I can write down $-kx = ma$. Therefore, the acceleration becomes $-\frac{k}{m}x$.

Now, to find out the velocity, let us use the acceleration. So, a is dv/dt. Let us multiply it by dx and divide by dx. So, dx/dt, I can write down as v. So, acceleration can also be written as vdv/dx. So, for the second part,

Let me use this relation vdv/dx equal to the acceleration which we have find out $-\frac{k}{m}x$. Now, we can use the variable separable rule. So, we have $vdv = -\frac{k}{m}xdx$, and let us integrate it. So, we have $\frac{v^2}{2} = -\frac{kx^2}{2m} + c$. This constant I can find out from the initial condition. It is given that at $x = 0$, the velocity v is v_0.

Therefore, I have $c = v_0^2/2$. Thus, I get $\frac{v^2}{2} = -\frac{kx^2}{2m} + \frac{v_0^2}{2}$ or v I can write down as$\pm\sqrt{\frac{-k}{m}x^2 + v_0^2}$. In the third part, we have been asked the value of x when the blocks comes to rest for the first time. So, when the block comes to rest that means v has to be 0.

In part second, let us put $v = 0$. So, we have $\frac{kx^2}{m} = v^2$ or $x = v_0\sqrt{m/k}$. Let us look at it by in picture. So, on the x-axis, let us say I have x and on the y-axis, I have v. At $x = 0$, its velocity is maxima, it is v_0 given in the question and at $x = v_0\sqrt{m/k}$, its velocity is 0 and because of this equation $v = \sqrt{\frac{-k}{m}x^2 + v_0^2}$, it will be a ellipse. So, this is $-v_0$. Here, this point is $v_0\sqrt{m/k}$, and this point will be $-v_0\sqrt{m/k}$.



Now, let us look at another problem based on projectile motion. The question statement is following. A projectile of with W is launched from origin O, the initial velocity v_0 makes an angle θ with the horizontal.

The projectile lands at A, a distance R from O as measured along the inclined plane. And we have been asked the following. Assuming v_0 and θ are known, find the rectangular component of the velocity and position of the projectile as a function of time. And in the second part, we have been given the value of v_0 and θ.

So, given v_0 equal to $20\, m/s$ and $\theta = 30^0$ determine the maximum height h and distance r. In this question, we have to use the rectangular or Cartesian coordinate because in the problem statement itself, we have been asked to find out the rectangular component of the velocity. So, let us use the Newton's law in the Cartesian coordinate. So, we have $\sum F_x = ma_x$ and $\sum F_y = ma_y$.

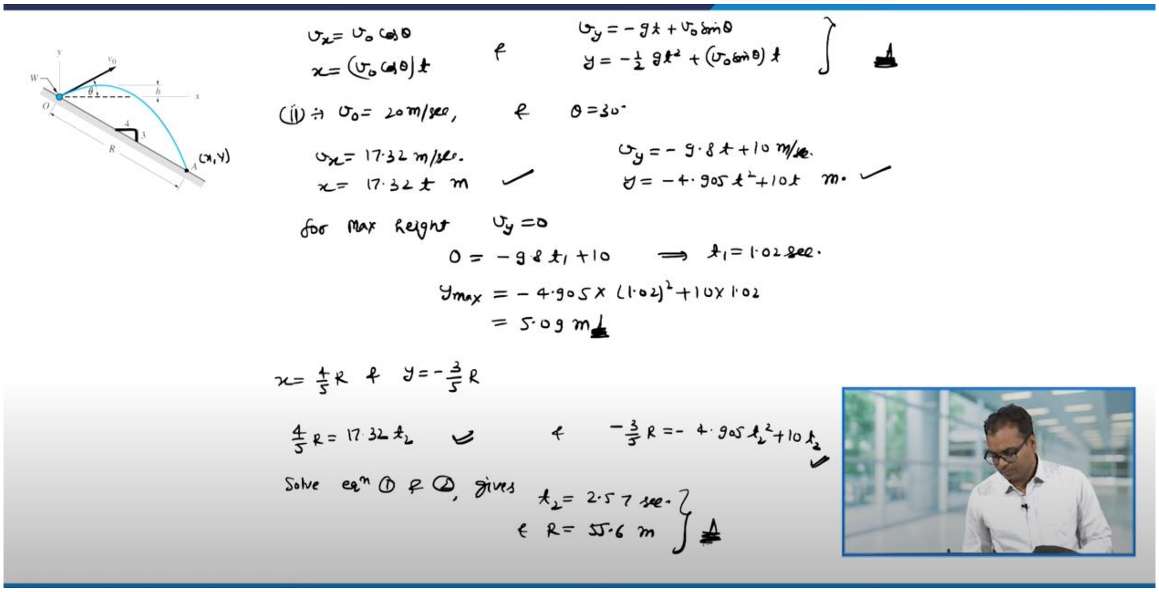
Note that there are no force acting along the x direction. The gravity acts along the y direction. Therefore, we have 0 equal to mass. The weight is given in the question. It is W. So, mass will be $\frac{W}{g} a_x$ and this gives you $a_x = 0$.

And $\sum F_y = ma_y$ gives you the weight will act in the minus y direction. So, $-W = \frac{W}{g} a_y$. This gives you $a_y = -g$. This is something that we already know that the acceleration will be g towards the ground and in the x direction there is no acceleration. Now, let us use $a_x = 0$. So, we have $a_x = 0$.

Therefore, v_x will be $\int a_x dt$ and since a_x is 0, it will be $0 + c_1$. Now, x will be $\int v_x dt$. So, it will be $c_1 dt$. So, that will be $c_1 t + c_2$. Similarly, a_y is given as$-g$. Therefore, v_y will be $\int a_y dt$. So, that will be$\int -gdt$. That will be$-gt + gt$. Constant, let us say c_3 and y will be $\int v_y dt$. So, that will be $\int gt + c_3 dt$ that will be$-\frac{1}{2}gt^2 + c_3 t + c_4$.

Now, we have to find out what are these constants c_1, c_2, c_3, and c_4 from the initial condition. Now, you know that at $t = 0$, $x = 0$ and y is also 0 because we have fixed our coordinate at $x = 0, y = 0$. And v_x at $x = 0$ is $v_0 cos\theta$ and v_y is $v_0 sin\theta$. From here, you can clearly see that c_1 will be $v_0 cos\theta$ and c_2 will be 0, c_3 will be $v_0 sin\theta$ and c_4 will be 0. Let us put in these equations. So, we get $v_x = v_0 cos\theta$ because v_x was c_1 and c_1 was $v_0 cos\theta$ and x comes out to be $v_0 cos\theta t$ and v_y is $-gt + v_0 sin\theta$. Let us look at v_y, v_y was $-gt + c_3$ and c_3 was $v_0 sin\theta$.

So, therefore, that becomes $-gt + v_0 sin\theta$ and y is $-\frac{1}{2}gt^2 + v_0 sin\theta t$. This is the expression for the velocity and position of the projectile as a function of time. Now, let us put the value $v_0 = 20\, m/s$ and $\theta = 30^0$ to determine the maximum height h and the distance r. So, let us put $v_0 = 20\, m/s$ and $\theta = 30^0$.



So, we get $v_x = v_0 cos\theta, v_0$ and θ is given. This comes out to be $17.32\ m/s$. Similarly, v_y comes out to be -9.8$t + 10\ m/s$ and x comes out to be $17 - 32t$ and y comes out to be$-4.905t^2 + 10t\ meter$. Now, in the question statement, we have been asked to find out the maximum height. We know that at the maximum height, its vertical velocity is going to be 0. So, let us put $v_y = 0$ and find out the time that it takes to reach the maximum height. So, let us put $v_y = 0$ for maximum height. We have $v_y = 0$.

So, we have 0 equal to -9.8. Let us say it takes a time t_1 to attain the maximum height. So, plus 10, this gives you $t_1 = 1.02$. Now, to find out the maximum height y_{max} , let us use that equation because now t is known. So, y_{max} becomes $-4.905 \times 1.02^2 + 10 \times 1.02$ and that gives you $5.09\ m$.

In the problem statement, we have been also asked to find out the distance r. So, therefore, let us say the coordinate of point A is x and y, then x and y I can write down in terms of r because this length is 4, then this length is 3, it is given. So, therefore, x will be $\frac{4}{5}R$ and y will be $-\frac{3}{5}R$. Now, let us put this value of x and y in this and this equation. So, we have $\frac{4}{5}R = 17.32t_2$ and let us say the time that it takes to reach at point A is t_2.

So, it will be t_2. Similarly$-\frac{3}{5}R = 4.905t_2^2 + 10t_2$. And these are two equations and there are two unknown R and t_2. So, you can solve equation number 1 and 2 and this gives you $t_2 = 2.57\ sec$ and $R = 55.6\ meter$.

So, with this, let me stop here. See you in the next class. Thank you.