MECHANICS
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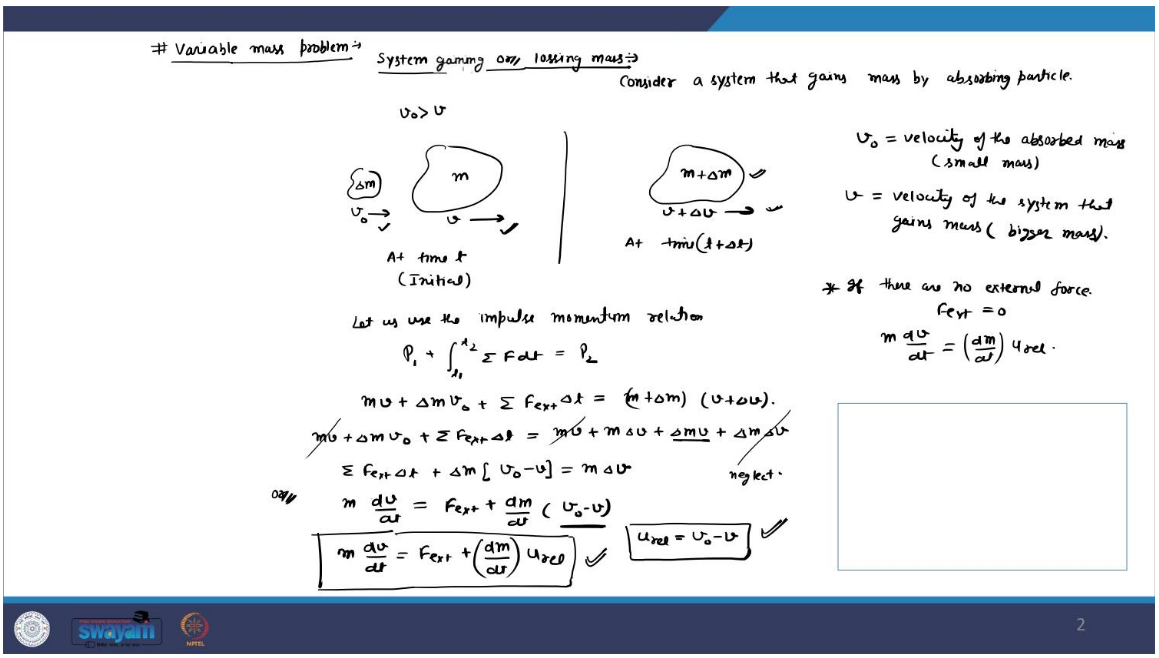
Lecture 40
Variable mass

Hello everyone, welcome to the lecture again. Today we are going to study the variable mass problem. So, herein we have a system which is either gaining the mass or losing the mass. Let us consider a system which is gaining the mass. So, we are going to derive the equation for that and the same equation is applicable for the system that is losing the mass. So, we are going to consider a system that gains mass let's say by absorbing particle.



So, let's say I have a mass m which is moving with velocity v and I have another mass which is let's say Δm moving with velocity v_0. This is the situation at time t, let us call it the initial time and at time $t + dt$, if your $v_0 > v$, then they combined. So, we have mass $m + \Delta m$ and it moves with a velocity $v + \Delta v$. Now, here note that v_0 is the velocity of the absorbed mass or the small mass or a small particle and v is the velocity of the system

that gains mass or you can call it bigger mass. Now, since all this thing happens in very small time dt, therefore, we can use the impulse momentum relation. So, let us use the impulse momentum relation So, it is following if P_1 is the initial momentum and the impulse is Fdt, then the final momentum is P_2. $P_1 + \int_{t_1}^{t_2} Fdt = P_2$. Now, here P_1 which is the initial momentum is the moment of this particle and that particle. So, we have $mv + \Delta m v_0$. So, this is the moment of the smaller and bigger particle initially plus $\sum F\, dt$ equal to final momentum is $(m + \Delta m)(v + \Delta v)$.

$$mv + \Delta m v_0 + \sum F_{ext} \Delta t = (m + \Delta m)(v + \Delta v)$$

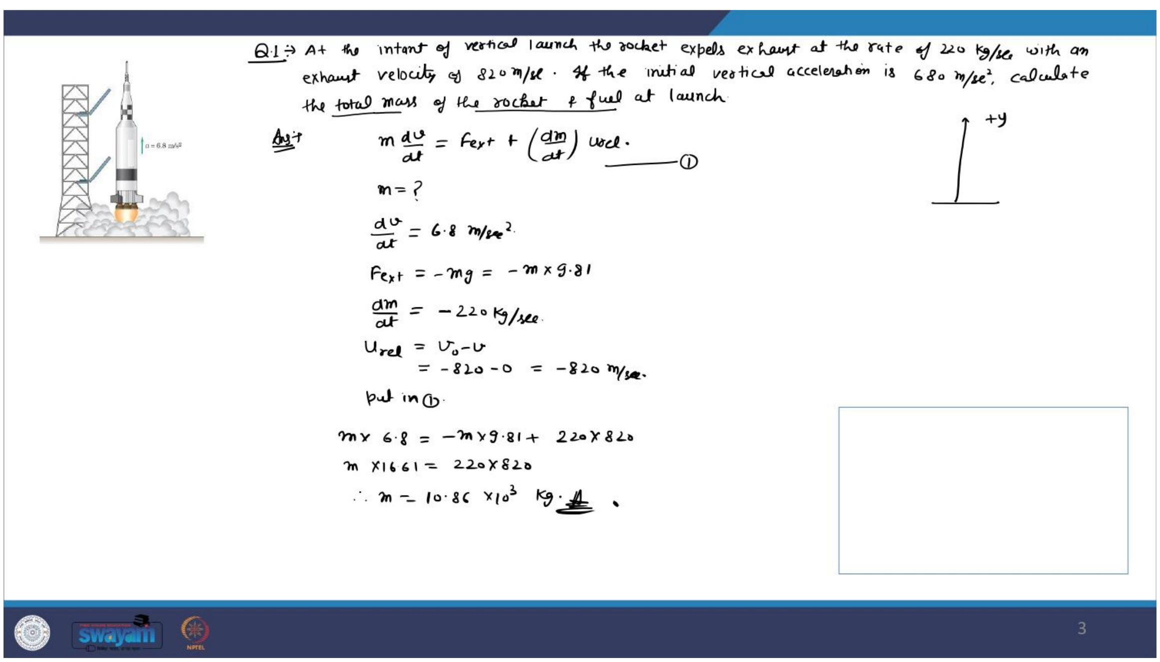
So, let us expand it. So, we have

$$mv + \Delta m v_0 + \sum F_{ext} \Delta t = mv + m\Delta v + \Delta m v + \Delta m \Delta v\ .$$

Now, since Δm and Δv, they both are small. So, therefore, their product will also be small. So, therefore, this we can neglect because this is second order in delta. So, let us neglect this term. Now, mv will get cancelled with this mv. So, therefore, I can write down $\sum F_{ext} \Delta t + \Delta m(v_0 - v) = m\Delta v$ or I can rewrite this as $m\frac{dv}{dt} = F_{ext} + \frac{dm}{dt}(v_0 - v)$. So, what I do is I divide it by Δt and then those small Δ I write down as d. Let us define this quantity u relative $u_{rel} = v_0 - v$. Then this equation I can rewrite as $m\frac{dv}{dt} = F_{ext} + \left(\frac{dm}{dt}\right) u_{rel}$. So, note that here $u_{rel} = v_0 - v$ and v_0 is the velocity of the small particle of the absorbed particle minus the velocity of the bigger particle. Now, let's say there are no external forces for example, gravity etcetera is not there. So, if there are no external force. In that case, $F_{ext} = 0$ and the same equation we can rewrite as

$$m\frac{dv}{dt} = \left(\frac{dm}{dt}\right) u_{rel}.$$

So, based on these concepts, now let us look at few problems. The first problem statement

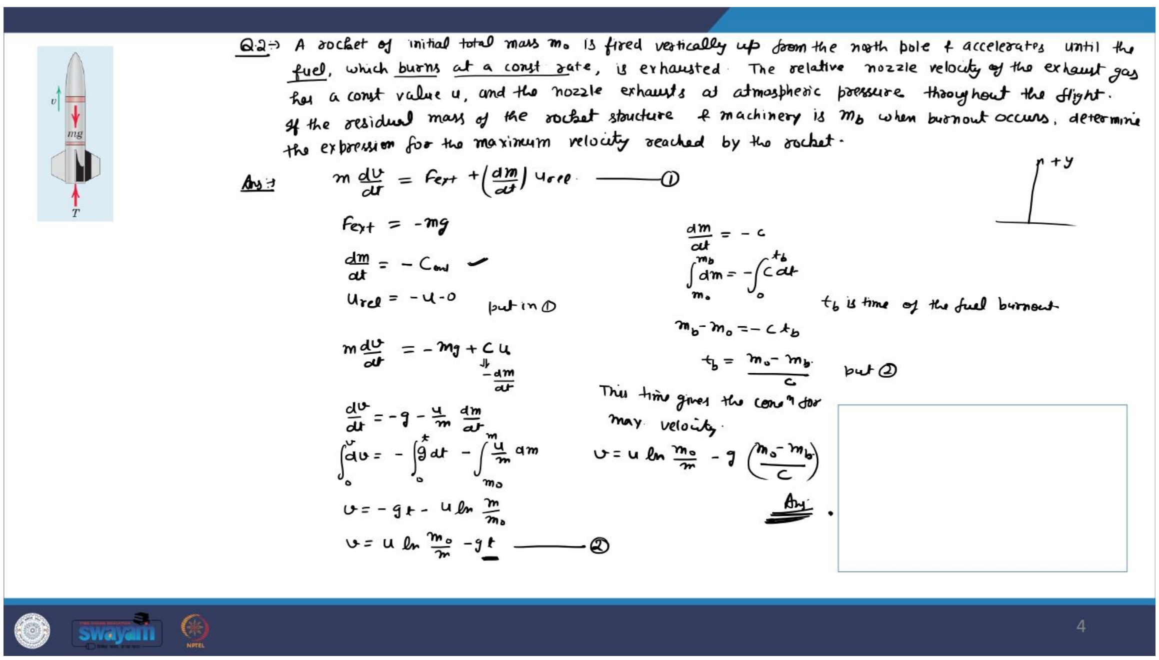


is following. At the instant of vertical launch, the rocket expels exhaust at the rate of

220 kg/s with an exhaust velocity of 820 m/s, if the initial vertical acceleration is 6.80 m/s^2, Calculate the total mass of the rocket and fuel at launch. So, clearly this is a variable mass problem. So, therefore, let us use this formula

$$m\frac{dv}{dt} = F_{ext} + \left(\frac{dm}{dt}\right) u_{rel} ------- (1).$$

In the question, we have been asked what is the total mass of the rocket and the fuel. So, therefore, m is something that we have to find out. The acceleration, the initial vertical acceleration is given. So, acceleration is $\frac{dv}{dt} = 6.8\ m/s^2$. The $F_{ext} = -mg$ and it is going to act downward. So, therefore, $F_{ext} = -m \times 9.81$ and $\frac{dm}{dt}$ is the rate at which the mass is changing. It is given that it is a 220 kg/s and since the mass is decreasing, so therefore, $\frac{dm}{dt} = -220\ kg/s$. Now, what is u_{rel}, u relative velocity? It was $u_{rel} = v_0 - v$. v_0 is the velocity of the small particle. Herein, it is exhaust and it is in the $-y$ direction. So, therefore, initially, the $v = 0$. So, therefore, $u_{rel} = -820 - 0 = -820\ m/s$. Let us put everything in equation number 1. So, we have $m \times 6.8 = -m \times 9.81 + 220 \times 820$. So, that gives $m \times 1660 = 220 \times 820$. This gives $m = 10.86 \times 10^3\ kg$.



Now, let us look at another problem statement. A rocket of initial total mass m_0 is fired vertically up from the North Pole and accelerates until the fuel which burns at a constant rate is exhausted. The relative nozzle velocity of the exhaust gas has a constant value u and the nozzle exhaust at atmospheric pressure throughout the flight. If the residual mass of the rocket structure and machinery is m_b. When burnout occurs, determine the expression for the maximum velocity reached by the rocket and it is given that neglect the atmospheric resistance and the variation of gravity with altitude. So, again here we can use the variable mass concept. So, for that we have the equation $m\frac{dv}{dt} = F_{ext} + (\frac{dm}{dt})\ u_{rel} - - - - - - - - (1)$. Now, here let us identify you know each and every term. So, we have F_{ext}. The external force is the gravity only which is going to act downward. So, let me take this as $+y$ direction. So, $F_{Ext} = -mg$. Now, $\frac{dm}{dt}$, it is given that the fuel burns at a constant rate. So, therefore, it will be minus because it is burning constant rate. Let us call it C or constant. $dmdt = -C$. Now, u_{rel} is is the relative velocity. So, the fuel velocity minus the velocity of the rocket, the fuel is going downward. So, minus u minus the velocity of the rocket initially it is 0. So, $u_{rel} = -u- = -u$. Let us put this in equation number 1. So, we have $m\frac{dv}{dt} = -mg + Cu$. And note that $C = -\frac{dm}{dt}$. So, I can write down as $\frac{dv}{dt} = -g - \frac{u}{m}\frac{dm}{dt}$. Therefore, $\int_0^v dv = -\int_0^t gdt - \int_{m_0}^{m}\frac{u}{m}dm$.

So, therefore, $v = -gt - u\,ln\left(\frac{m}{m_0}\right)$ or $v = u\,ln\left(\frac{m}{m_0}\right) - - - - - - - - (2)$.

Now, let us find out this t in terms of the given parameter. It is already given that $\frac{dm}{dt} = -C$. Let us integrate this. So, we have $\int_{m_0}^{m_b} dm = -\int_0^{t_b} C\,dt$. Now, the mass initially it is m_0 and later it becomes m_b when the complete burnout happens and let's say the time to complete burnout is t_b. So, here as I said t_b is the time of the fuel burnout. So, we get $m_b - m_0 = -Ct_b$ or $t_b = (m_0 - m_b)/C$. Let us put it in equation number 2. So, this time gives the condition for maximum velocity because that is when the complete burnout happens. So, let us put in 2. So, we have $v = u\ln\left(\frac{m_0}{m}\right) - g\left(\frac{m_0 - m_b}{C}\right)$.

Q.3 → The end of a chain of length L and mass ρ per unit length which is piled on a plateform is lifted vertically with a const velocity v by a variable force P. Find P as a fn of the height x of the end above the plateform.

$+x$

Ans:→ $m\frac{dv}{dt} = F_{ext} + \left(\frac{dm}{dt}\right)u_{rel}$ ——— (1)

v is const.

$\frac{dv}{dt} = 0$

$F_{ext} = P - \rho x g$

$m = \rho x$

$\therefore \frac{dm}{dt} = \rho\frac{dx}{dt} = \rho v$

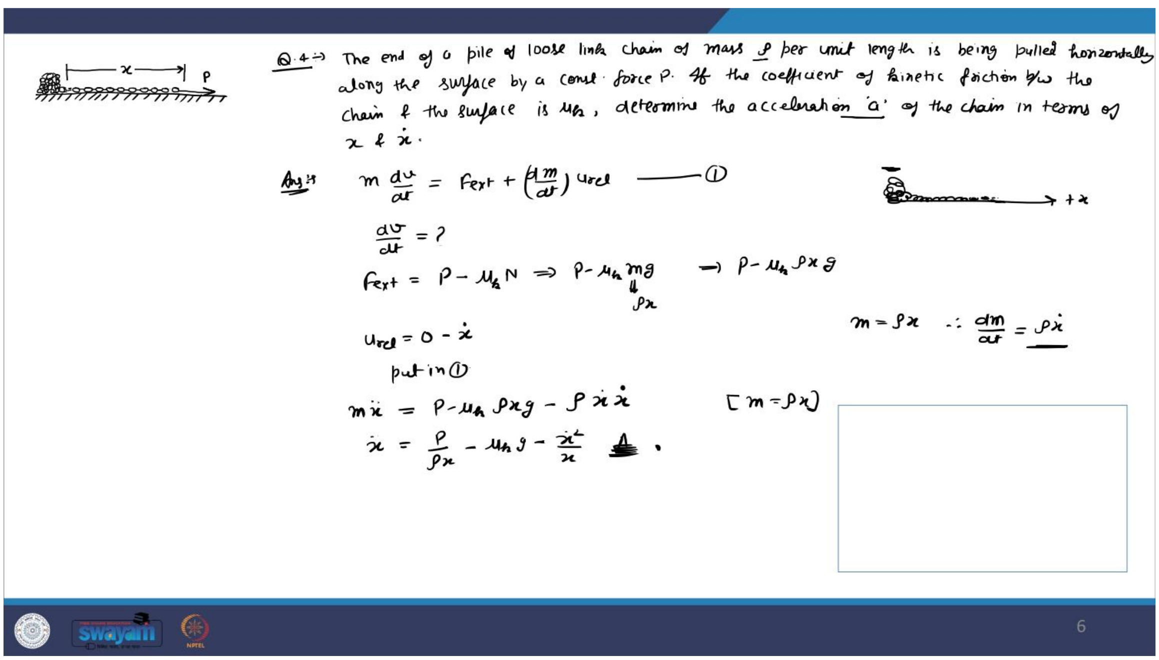
$u_{rel} = 0 - v = -v$

put in (1)

$0 = P - \rho x g - \rho v \cdot v$

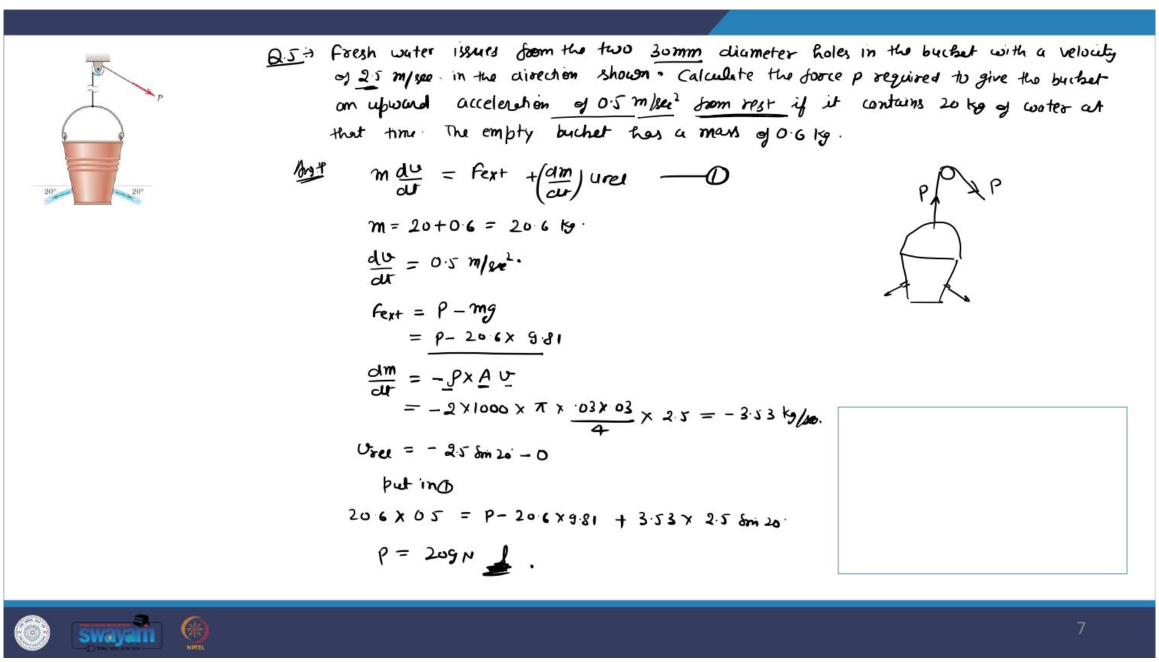
$\therefore P = \rho[gx + v^2]$ Ans.

Now, let us look at another problem statement. The end of a chain of length l and mass ρ per unit length which is piled on a platform is lifted vertically with a constant velocity v by a variable force P. Find P as a function of the height x of the end above the platform. So, again this is also a variable mass problem. Therefore, let us use the equation $m\frac{dv}{dt} = F_{ext} + (\frac{dm}{dt})\ u_{rel} -------- (1)$. Let us take this as positive x direction. And it is given that the chain is lifted vertically with a constant velocity v. So, it is given that v is constant. Therefore, the acceleration, which is $\frac{dv}{dt} = 0$. Now, let us look at the F_{ext} force. So, F_{ext} is P, it will be plus P because it is acting in $+x$ direction minus the weight of this chain, so which is mass times the g, so mass is ρx into g because ρ is the mass per unit length multiplied by the length will give you the total mass. $F_{ext} = P - \rho x g$ And since $m = \rho x$, this is your m, therefore $dmdt = \rho\frac{dx}{dt} = \rho v$ or where v is the velocity with which we are lifting it. Now, let us look at u_{rel}. So, u relative velocity, it is the velocity of the smaller mass. So, the smaller mass is kept on the ground on the platform. So, therefore, this is 0 minus the velocity of the bigger mass. It is moving with a velocity v. So, therefore, $u_{rel} = -v = -v$. Now, put everything in equation number 1. So, we have $0 = P - \rho x g - \rho v.v$. So, we have $P = \rho(gx + v^2)$.



Now, let us look at another problem on the same concept. So, question number 4, the end of a pile of loose link chain of mass rho per unit length is being pulled horizontally along the surface by a constant force P. If the coefficient of kinetic friction between the chain and the surface is μ_k, determine the acceleration a of the chain in terms of x and $\dot{x}$. So, obviously, it is a variable mass problem. So, therefore, we can use

$m\frac{dv}{dt} = F_{ext} + (\frac{dm}{dt})\ u_{rel}$ −−−−−−−− (1) and let us choose this as our $+x$ direction. So, in the question, we have been asked to find out the acceleration a. So, that means $\frac{dv}{dt}$, something that is asked in the question. Now, the F_{ext} will be P because P is acting in the plus x direction minus the frictional force which is $\mu_k N$ where N is the normal reaction force and it will be equal to mg. $F_{ext} = P - \mu_k N$. It is given that ρ is the mass per unit length. So, therefore, $m = \rho x$. So, therefore, $F_{ext} = P - \mu_k \rho x g$. So, as I said $m = \rho x$. Therefore, $\frac{dm}{dt} = \rho \dot{x}$. Now, let us look at u_{rel}, relative velocity. It is the velocity of the smaller mass minus the bigger mass. So, initially the smaller mass is kept into the rest. So, this is at rest. Therefore, it will be 0 minus the velocity of the bigger mass. It is moving with velocity v. So, therefore, $u_{rel} = 0 - \dot{x}$. Let us put everything in equation number 1. So, we have $m\ddot{x} = P - \mu_k \rho x g - \rho \dot{x}\dot{x}$. Therefore, $\ddot{x} = \frac{P}{\rho x} - \mu_k g - \frac{\dot{x}^2}{x}$.



Now, let us look at another problem. And the statement is following, fresh water issues from the two 30 mm diameter hole in the bucket with a velocity of 2.5 m/s in the direction shown. Calculate the force P required to give the bucket an upward acceleration of 0.5 m/s^2 from rest if it contains 20 kg of water at that time and it is given that the empty bucket has a mass of 0.6 kg. So, again let us use the variable mass equation

$m\frac{dv}{dt} = F_{ext} + (\frac{dm}{dt})\ u_{rel}$ −−−−−−−− (1). Here m is given, m is the total mass. So, 20 kg is the water and 0.6 is the mass of the bucket. So, therefore, it will be $m = 20 +$

$0.6 = 20.6\,kg$. $\frac{dv}{dt}$ is also given. It accelerate with a velocity of $0.5\,m/s^2$. So, $\frac{dv}{dt} = 0.5\,m/s^2$. Now, let us calculate F_{ext}. So, we have the situation, we have this bucket and then it is pulled by this rope and the force is P. The same force is going to act over here. So, therefore, $F_{ext} = P - mg$, where m is the mass of the water plus the bucket. So, it will be $F_{ext} = P - 20.6 \times 9.81$. Now, let us look at $\frac{dm}{dt}$, the rate at which the mass is changing. It will be minus because the water is coming out. Density times the volume and the volume is Av where A is the area, v is the velocity and ρ is the density. $\frac{dm}{dt} = -\rho \times Av$. Since there are two holes, so it will be $-2 \times 1000 \times \pi \times 0.03 \times \frac{0.03}{4} \times 2.5 = -3.53\,kg/s$. Now, let us calculate the v_{rel} along the y direction. So, along the y direction, the velocity of the water is $-2.5 sin20°$ minus the velocity of the bigger mass which is bucket here. Note that initially the bucket was at rest. So, it is given that it is rest. So, therefore, minus 0. $v_{rel} = -2.5 sin20° - 0$. Now, let us put everything in equation number 1. So, we have $20.6 \times\ 0.5 = P - 20.6 \times 9.81 + 3.53 \times 2.5 sin20°$. This gives P=209 N.

With this, let me stop here. See you in the next class. Thank you.