MECHANICS
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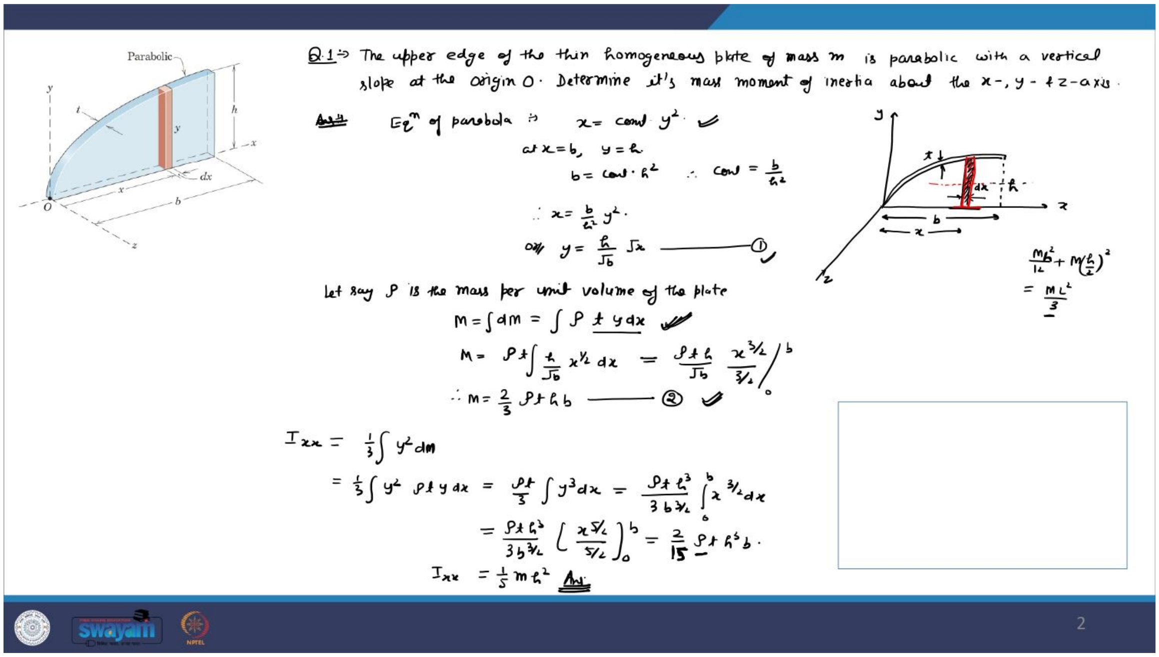
Lecture 44
Moment of inertia: examples

Hello everyone, welcome to the lecture again. In the last class, we learned about the definition of moment of inertia, moment of inertia of some standard bodies, the parallel axis theorem and the perpendicular axis theorem. Today, we are going to utilize those concepts and going to solve some examples wherein we will find the moment of inertia.



So, this is the first problem statement. The upper edge of the thin homogeneous plate of mass m is parabolic with a vertical slope at the origin O, determine its mass moment of inertia about the x, y and z-axis. So, here we have this x-axis, y-axis and the z-axis. We have this plate which is parabolic. It has a thickness of t. So, t is the thickness and the height is h. So, to find out the moment of inertia, let us take a portion of this plate which is rectangular in shape. Let's say it is at a distance of x and the width of this plate is dx. So, from the equation because this is the equation of the parabola, we know that how x varies as a function of y. So, this will be $x = constant. y^2$. Now, this constant also we can find out from the boundary condition. It is given that if $x\ is\ equal\ to\ b$, then y becomes

h. So, at $x = b$, we have $y = h$. Let us put it over here. So, we have $b = constant.h^2$ and this gives that $constant = \frac{b}{h^2}$. So, let us put it above.

So, we get $x = \frac{b}{h^2}y^2$ or $y = \frac{h}{\sqrt{b}}\sqrt{x}$ ------(1)

So, this is how y and x varies. Now, let us say that the mass per unit volume of this plate is ρ and let us find out then what will be the mass of this object. So, let us say that ρ is the mass per unit volume of the plate. Then, total mass $m = \int dm$. Now, if the mass per unit volume is ρ, then we have to multiply this ρ by the volume to get the mass. So, the volume of this plate will be $tydx$ because ydx is the area, t is the thickness. So, you multiply this to get the volume and then multiply by ρ to get the mass. So, therefore, mass m will be ρ is constant, t is also constant and let us write down this y in terms of x to find out this integral. So, the value of y, I can put from equation number 1, it is $\frac{h}{\sqrt{b}}\sqrt{x}$

and this will be, $m = \rho t \int \frac{h}{\sqrt{b}}\sqrt{x}dx = \frac{\rho th}{\sqrt{b}} \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \Big|_0^b$

Therefore, $m = \frac{2}{3}\rho thb$ ------(2)

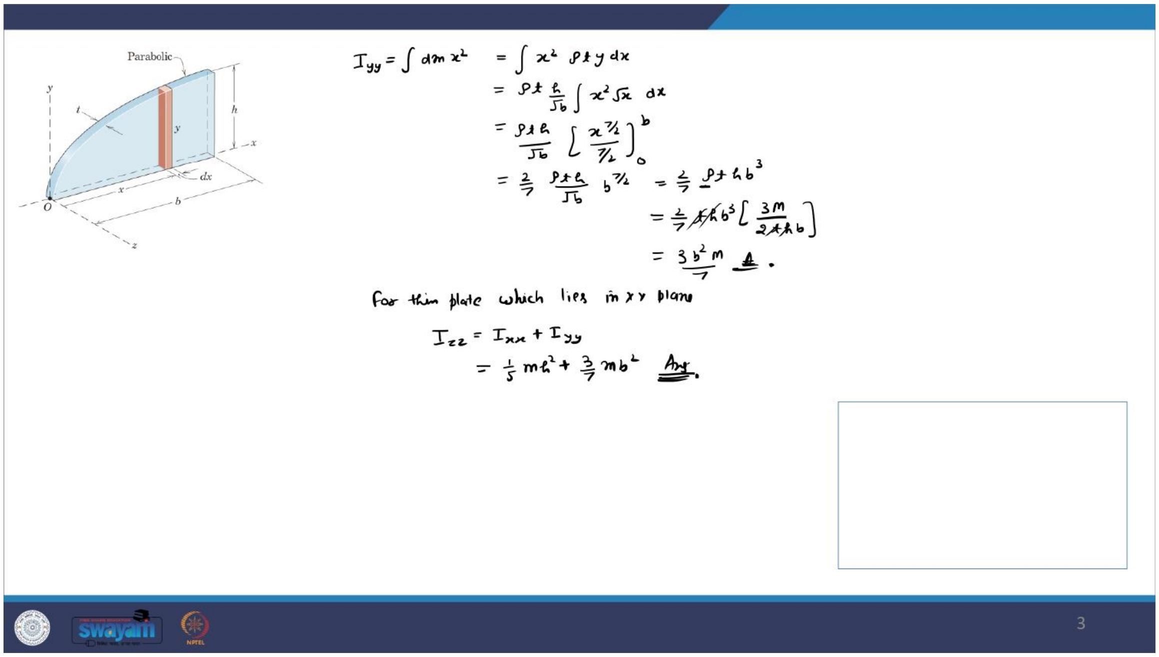
So, now we have the mass. Let us find out what is the moment of inertia about the x-axis first. So, about the x-axis, I am writing it as I_{xx}. That means the moment of inertia about the x-axis. So, to find out the moment of inertia of the x-axis, we have to find out what is the moment of inertia of this part about the x-axis. Now, this is a rod and we have in the last class we have find out what is the moment of inertia of a rod about you know one end. So, there we can use the parallel axis theorem. We know the moment of inertia about this point is $\frac{ml^2}{12} + m\left(\frac{h}{2}\right)^2$. So, Let me just remind you. So, the moment of inertia about this line was $\frac{ml^2}{12} + mx^2$. So, $x = \frac{h}{2}$. So, this will also be h because there is no l, there is h only. So, this will give you $\frac{ml^2}{3}$. So, herein we have used the parallel axis theorem. This is the moment of inertia of this part about the x-axis. So, let me write it down. So, it will be

$$I_{xx} = \frac{1}{3}\int y^2 dm$$

$$= \frac{1}{3}\int y^2 \rho tydx = \frac{\rho t}{3}\int y^3 dx = \frac{\rho th^3}{3b^{\frac{3}{2}}}\int_0^b x^{\frac{3}{2}}\,dx$$

$$= \frac{\rho th^3}{3b^{\frac{3}{2}}}\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right)_0^b = \frac{2}{15}\rho th^3 b$$

So, this is the answer. Note that here, In the question statement, ρ is not given, the mass is given. So, this value of ρ also I can put from equation number 2, okay. So, I can eliminate ρ in terms of m from equation number 2 . So, if you do that, then you get $I_{xx} = \frac{1}{5}mh^2$.



Now, let us find out the moment of inertia about the y-axis. So,

$$I_{yy} = \int dmx^2 = \int x^2 \rho t y dx$$

So, this will be $= \rho t \frac{h}{\sqrt{b}} \int x^2 \sqrt{x} dx$

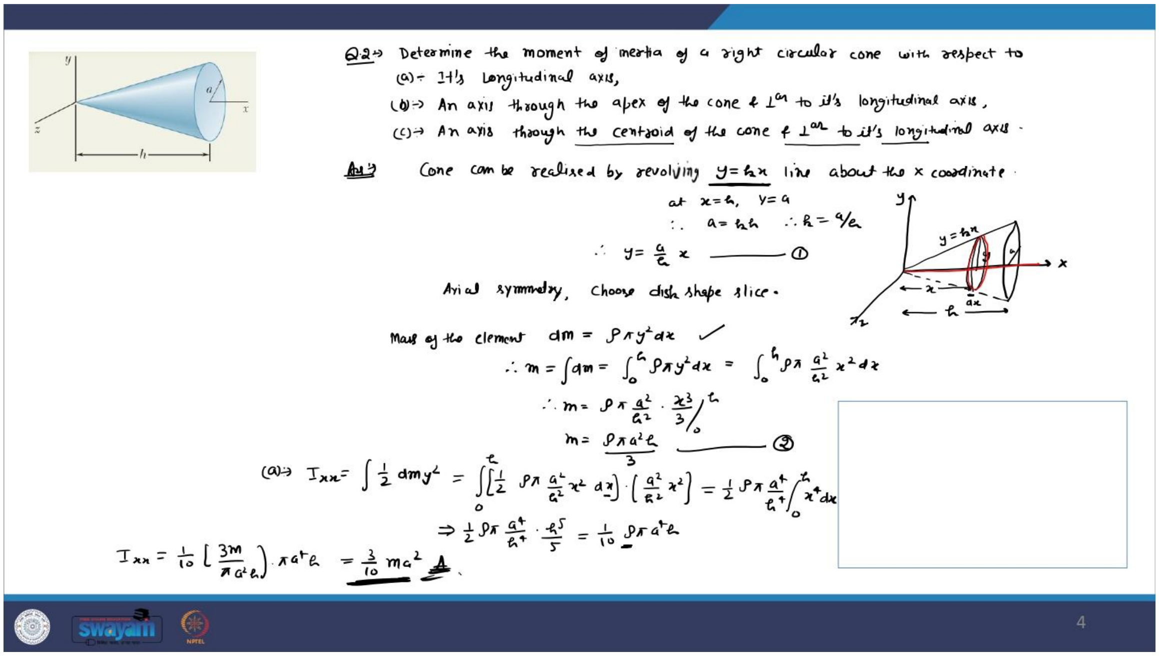
So, $I_{yy} = \frac{\rho th}{\sqrt{b}} \left(\frac{x^{\frac{7}{2}}}{\frac{7}{2}} \right)_0^b = \frac{2}{7} \frac{\rho th}{\sqrt{b}} b^{\frac{7}{2}} = \frac{2}{7} \rho thb^3$

Now, again here I have ρ. The value of ρ I can put from equation number 2. So, $\rho = \frac{3m}{2tbh}$. So, let me put it over here. So, this is equal to $\frac{2}{7} thb^3 \left(\frac{3m}{2thb} \right)$. So, t will get cancelled, h will also get cancelled and we will have $\frac{3b^2m}{7}$. So, this is the moment of inertia about the y-axis. Now, let us look at the moment of inertia about the z-axis. Now, note that this plate, it lies in the xy plane. Therefore, the moment of inertia about the z-axis will be the sum of the moment of inertia about the x and the y-axis. So, for thin plate, which lies in xy plane, the moment of inertia about the z -axis will be

$$I_{zz} = I_{xx} + I_{yy}$$

$$I_{zz} = \frac{1}{5} mh^2 + \frac{3}{7} mb^2$$

Now, let us look at another problem, determine the moment of inertia of a right circular cone with respect to (a) its longitudinal axis, (b) an axis through the apex of the cone and perpendicular to its longitudinal axis and (c) an axis through the centroid of the cone and perpendicular to its longitudinal axis. So, herein we can clearly see that how a cone is formed. A cone can be realized by revolving $y = kx$ line about the x coordinate.



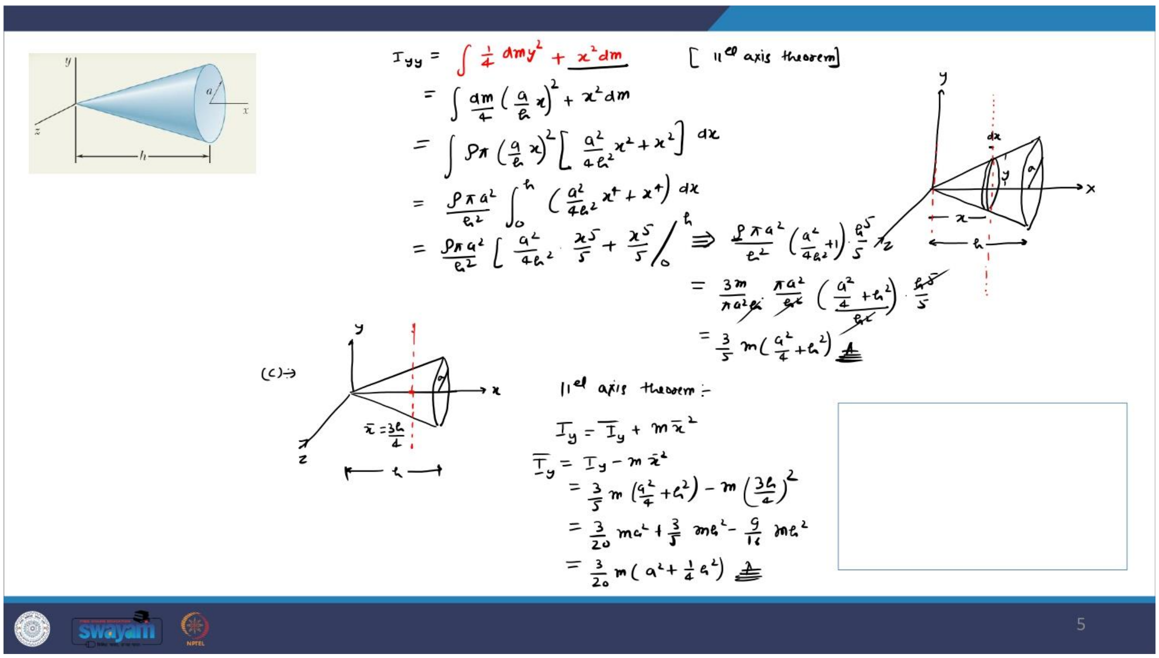
So, for example, I have the axis x and y and if I have $y = kx$ line, you rotate it about the x-axis, then you get the cone, okay. So, you have the x-axis, the y-axis and the z-axis, you have a line, let us say $y = kx$, you rotate it about the x-axis and you will get this cone. Herein, the boundary conditions are given that the height of the cone is h and its radius is a. So, therefore, I can find out what is the value of k because here at $x \, equal \, to \, h$, y should be a. So, at $x = h$, $y = a$. Therefore, $a = kh$ and therefore, $k = a/h$. So, therefore, we have the equation of the line that form the cone and it is

$y = \frac{a}{h}x ------(1)$.

Now, to find out the moment of inertia of this, we have to look for the symmetry and clearly, there is axial symmetry. So, therefore, let us choose a disc and let's say at some x, it's you know, diameter is $2y$ or radius is y and the thickness of the disc is dx. So, as I said, because there is a axial symmetry, I choose disc shape slice. Now, let us find out what is the moment of inertia of this disc and then we will integrate it from 0 to you know x which is h here. So, the mass of this element $d\,m$ is the density times the volume. So, density is let us say ρ and the volume is area multiplied by the thickness. So, area is $\pi y^2 dx$. This implies $dm = \rho\pi y^2 d$ Therefore, the mass $m = \int dm = \int_0^h \rho\pi y^2 dx$. And this integral I can find out. So, for that let me first put the value of y. So, I will have $\int_0^h \rho\pi \frac{a^2}{h^2} x^2 dx$. So, therefore,

$m = \rho\pi \frac{a^2}{h^2} \frac{x^3}{3}\Big|_0^h$.Therefore, $m = \frac{\rho\pi a^2 h}{3} ------(2)$.

Now, the first part, we have to find out the moment of inertia about the longitudinal axis which is the x-axis here. So, I_{xx}, it will be integral moment of inertia of this disc about the x-axis is $\frac{1}{2}my^2$. So, it will be $I_{xx} = \int \frac{1}{2}dmy^2$. So, let me put the value. So, it is $\int_0^h \left(\frac{1}{2}\rho\pi\frac{a^2}{h^2}x^2dx\right)\left(\frac{a^2}{h^2}x^2\right)$. So, this I can rewrite as $\frac{1}{2}\rho\pi\frac{a^4}{h^4}\int_0^h x^4dx$. And this integral is $\frac{1}{2}\rho\pi\frac{a^4}{h^4}\frac{h^5}{5} = \frac{1}{10}\rho\pi a^4h$.So, this is the answer, but here I have ρ, ρ I can replace with equation number 2 . So, $\rho = \frac{3m}{\pi a^2h}$. So, this I can put it back. So, I get $I_{xx} = \frac{1}{10}\left(\frac{3m}{\pi a^2h}\right)\pi a^4h = \frac{3}{10}ma^2$.



Now, let us find out the moment of inertia about the y-axis. So, let me make this figure again. We have the x-axis, the y-axis and the z-axis. We have this cone and then we took this disc of thickness dx and its height was y and the radius of this cone at the end was a and this was h. So, this is the situation that we have. Now, we want to find out the moment of inertia about the y-axis. So, to find out the moment of inertia about the y-axis, I have to first find out what is the moment of inertia of this disc about this axis and then I can use the parallel axis theorem. So, the moment of inertia of this disc about this line is $\frac{1}{4}dmy^2$. And plus we have to find out what is the moment of inertia about this axis. So, I have to add you know x^2dm here because this distance is x. So, therefore, $I_{yy} = \frac{1}{4}dmy^2 + x^2dm$. So, note that here I have used the parallel axis theorem. So, this comes from the parallel axis theorem. Now, let us solve it. So, we have dmy^2. $y^2 = \left(\frac{a}{h}x\right)^2$. So, $\int\frac{dm}{4}\left(\frac{a}{h}x\right)^2 +$

$x^2 dm$. So, this I can write it as $\int \rho\pi \left(\frac{a}{h}x\right)^2 \left(\frac{a^2}{4h^2}x^2 + x^2\right) dx$. So, this I can solve now. It is $\frac{\rho\pi a^2}{h^2}\int_0^h \left(\frac{a^2}{4h^2}x^4 + x^4\right) dx$ and this will be $\frac{\rho\pi a^2}{h^2}\left(\frac{a^2}{4h^2}\frac{x^5}{5} + \frac{x^5}{5}\right)_0^h$ and this will be equal to $\frac{\rho\pi a^2}{h^2}\left(\frac{a^2}{4h^2} + 1\right)\frac{h^5}{5}$. Now, again this value of ρ I can write down in terms of m. So, let us do that. So, it will be $\rho = \frac{3m}{\pi a^2 h}$. So, $\frac{3m}{\pi a^2 h}\frac{\pi a^2}{h^2}\frac{\left(\frac{a^2}{4}+h^2\right)}{h^2}\frac{h^5}{5}$. So, we get $\frac{3}{5}m(\frac{a^2}{4} + h^2)$. This is the moment of inertia about the y-axis. In the third part, so part c, we have been asked to find out the moment of inertia through the centroid which is perpendicular to the longitudinal axis. So, that means we have to find out the moment of inertia. So this was x-axis, y-axis and the z-axis. We have this cone and we have to find out its moment of inertia about the centroid which is perpendicular to the longitudinal axis. So, we have to find out the moment of inertia about this red line. So, note that the centroid in the cone is situated at a distance of $\frac{3h}{4}$ if h is the height of the cone. Now, to find out this moment of inertia, we can use the parallel axis theorem because we have already calculated what is the moment of inertia about the y-axis. So, the moment of inertia about the y-axis is the moment of inertia about this red line.

So, let us call it $\overline{I_y}$ plus m into the distance. So, $\bar{x}$. So, all of them we know. So,

$$I_y = \overline{I_y} + m\bar{x}^2$$

$$\overline{I_y} = I_y - m\bar{x}^2$$

$$= \frac{3}{5}m\left(\frac{a^2}{4} + h^2\right) - m\left(\frac{3h}{4}\right)^2$$

$$= \frac{3}{20}ma^2 + \frac{3}{5}mh^2 - \frac{9}{16}mh^2$$

$\overline{I_y} = \frac{3}{20}m\left(a^2 + \frac{1}{4}h^2\right)$. So, this is the answer.

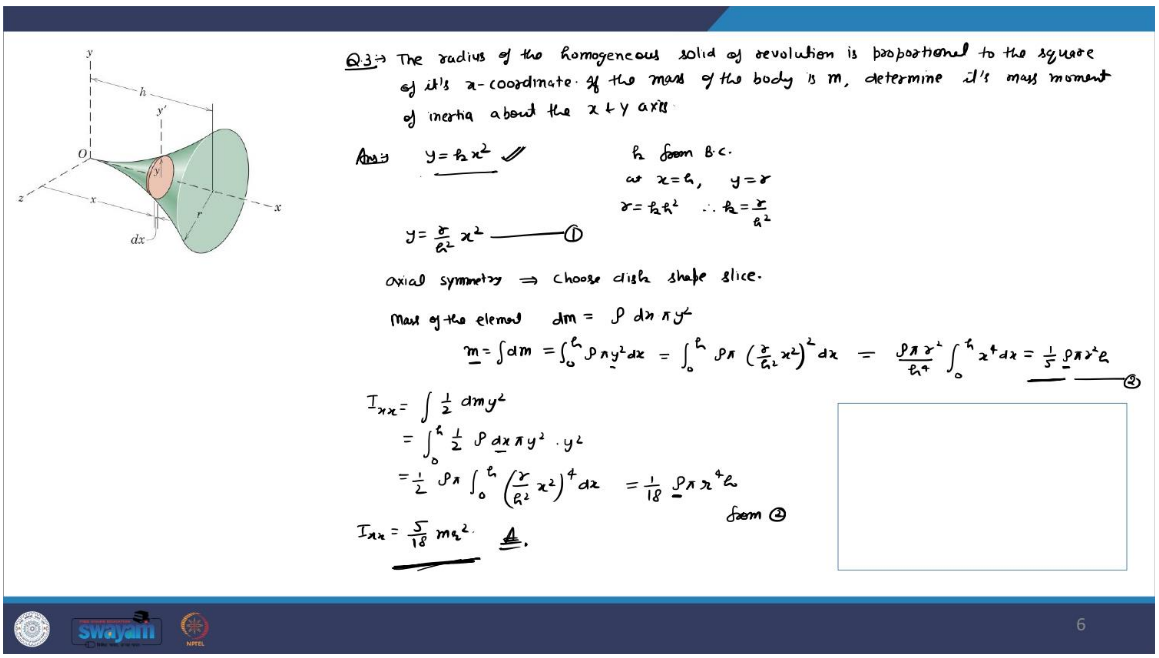
Now, let us look at one more problem. The radius of the homogeneous solid of revolution is proportional to the square of its x coordinate if the mass of the body is m, determine its mass moment of inertia about the x and y-axis. In this question, it is given that the y varies as the square of the x coordinate. So, let us take $y = kx^2$ and k, I can find out from the boundary condition. It is given that at $x = h$, $y = r$. Let us put it in the equation. So, we have $r = kh^2$. Therefore, $k = \frac{r}{h^2}$. So, if I put it over here, then I get

$y = \frac{r}{h^2}x^2 - - - - - (1)$.

This tells me how y varies as a function of x. Now, to find out the moment of inertia of this, I have to choose some element and because of the axial symmetry. So, this structure is symmetrical about the x-axis. Therefore, let us choose disk shape slice. And the mass of this element will be let us call it dm.

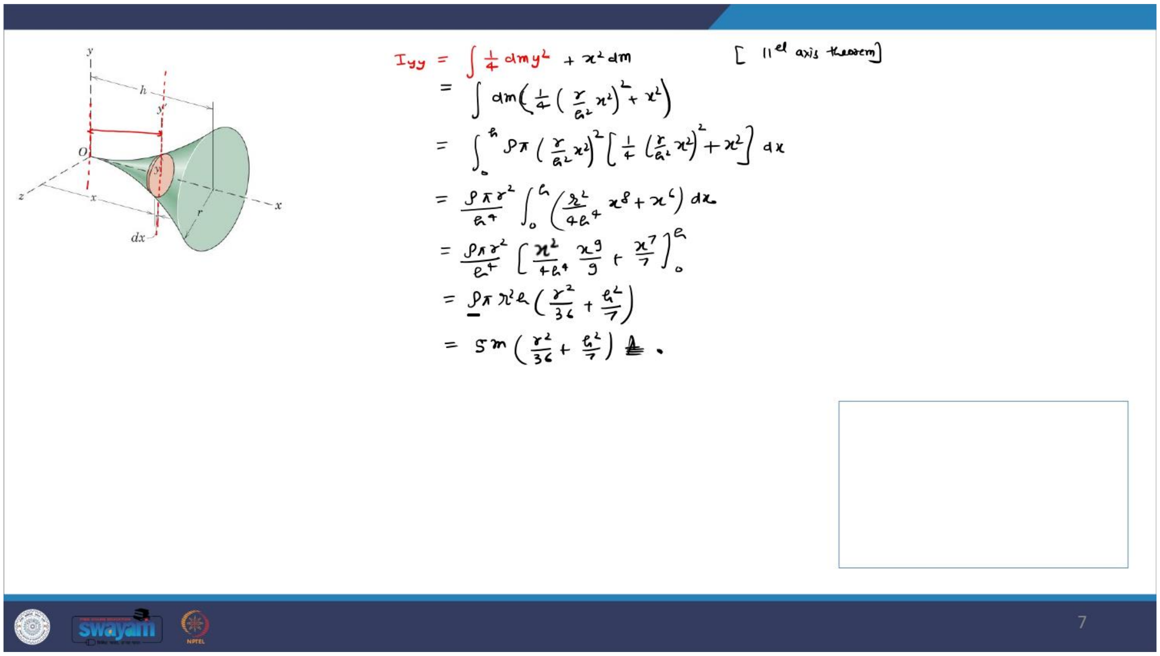
So, that will be again the ρ times the volume and the volume will be dx times the area which is πy^2. This implies $dm = \rho dx \pi y^2$. Therefore, the total mass of this object will be $m = \int dm = \int_0^h \rho \pi y^2 dx,$ let us put the value of y from equation number 1. So, that will be $\int_0^h \rho\pi \left(\frac{r}{h^2}x^2\right)^2 dx$. So, here $\frac{\rho\pi r^2}{h^4}\int_0^h x^4 dx$. $m = \frac{1}{5}\rho\pi r^2 h - - - - - (2)$

This we can use to convert ρ to m. Now, the moment of inertia of this thin disc about the x-axis will be $\frac{1}{2}dmy^2$ because dm is the mass of this disc and y is its radius and the x-axis passes through its center and perpendicular to the disc. So, therefore, it will be $I_{xx} = \int \frac{1}{2}dmy^2$. Let us put the value of dm.



$dm = \rho dx \pi y^2$. Now, we have to integrate this with respect to x. So, therefore, let us put the value of y also. This implies $\int_0^h \frac{1}{2}\rho dx \pi y^2\, y^2$ So, that will be $\frac{1}{2}\rho\pi \int_0^h \left(\frac{r}{h^2}x^2\right)^4 dx$. And this comes out to be $\frac{1}{18}\rho\pi r^4 h$. Now, we can put the value of ρ from equation number 2. So, from 2, we get $I_{xx} = \frac{5}{18}mr^2$.

Now, let us find out the moment of inertia about the y-axis. So, I_{yy}. For that, we can first write down the moment of inertia about this axis and then use the parallel axis theorem to find out the moment of inertia about the y-axis. So, the moment of inertia about the diameter is equal to integral $\frac{1}{4}dmy^2$ and then we can use the parallel axis theorem. So, plus $x^2 dm$. Again, it comes from the parallel axis theorem. So, now we have to just put the values.



So, we have

$$I_{yy} = \int \frac{1}{4} dmy^2 + x^2 dm$$

$$= \int dm \left(\frac{1}{4} \left(\frac{r}{h^2} x^2 \right)^2 + x^2 \right)$$

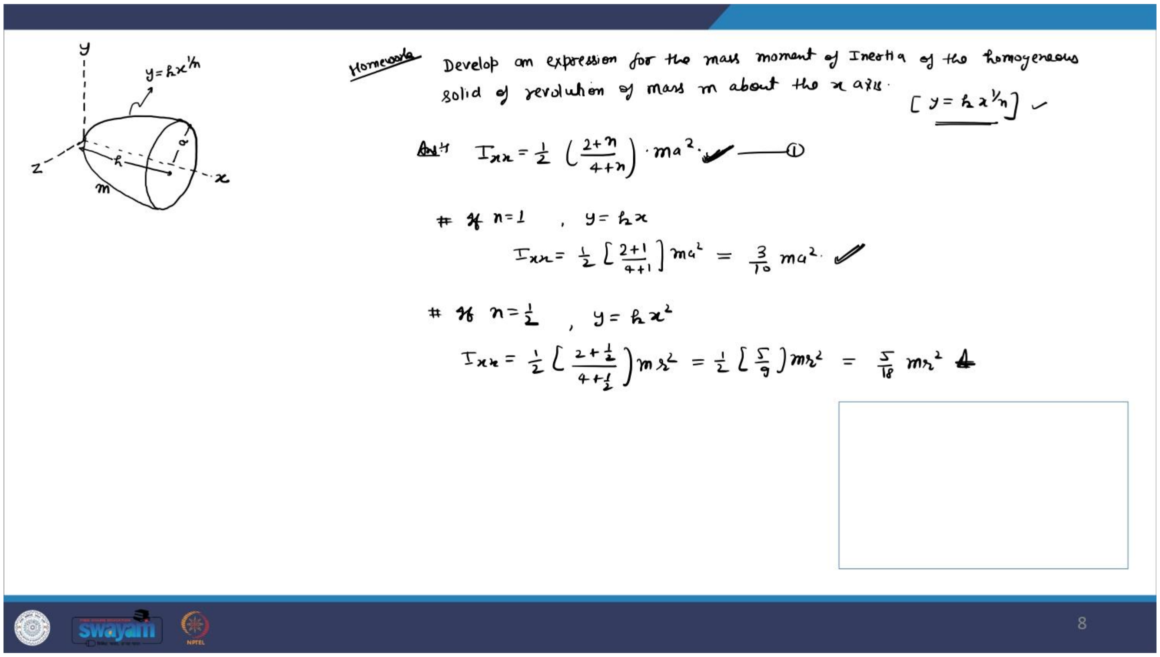
$$= \int_0^h \rho\pi \left(\frac{r}{h^2} x^2 \right)^2 \left(\frac{1}{4} \left(\frac{r}{h^2} x^2 \right)^2 + x^2 \right) dx$$

$$= \frac{\rho\pi r^2}{h^4} \int_0^h \left(\frac{r^2}{4h^2} x^8 + x^6 \right) dx$$

$$= \frac{\rho\pi r^2}{h^4} \left(\frac{x^2}{4h^4} \frac{x^9}{9} + \frac{x^7}{7} \right)_0^h$$

$$= \rho\pi r^2 h \left(\frac{r^2}{36} + \frac{h^2}{7} \right)$$

$$= 5m \left(\frac{r^2}{36} + \frac{h^2}{7} \right)$$



Now, let me give you a homework problem, and the statement is following, develop an expression for the mass moment of inertia of the homogeneous solid of revolution of mass m about the x-axis and the equation is given. Equation is $y = kx^{\frac{1}{n}}$. Its answer is $I_{xx} = \frac{1}{2}\left(\frac{2+n}{4+n}\right) ma^2 ------(1)$.

So, you can take it as a homework problem and you can find out the moment of inertia about x-axis and it will be this. Now, let us say if $n = 1$, in that case, this equation becomes $y = kx$ and the moment of inertia about the x-axis for this case $I_{xx} = \frac{1}{2}\left(\frac{2+1}{4+1}\right) ma^2 = \frac{3}{10}ma^2$. This was our second problem. So, in the second problem, we find out the moment of inertia for a structure which is made by revolving $y = kx$ about the x-axis. So, here the moment of inertia about the x-axis was $\frac{3}{10}ma^2$. This is exactly what we have. Now, let us look at the case if $n = \frac{1}{2}$. In that case, the equation becomes $y = kx^2$. This was our third problem. So, herein, I have $y = kx^2$ and the moment of inertia about the x-axis was $\frac{5}{18}mr^2$. So, let us see that. Let us put n equal to half in equation number 1. So, I have $I_{xx} = \frac{1}{2}\left(\frac{\left(2+\frac{1}{2}\right)}{\left(4+\frac{1}{2}\right)}\right)mr^2$. So, here a is r. So, that is equal to $\frac{5}{18}mr^2$. This is exactly what we got when the equation was kx^2.

With this, let me stop here. See you in the next class. Thank you.