MECHANICS

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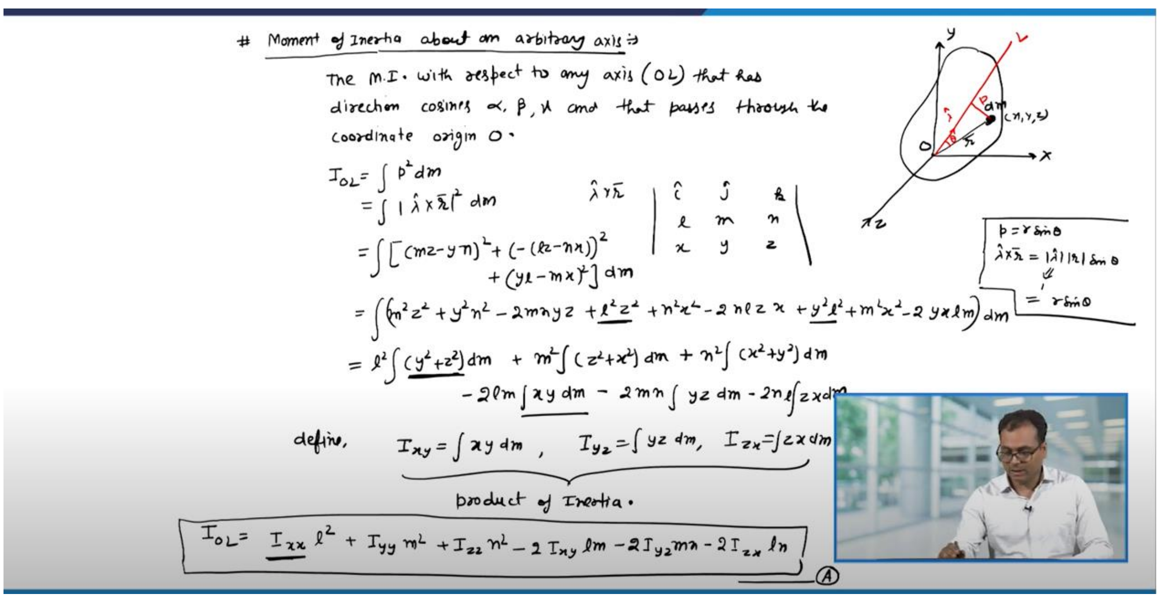
Lecture: 46

Product of inertia and principal axes of inertia

Hello everyone, welcome to the lecture again. So far, we have seen how we can find out the moment of inertia about the $x,\ y$ and $z-axis$. Today, we are going to learn how we can find out the moment of inertia about an arbitrary axis. For that, we require the concept of product of inertia. We will also learn what is principal axis of inertia.



So, moment of inertia is about an arbitrary axis. Let us say I have a rigid body and let us fix some point O, then these are the x-axis, y-axis and the z-axis. Let us take a small mass dm and let us say this mass dm , its position vector is r and of course, in Cartesian coordinate, its coordinates are $x,\ y$ and z , okay. Let us say, I want to find out the moment of inertia of this rigid body about the axis OL, okay.

To find out the moment of inertia, let us say that the unit vector along OL is $\hat{\lambda}$ and this $\hat{\lambda}$ is making an angle θ with $\vec{r}$ and the perpendicular distance from mass dm to OL is p. Then, the moment of inertia with respect to any axis, okay? Here in that axis is OL and let us say its direction cosines are known because I know at what angle this axis is. So, that is given to me.

So, that has direction cosine, let us say α , β and γ and this axis passes through the coordinate origin O. Now, the moment of inertia of this rigid body about OL will be $\int p^2\, dm$ and from the geometry, this P I can write down as $|\hat{\lambda} \times \vec{r}|^2 dm$. You can see it clearly because again from the geometry my p will be $r\, sin\theta$ and $\lambda \times r$ will also be $r\, sin\theta$ because $\hat{\lambda} \times \vec{r}$ is $\hat{\lambda}\, r$ and then the angle between them. So, $sin\theta$ and this is equal to 1 because that is a unit vector.

So, we again get $rsin\theta$. So, now this $\hat{\lambda}$ has a direction cosine of l, m and n and this is just a vector product. So, let us find out what is $\hat{\lambda} \times r$. So, $\hat{\lambda} \times r$ will be i, j and k, l, m and n and $\vec{r}$ is x, y and z . So, this will be equal to $\int \left[(mz - yn)^2 + \left(-(lz - nx)\right)^2 + (yl - mx)^2\right] dm$ because there is mode.

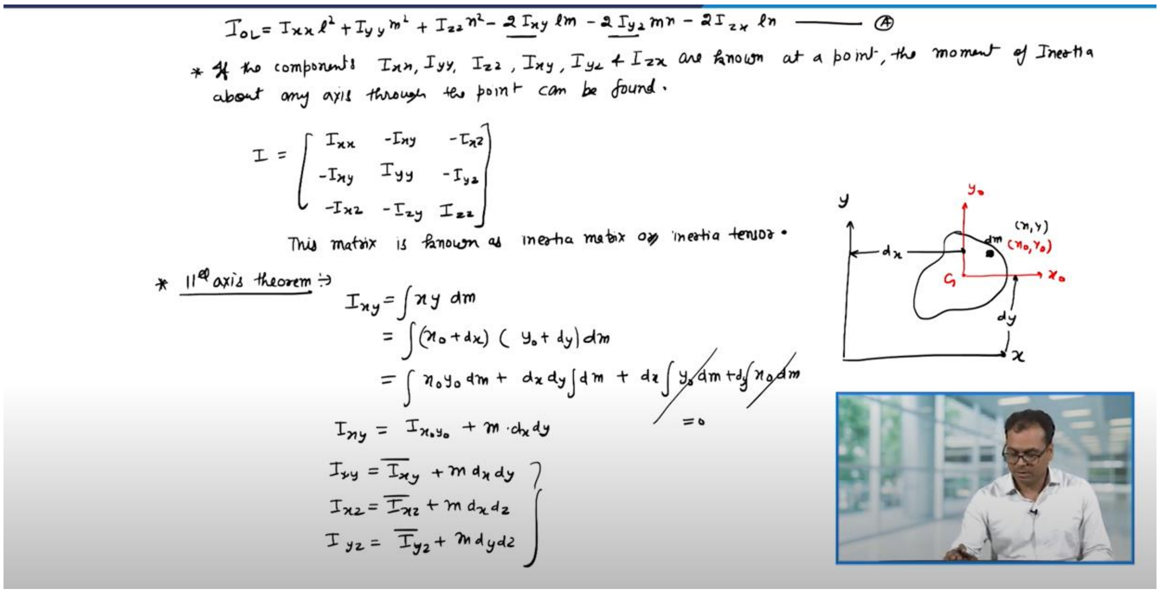
So, there will be this plus sign. Now, let us expand this. So, we have $\int (m^2z^2 + y^2n^2 - 2mnyz + l^2z^2 + n^2x^2 - 2nlzx + y^2l^2 + m^2x^2 - 2yxlm)dm$. Now, let us collect the coefficient of l^2, m^2, n^2 and note that the l, m and n are constant.

So, we can take it outside the integral because the orientation of the OL is fixed. So, therefore, $l, m\ and\ n$ are constant. So, let me collect the coefficient of l^2. I have taken it outside the $m^2 \int (z^2 + x^2)dm + n^2 \int (x^2 + y^2)dm - 2lm \int xydm - 2mn \int yzdm - 2nl \int zxdm$.

Let me define this quantity as $I_{xy} = \int xydm$. Similarly, $I_{yz} = \int yzdm$ and $I_{zx} = \int zxdm$. So, these quantities are called the mass product of inertia or the product of inertia. So, the moment of inertia of this rigid body about OL becomes $I_{xx}l^2 + I_{yy}m^2 + I_{zz}n^2$.

So, here I_{xx} is the moment of inertia of this body about the x-axis. I_{yy} Is the moment of inertia about the y-axis? So, I_{xx} is this $-2I_{xy}lm - 2I_{yz}mn - 2I_{zx}ln$. Let me call this equation number A.

So, what this equation tells me? This equation tells me that if I know I_{xx}, which is the moment of inertia of this body about the x-axis, I_{yy} is the moment of inertia about the y-axis. And I_{zz} along with I_{xy}, I_{yz}, and I_{zx}, which are the product of inertia, then I can find out the moment of inertia of this body about OL. Let me write down this statement. Let me again first write down what is I_{OL}.



So, this is the same equation that I am writing here. It is $I_{xx}\ l^2 + I_{yy}m^2 + I_{zz}n^2 - 2I_{xy}ln - 2I_{yz}mn - 2I_{zx}ln$. I call this equation number A. So, this equation tells me as I said, if the component $I_{xx}, I_{yy}, I_{zz}, I_{xy}, I_{yz}$ and I_{zx} are known at a point. So, that point is O here.

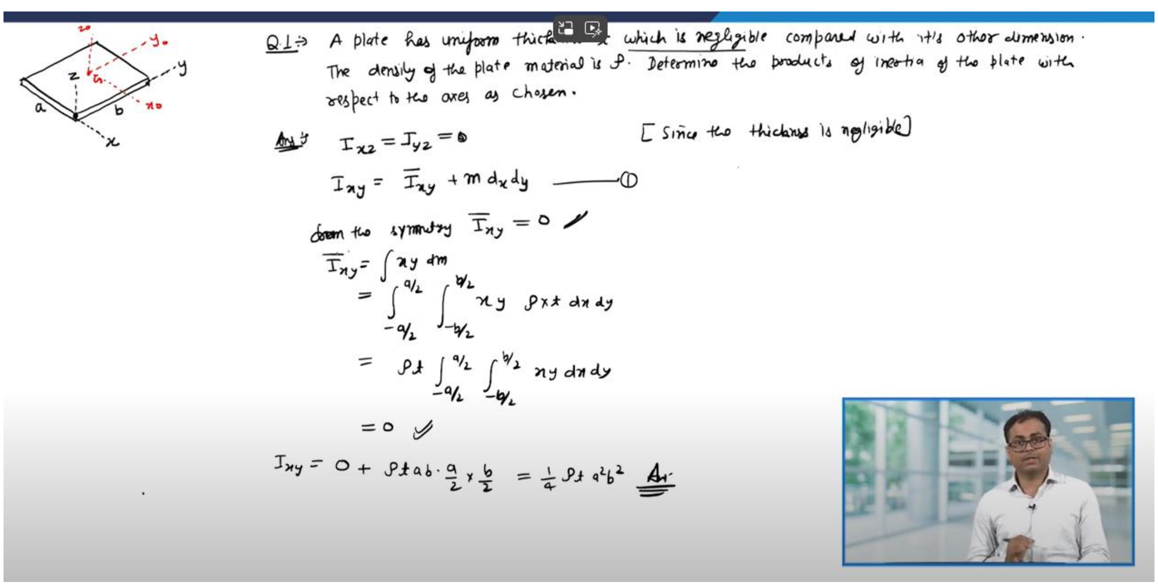
Then the moment of inertia about any axis through the point can be found, okay? And it can be found by equation number A. Now, this equation I can write down in the matrix form. So, I can write down as I_{xx}, I_{yy}, I_{zz} and then I have these two xy. So, let me write down $-I_{xy}$ here and $-\ I_{xy}$ there and similarly, I have two yz. So, let me write down $-I_{zy} - I_{yz}$ or zy and$-I_{xz} - I_{x}z$. So, I have just arranged them in the matrix. Now, this matrix is known as the inertia matrix or the inertia tensor. Before we discuss further, let us look at the product of inertia and let me say that the calculation of the product of inertia is similar to the moment of inertia calculation and also the parallel axis theorem holds here. So, let me show you the parallel axis theorem for product of inertia.

So, for that, let us say I have a rigid body and this is the x-axis and this one is the y-axis. Let us say I make another axis which passes through G, the center of mass. So, let me call it x_0 and y_0 . Let us take a small mass dm.

And in x_0, y_0 coordinate system, its coordinate are x_0 and y_0. Similarly, in xy coordinate system, the coordinates are x and y. Let us say this x_0 and y_0 are situated at a distance of dx and dy. So, this is dx and this one is dy. Now, the product of inertia I_{xy} will be $\int xydm$ and this I can write down as integral x is equal to $x_0 + dx$. Similarly, $y = y_0 + dy$ and dm.

So, this I can expand $x_0y_0dm + dxdy\int dm + dx\int y_0dm + dy\int x_0dm$. But note that this x_0 and y_0, they are measured from the center of mass. Therefore, $\int y_0dm$ will be 0. It is the first moment.

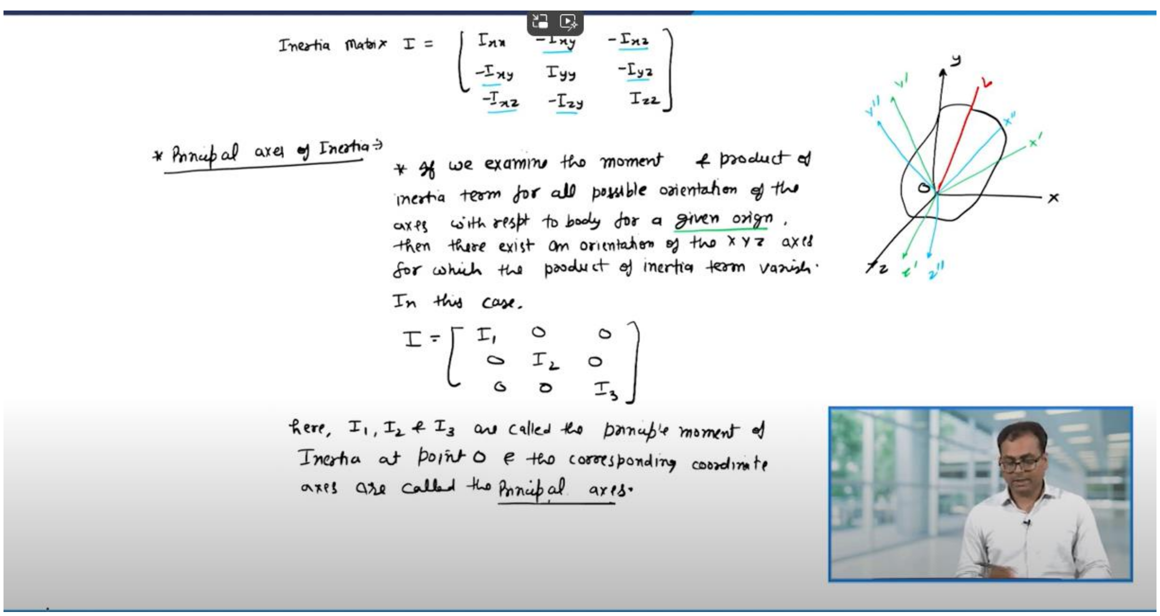
So, this and this will be 0 because that is the definition of the centre of mass. So, therefore, I am left with $\int x_0y_0dm$. So, this is nothing but the product of inertia about $x_0y_0 + mdxdy$. And I, x_0y_0 is the product of inertia about the center of mass. So, I can rewrite this as $I_{xy} = I_{xy}$ about the center of mass plus $mdxdy$. And the same thing is true for the product of inertia about xz. So, that will be $I_{xz} + mdxdz$ and, similarly $I_{yz} = I_{yz} + mdydz$. So, it holds.



Now, let us look at a very simple example about the product of inertia. So, the problem statement is following: a plate has uniform thickness t which is negligible compared with its other dimension. The density of the plate material is rho. Determine the products of inertia of the plate with respect to the axis as chosen. So, here we have to find out the product of inertia of this plate about the x, y and z axis. So, first of all, it is given that the thickness is negligible.

Since the thickness is negligible, therefore, so let me write down, since the thickness is negligible. Therefore, $I_{xz} = I_{yz} = 0$ because when you calculate xz, there will be $\int xzdm$ and since z is almost 0, so therefore, the integral will be 0 and the same thing is true for I_{yz}. Now, let us see what is I_{xy}. So, I_{xy}, I can find out if I know the product of inertia about the center of mass, then I can just do $\overline{I_{xy}} + mdxdy$. So, let me put an axis at the center of mass. So, this is my y, let us say y_0, similarly x_0 and you have z_0. Now, from the symmetry, $\overline{I_{xy}}$ will be 0. This also we can see mathematically.

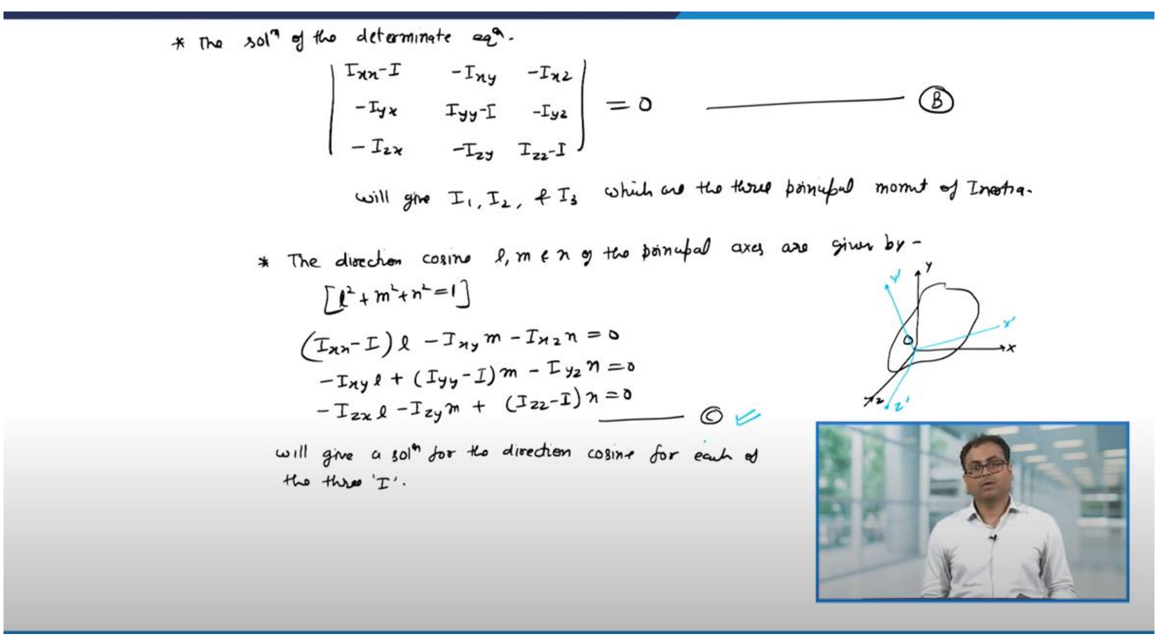
So, I_{xy} is nothing but $\int xydm$ and that will be equal to, so let me take the limit $-\frac{a}{2}$ to plus$\frac{a}{2}$ and for the y, It is$-\frac{b}{2}$ to $+\frac{b}{2}$ $x\,y$ and mass dm. So, mass will be the density times the volume. So, thickness is t which is small and then $dxdy$. So, that will be equal to $\rho t, -\frac{a}{2}$ to$\frac{a}{2}$, $-\frac{b}{2}$ to $\frac{b}{2}$ $xydxdy$. Which is of course 0. Let us put this in equation number 1. So, I have $I_{xy} = 0 + M$, mass is ρ times the volume. So, tab and then dx. dx Is the distance of this G from x. So, that will be dx and this will be dy. So, dx is $a/2$, and dy is $b/2$. So, that comes out to be $\frac{1}{4}\rho t a^2 b^2$.



Now, let us come back to the discussion of the moment of inertia about an arbitrary axis. We saw that the inertia matrix I was $I_{xx} - I_{xy} - I_{xz} - I_{xy}I_{yy} - I_{yz} - I_{xz} - I_{zy}$ and I_{zz}. Now, let me write down the definition of the principal axis of inertia.

And let me remind you the situation was following, we had this rigid body, we fix some point O, then the Cartesian axis were x, y and z and we wanted to find out what is the moment of inertia of this body about an arbitrary axis, let us say OL. So, the statement is following, if we examine the moment and product of inertia term for all possible orientation of the axis with respect to the body for a given origin. So, that given origin is O here, then there exists an orientation of the x, y, z axis for which the product of inertia term vanishes okay. So, what you do is we have this point O we call this given origin Now, you find out the moment of inertia for all the possible orientation of this x, y and z axis.

So, let us say one possible orientation is this x', y', z' . Another possible orientation may be x'', y'', z'' and so on. Then there will always exist some orientation of the x, y, z for which the product of inertia term will not be there, okay. So, all this term will become 0, okay. So, that means for that particular orientation, we have the inertia matrix I equal to Ix x, let me call it I_1, I_{yy}, and I_{zz} for that new orientation, and there would not be any product of inertia term, okay? Here, this I_1, I_2 and I_3 are called the principal moment of inertia at point O and the corresponding coordinate axis are called the principal axis. Now, the question is how do I find out the principal moment of inertia and the principal axis?



Now, this is something that I am not proving it, but it can be shown that the solution of the determinant equation. So, I have this matrix and I have that matrix. So, let us make a determinant $I_{xx} - I, -I_{xy}, -I_{xz}, -I_{yx},\ I_{yy} - I, -I_{yz}, -I_{zx}, -I_{zy}, I_{zz} - I$ equal to 0 will give you I_1, I_2 and I_3 which are the three principal moment of inertia. Let me just call this equation number B.

Now, the direction cosine l, m, and n of the principal axis are given by, so first of all, this $l, m,$ and n will satisfy the relation; know that $l^2 + m^2 + n^2 = 1$, okay? Also, we have $I_{xx} - I \times l - I_{xy}m - I_{xz}n = 0 - I_{xy}l + I_{yy} - Im - I_{yz}n = 0$ and $-I_{zx}l - I_{zy}m + I_{zz} - In = 0$. Let me call this as equation number 3, then this equation will give a solution for the direction cosine for each of the three i's that is I_1, I_2 and I_3.

So, let me just repeat again. You have a rigid body. Let us say there is some point O and we have the x-axis, y-axis and the z-axis. And of course, about this x, y and z axis, this body has some product of inertia. Now, if you orient your x, y and z axis, then you will always find some x', y' and z' such that the product of inertia of this body will vanish and you will only have the moment of inertia.

These moment of inertia are called the principal moment of inertia and the direction cosines of this x', y' and z' we can find out. With this, let me stop here. See you in the next class. Thank you.