MECHANICS

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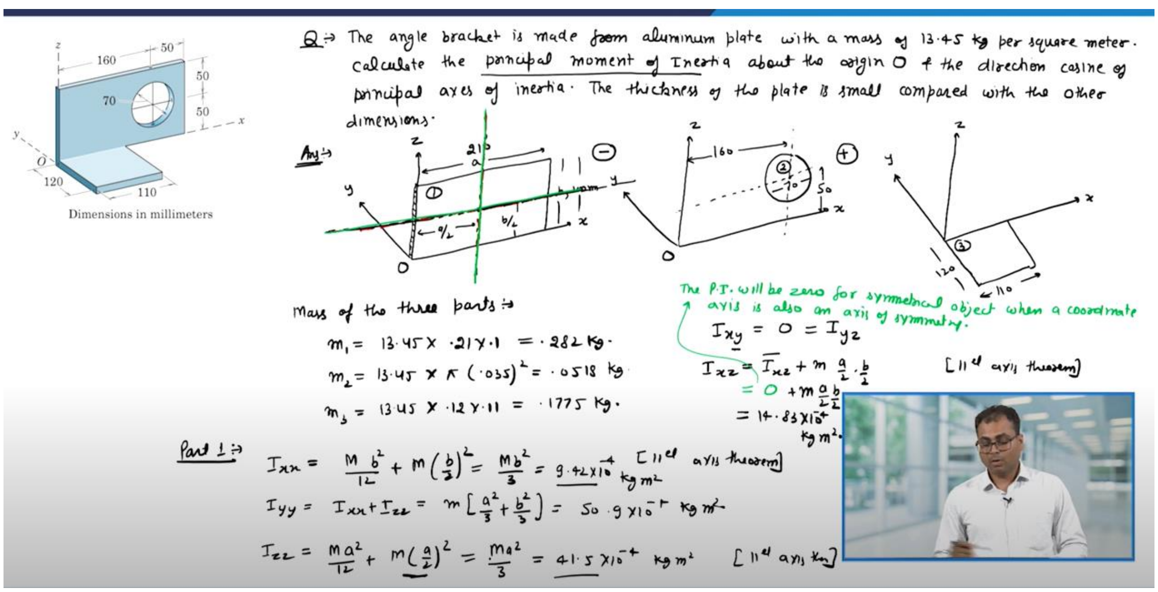
Lecture: 48

Principal axes of inertia: examples-II

Hello everyone, welcome to the lecture again. In the last lecture, we saw one example wherein we find out the inertia matrix and principal moment of inertia of an assembly made of thin plates.



Today, we are going to continue the discussion on the same concept and we will find out the inertia matrix, the principal moment of inertia and principal axis of inertia of another assembly. So, the problem statement is following. The angle bracket is made from aluminum plate with a mass of 13.45 kg/m^2. Calculate the principal moment of inertia about the origin O and the direction cosine of the principal axis of inertia. It is given that the thickness of the plate is small compared with the other dimensions. So, this angle bracket can be divided into three part

So, here this is x-axis, the y-axis and the z-axis. Let us say this is part 1, okay. The entire plate on the $x - z$ plane, then part 2 is that circular plate, okay. And part 3 will be the another plate on the $y - x$ plane. So, that is this.

So, this is part 1, part 2. And part 3, the entire assembly can be think of part 1 minus part 2 plus part 3, okay. So, let me put the dimensions. So, let us say this is a , this is b . So, therefore, let us say this dotted line, it passes through the centre of mass of this plate then that length will be $\frac{b}{2}$ and this will be $\frac{a}{2}$.

Now, the value of a and b is of course known to us. Now, let us look at the second part. So, this part. So, again this dotted line passes through the center of this circle. It is given that the diameter of this is 70, okay. And it is situated at a distance of 50 mm, okay. And this length is 160, okay. Now, for the third part, this length is 120 and this is 110. Now, we have to find out the principal moment of inertia of this bracket.

For that, we need to know the moment of inertia and product of inertia of all these three parts. For that, let us first look at the mass of the three parts. So, for the first part, let us say the mass is m_1. So, that will be density multiplied by the area. So, density is 13.45 into the length.

So, $a = 210\,mm,\ and\, b = 100\,mm$. So, that will be equal to 0.21 into 0.1, and that will be 0.2821 kg. The mass of the second part, m_2, which is removed, will be equal to the density multiplied by the area. So, $\pi.(r)^2$ $(0.035)^2$ and that will be equal to 0.0518 kg. The mass of the third part will be the density multiplied by the area.

So, 0.12 into 0.11, and that is equal to 0.1775 kg. So, now, we know the mass of all the three parts. Let us calculate the moment of inertia and product of inertia of part 1 first. So, part 1, the moment of inertia about the x-axis I_{xx} will be equal to the moment of inertia about this axis plus the $M\left(\frac{b}{2}\right)^2$.

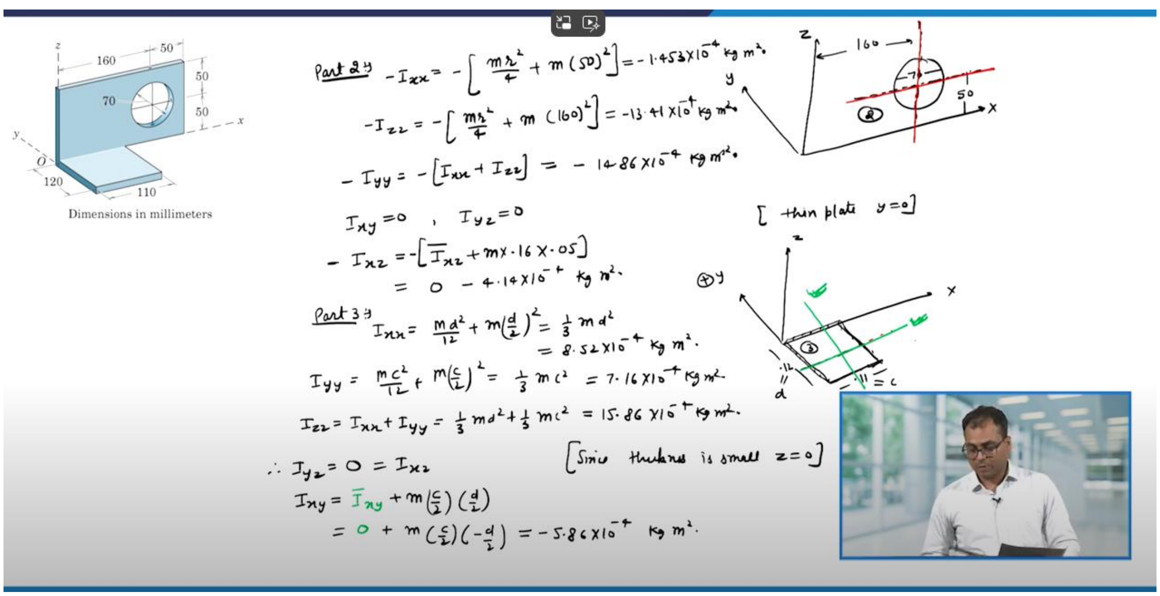
So, we can use the parallel axis theorem. The moment of inertia about this dash axis will be $\frac{Mb^2}{12} + M\left(\frac{b}{2}\right)^2$. So, herein, we have used the parallel axis theorem. So, this is equal to $\frac{Mb^2}{3}$. Now, the value of M is known value of $b = 100\,mm$.

So, therefore, this comes out to be $9.42 \times 10^{-4}\ kg\,m^2$. Now, the moment of inertia about the y axis we can calculate by using the perpendicular axis theorem because this plate lies in the $xz\ plane$. So, therefore, let us first calculate the moment of inertia about the z axis. The moment of inertia about the z-axis can be calculated using the parallel axis theorem.

So, for that let us first calculate the moment of inertia about the dashed red line and then use the parallel axis theorem.

So, it will be $\frac{Ma^2}{12} + M\left(\frac{a}{2}\right)^2 = \frac{Ma^2}{3}$. Now, again the value of M and a is known. So, therefore, it comes out to be $41.5 \times 10^{-4}\ kg\ m^2$. Again, we have used the parallel axis theorem. Now, $I_{yy} = I_{xx} + I_{zz} = M\left(\frac{a^2}{3} + \frac{b^2}{3}\right)$ and that is equal to this plus that. So, it comes out to be $50.9 \times 10^{-4}\ kg\ m^2$. Now, note that the thickness of the plate is very small. So, this thickness that you see here is very-very small. Therefore, y will be 0 in terms of thickness. So, the product of inertia $I_{xy} = Mxy$, but then y is small. So, therefore, that comes out to be 0.

Similarly, $I_{yz} = 0$. Now, let us find out I_{xz}. So, for the I_{xz}, you can use the parallel axis theorem. We can first find out the moment of inertia about these axes, let me call them $\bar{I}_{xz} + M\frac{a}{2}\frac{b}{2}$. So, this comes from the parallel axis theorem. Now, the $\bar{I}_{xz} = 0$ because the plate is symmetrical about this axis. So, let me mention this clearly the product of inertia will be zero for symmetrical object when a coordinate axis is also an axis of symmetry, okay. So, you can see here that the green axis, which is the axis of symmetry; therefore, the product of inertia about it will be $0 + M\frac{a}{2}\frac{b}{2}$, okay. And the value of a, b, and M is known. So, that comes out to be $14.83 \times 10^{-4}\ kg\ m^2$.



Now, let us look at part 2. So, the part 2 was the ring. So, it was given that this is 50, this is 160 and the radius is 35. So, the diameter is 70. So, this was our part 2. So, for part 2, everything will be negative because this part has been removed. So, let us look at the moment of inertia about the x-axis minus of that, that will be minus. Again, we can use the parallel axis theorem.

So, we can first calculate the moment of inertia about this line, and then we can use the parallel axis theorem. So, that will be $\frac{mr^2}{4} + m(50)^2$. So, r is also known, m is also known. So, that comes out to be $-1.453 \times 10^{-4}\ kg\, m^2$

Now, the moment of inertia about the z-axis I_{zz} can be calculated. If we find out the moment of inertia about this axis and then use the parallel axis theorem, that will be $-\left[\frac{mr^2}{4} + m(160)^2\right] = -13.41 \times 10^{-4}\ kg\, m^2$. Now, since this plate lies in the x-z plane, therefore, the moment of inertia about the y-axis will be the sum of the moment of inertia about the x and the z-axis. So, let me just write down that will be $-[I_{xx} + I_{zz}] = -14.86 \times 10^{-4}\ kg\, m^2$.

Now, let us look at the product of inertia. So, the product of inertia $I_{xy} = 0$ because the plate is thin. So, thin plate therefore, y can be taken as 0. So $I_{xy} = mxy$, but y is 0. Therefore, $I_{xy} = 0$, and also $I_{yz} = 0$.

Now, I x z can be found out by using the parallel axis theorem. For that, we should know the product of inertia about the axis marked by the red, and let us say it is equal to $\bar{I}_{xz} + m\ (0.16)(0.05)$. So, minus that will be minus. Now, $\bar{I}_{xz} = 0$ because this disk is symmetrical about the x is marked by the red. So, therefore, this is equal to 0.

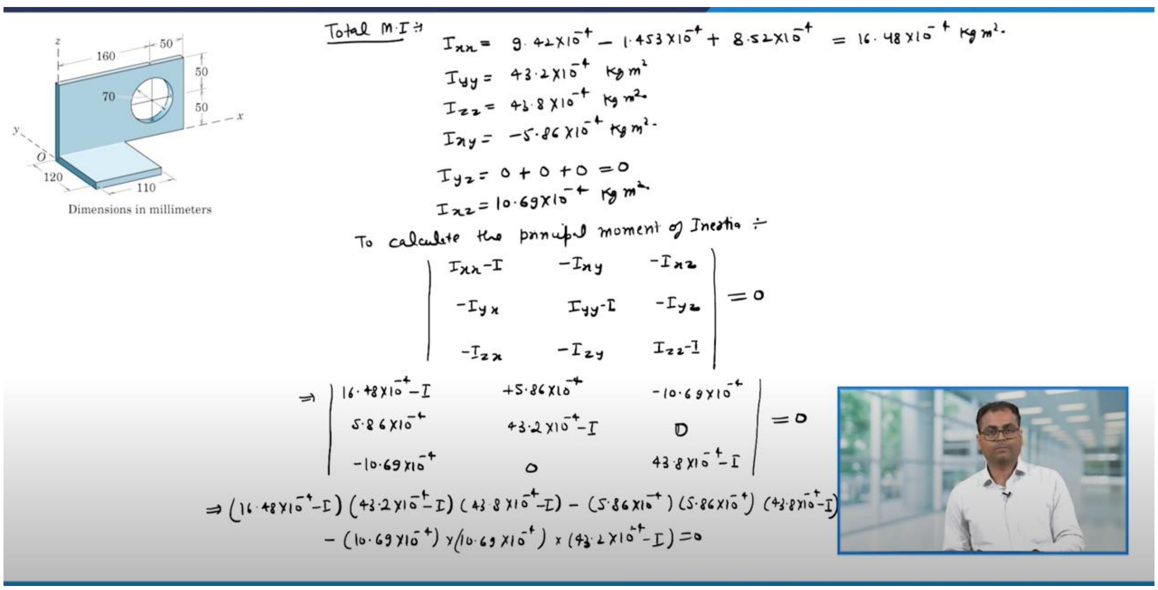
Then we have minus m is known here. So, that comes out to be $4.14 \times 10^{-4}\ kg\, m^2$. Now, let us look at part 3. So, part 3 is lying in the xy plane. So, that is our part 3 and this is 120 mm. So, 0.12 meters, and this one is 0.11. Let me call this is equal to d and this is equal to c. So, the moment of inertia about the x-axis will be equal to the moment of inertia about this axis plus $m\left(\frac{d}{2}\right)^2$. So, that is equal to $\frac{md^2}{12} + m\left(\frac{d}{2}\right)^2$. Again, we have used the parallel axis theorem and that comes out to be $\frac{1}{3}md^2$.

Now, the value of m and d is known. So, that comes out to be $8.52 \times 10^{-4}\ kg\, m^2$. Now, the moment of inertia about the y-axis will be $\frac{mc^2}{12} + m\left(\frac{c}{2}\right)^2 = \frac{1}{3}mc^2 = 7.16 \times 10^{-4}\ kg\, m^2$. Now, the moment of inertia about the z-axis can be find out using the perpendicular axis theorem.

So, that will be equal to $I_{zz} = I_{xx} + I_{yy} = \frac{1}{3}md^2 + \frac{1}{3}mc^2 = 15.86 \times 10^{-4}\ kg\, m^2$. Now, let us find out the product of inertia. Since the thickness of this split is very small, so, therefore, z will be 0. So, since thickness is small, so therefore, this time $z = 0$.

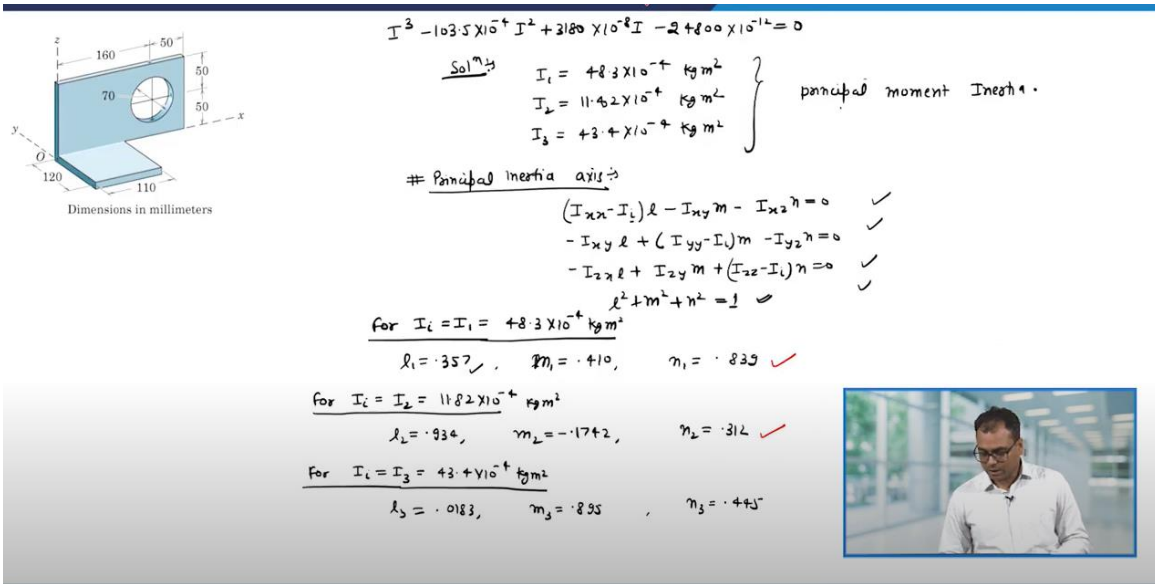
Therefore $I_{yz} = myz = 0$, and so is $I_{xz} = 0$. Now, I_{xy} can we find out using the parallel axis theorem? So, let us say the product of inertia about this is $\bar{I}_{xy}$, then $I_{xy} = \bar{I}_{xy} + m\left(\frac{c}{2}\right)\left(\frac{d}{2}\right)$. Now, $\bar{I}_{xy} = 0$ because this structure is symmetric about the, you know, the new x y, which is marked by the green axis.

So, that is equal to 0 plus m is known, and $\frac{c}{2}$. Note that d will be $-d$ here because this is your positive y. The direction of positive y is opposite to the d. So, therefore $-\frac{d}{2}$. Now, the value of m, c and d is known to us. Therefore, this comes out to be $-5.86 \times 10^{-4}\ kg\, m^2$. So, we have find out the moment of inertia and product of inertia of all three parts.



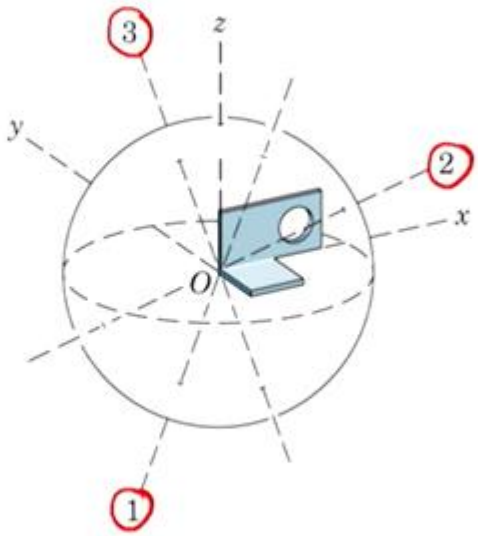
Now, let us write down the total moment of inertia. Okay. So, total moment of inertia, let us say I_{xx}. So, I_{xx} will be equal to the moment of inertia of the first part about the x-axis that is 9.42×10^{-4}. So $9.42 \times 10^{-4} - 1.453 \times 10^{-4} + 8.52 \times 10^{-4}$, okay? And this comes out to be $16.48 \times 10^{-4}\, kg\, m^2$. Similarly $I_{yy} = 43.2 \times 10^{-4}$, and $I_{zz} = 43.8 \times 10^{-4}\, kg\, m^2$. And the product of inertia $I_{xy} = -5.86 \times 10^{-4}\, kg\, m^2$ and I_{yz}. Let us look at I_{yz}. So $I_{yz} = 0 + 0 + 0 = 0$. And similarly $I_{xz} = 10.69 \times 10^{-4}\, kg\, m^2$. So, we have the moment of inertia and product of inertia of all the parts. Now, to calculate the principal moment of inertia, we have to use the following equation. $I_{xx} - I,\ -I_{xy},\ -I_{xz},\ -I_{yx},\ I_{yy} - I,\ -I_{yz},\ -I_{zx}, -I_{zy}\ and\ I_{zz} - I$ Equal to 0. Okay, let us put the values. So, we have $16.48 \times 10^{-4} - I$, then we have $+5.86 \times 10^{-4}$, then we have -10.69×10^{-4}, 5.86×10^{-4}, $43.2 \times 10^{-4} - I$, and this was 0, then -10.69×10^{-4}, 0, and $43.8 \times$

$10^{-4} - I$ equal to 0. Now, this equation we can solve by expanding this. So, that will be $(16.48 \times 10^{-4} - I)(43.2 \times 10^{-4} - I)(43.8 \times 10^{-4} - I) - (5.86 \times 10^{-4})(5.86 \times 10^{-4})(43.8 \times 10^{-4} - I) - (10.69 \times 10^{-4})(10.69 \times 10^{-4})(43.2 \times 10^{-4} - I) = 0$. So, this equation I can simplify. After simplification, we get $I^3 - 103.5 \times 10^{-4} I^2 + 3180 \times 10^{-8} I - 24800 \times 10^{-12} = 0$.



Now, this equation is in the form of I^3. Therefore, it will have three solutions of the I's and let me write down the solution. So $I_1 = 48.3 \times 10^{-4}\, kg\, m^2$. $I_2 = 11.82 \times 10^{-4}\, kg\, m^2$ And $I_3 = 43.4 \times 10^{-4}\, kg\, m^2$.

These are your principal moment of inertia. Now, let us find out the principal inertia axis corresponding to I_1, I_2, I_3, and for that, we can use the equation $(I_{xx} - I_i)l - I_{xy}m - I_{xz}n = 0$, $-I_{xy}\, l + (I_{yy} - I_i)\, m - I_{yz}n = 0$ and, $-I_{zx}l + I_{zy}m + (I_{zz} - I_i)\, n = 0$, and we also have $l^2 + m^2 + n^2 = 1$. Now, for $I_i = I_1 = 48.3 \times 10^{-4}\, kg\, m^2$, I can solve these equations, and I get $l_1 = 0.357,\ m_1 = 0.410,\ and\ n_1 = 0.839$. So, you can check that l square plus m square plus n square that will be equal to 1 here. Similarly, for $I_i = I_2 = 11.82 \times 10^{-4}\, kg\, m^2$, you can find out again the l, m and n. So, here $l_2 = 0.934,\ m_2 = -0.1742,\ and\ n_2 = 0.312$.



And similarly, for $I_i = I_3 = 43.4 \times 10^{-4}\ kg\ m^2$. $l_3 = 0.0183,\ m_3 = 0.895,\ and\ n_3 = 0.445$. So, these are the direction cosines of the principal inertia axis. Now, let me show you the orientation of the principal axis of inertia corresponding to $l_1,\ m_1,\ and\ n_1$ we have the first principal axis.

Corresponding to $l_2,\ m_2,$ and n_2 we have the second principal axis. And corresponding to $l_3,\ m_3,$ and n_3, we have the third principal axis. If we calculate the product of inertia of this structure about 1, 2 and 3, then it will be the 0. There will not be any product of inertia. Now, this has the implication when we study the angular momentum and rotation of the rigid bodies.

With this, let me stop here. See you in the next class. Thank you.