MECHANICS

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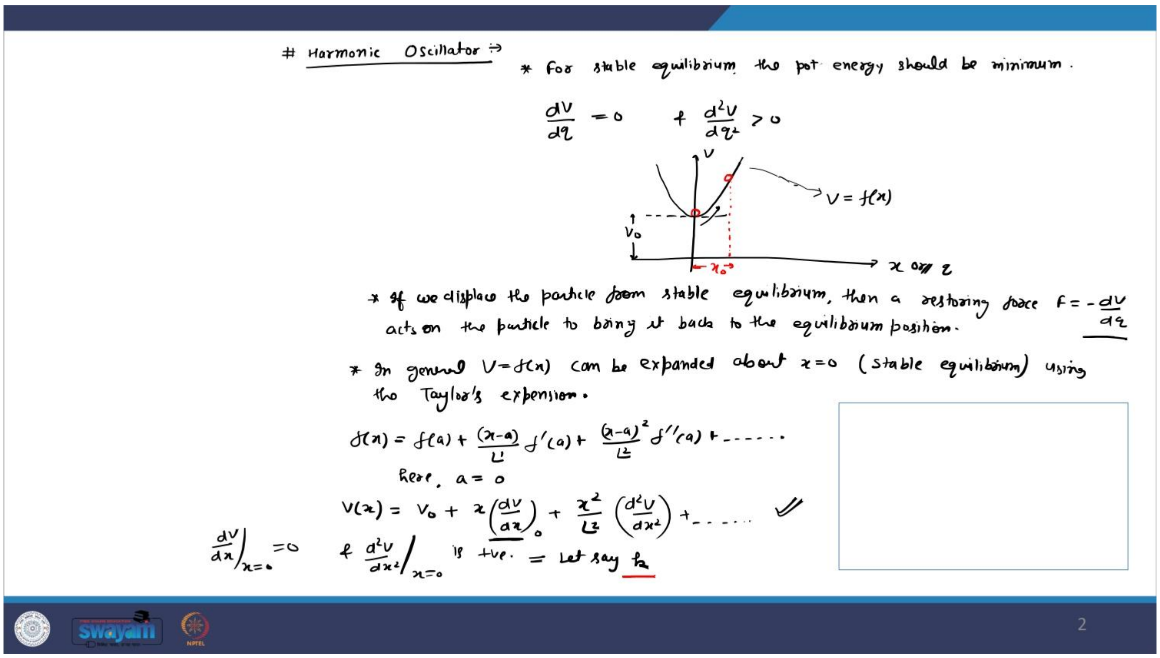
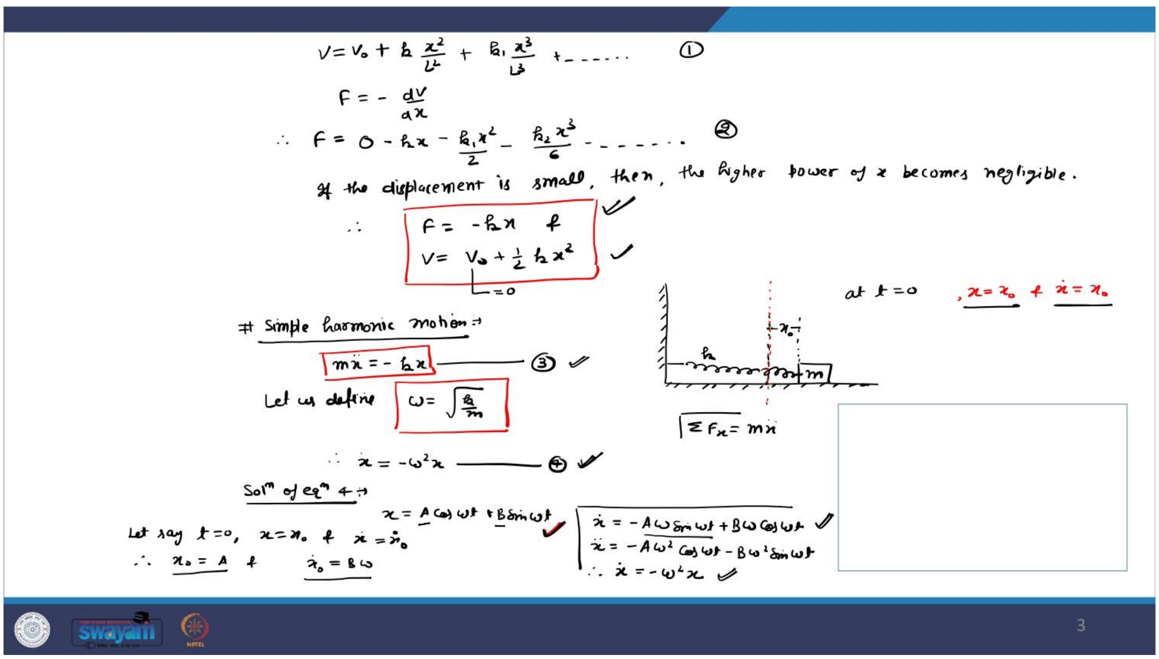
Lecture 49
Harmonic oscillator: simple harmonic motion

Hello everyone, welcome to the lecture again. Today, we are going to discuss the harmonic oscillator, particularly the simple harmonic motion.

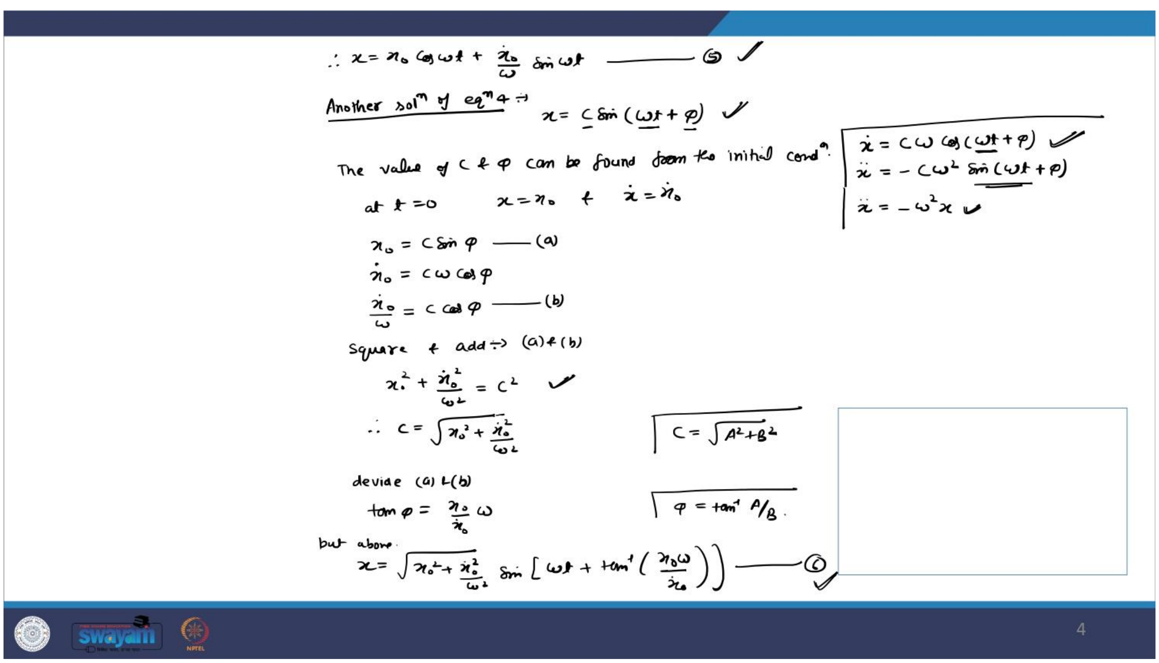


So, remember for stable equilibrium, we have discussed that the potential energy should be minimum. So, this we already know that for a stable equilibrium, the potential energy should be minimum. This implies that if V is the potential energy then $\frac{dV}{dq} = 0$, where q is the coordinate and $\frac{d^2V}{dq^2} > 0$. So, if let's say this is the x-axis or this is q and we have V, then the particles remains at minimum potential energy. So, let's say the particle is here and we have taken this as V_0. This can also be 0 depending upon where you set the reference and now let's say you displace this particle by an amount x_0. So, in that case a force is going to act on the particle to bring it back to the equilibrium. So, let me state it here if we displace the particle from a stable equilibrium, then a restoring force and the value of the restoring force will be $-\frac{dV}{dq}$, $F = -\frac{dV}{dq}$ will act on the particle to bring it back to the equilibrium position. Now, this $V = f(x)$ because V is varying as a function of x. Now, this function of $f(x)$, I can expand using the Taylor expansion about let's say $x = 0$. So, in general, this V which is equal to $f(x)$ can be expanded about $x = 0$. This is the position wherein we have stable equilibrium using the Taylor expansion. And in general, any function $f(x)$ using Taylor expansion can be written as: $f(x) = f(a) + \frac{x-a}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots$ wherein $f' = \frac{df}{dx}$. Here as I said we are expanding it around the stable equilibrium. So, therefore, your $a = 0$. Now, $V(x) = V_0 + x\left(\frac{dV}{dx}\right)_0 + \frac{x^2}{2!}\left(\frac{d^2V}{dx^2}\right) + \cdots$. Now, as I already said that for the stable equilibrium your $\frac{dV}{dq} = 0$. So, herein you have $\frac{dV}{dx}$ at $x = 0$ because that is where I have the stable equilibrium. So, therefore, this has to be 0 and $\frac{d^2V}{dx^2}|_{x=0} > 0$. And let's say its value is equal to k.

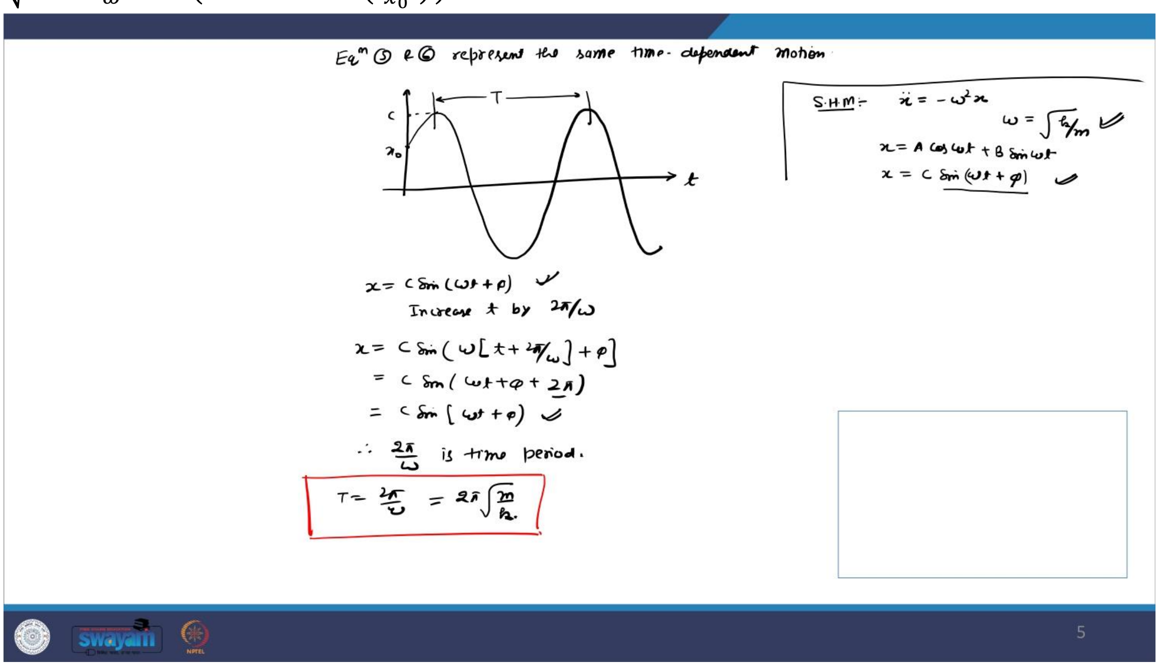


So, therefore, I have V equal to, so we are looking at this equation, we have $V = V_0 + k\frac{x^2}{2!} + k_1\frac{x^3}{3!} + \cdots ------(1)$. Now, we have also seen that the force that is acting on the particle when you move it from the stable equilibrium is $-\frac{dV}{dq}$. So, $F = -\frac{dV}{dx}$ and from here I can find out what is the force that is acting on the particle. So, therefore, let us differentiate equation number 1. V_0 is a constant value. So, therefore, that $F = 0 - kx - \frac{k_1x^2}{2} - \frac{k_2x^3}{6} \ldots ------(2)$. Now, assume that the displacement is small. So, here in when we move this particle, let's say we move it by a small amount in that case. So, let me write down if the displacement is small, then the higher power of x can be negligible because x is small. So, therefore, x^2 and x^3, they will be even smaller. So, we can drop this term under the assumption that the displacement is small. So, in that case, we have $F = -kx$ and your $V = V_0 + \frac{1}{2}kx^2$. Now, let us look at simple harmonic motion. Let's say

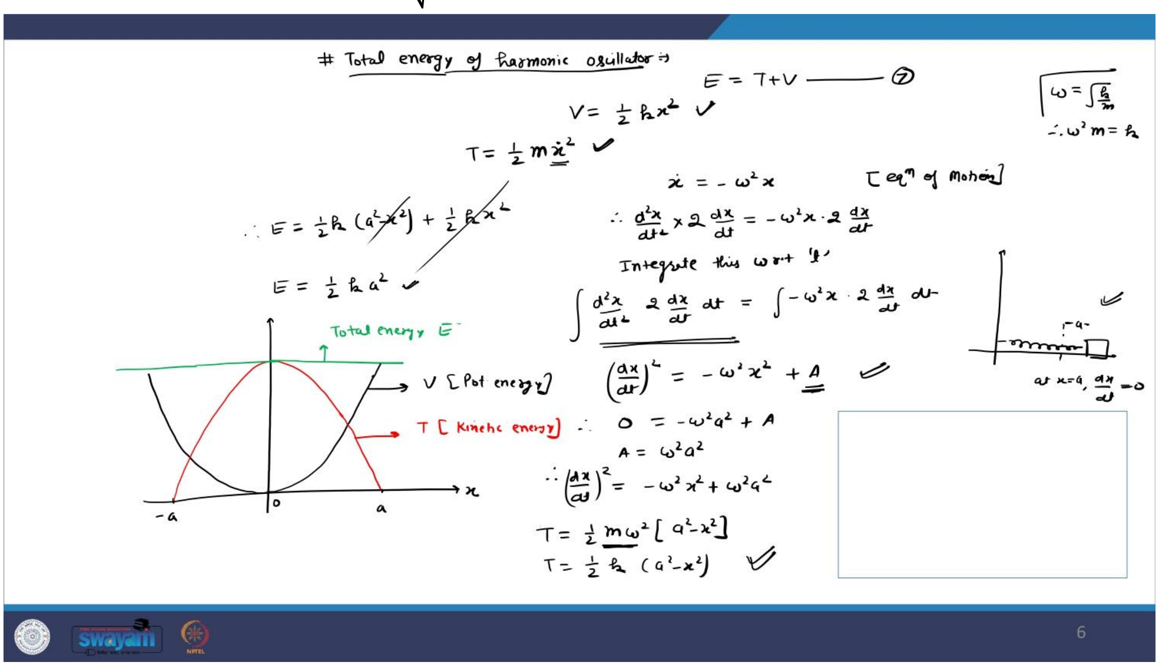
I have a spring and the spring constant of the spring is k and a mass m is attached to the spring and this spring is stretched by x_0. So, in that case, now we can analyze the motion. So, we have the equation of motion $\sum F_x = m\ddot{x}$. So, herein the force that is acting on the spring is $-kx$. So, we have $m\ddot{x} = -kx ----- (3)$. Now, also note that this $V_0 = 0$ by choosing an appropriate reference. So, for example, here in if this curve V touches the x-axis in that case $V_0 = 0$. So, to analyze this equation of motion, let us define a quantity $\omega = \sqrt{\frac{k}{m}}$. So, in that case, the equation of motion becomes $\ddot{x} = -\omega^2 x ----- (4)$. Now, its solution will be $x = A\,cos\omega t + B\,sin\omega t$. Now, we can check whether this solution is correct or not. So, for that, let us find out what is $\dot{x}$. $\dot{x} = \frac{dx}{dt}$. So, that will be $-A\omega\,sin\omega t + B\omega\,cos\omega t$. Similarly, $\ddot{x} = -A\omega^2\,cos\omega t - B\omega^2\,sin\omega t$ and from here I can take ω as a common factor. So, $-\omega^2 x$. Therefore, the solution $x = A\,cos\omega t + B\,sin\omega t$ satisfies equation number 4. Now, the values of this constant A and B can be determined from the initial condition. So, let's say at $t = 0$, this is the situation that we have. And let us put our reference axis over here. So, at $t = 0$, we have $x = x_0$. And let's say at this instant, the velocity of the mass is $\dot{x_0}$. So, this is the initial condition. Let us put it in the solution. So, we have at $t = 0$, we have $x = x_0$ and $\dot{x} = \dot{x_0}$. Therefore, if I put it in this solution, at $t = 0$, $sin\omega t = 0$ and $cos\omega t = 1$. So, therefore, $x_0 = A$ and let us put the value of $\dot{x}$ over here. So, at $t = 0$, again this term will be 0 and $cos\omega t = 1$. So, therefore, we have $\dot{x_0} = B\omega$ and put this in the solution.



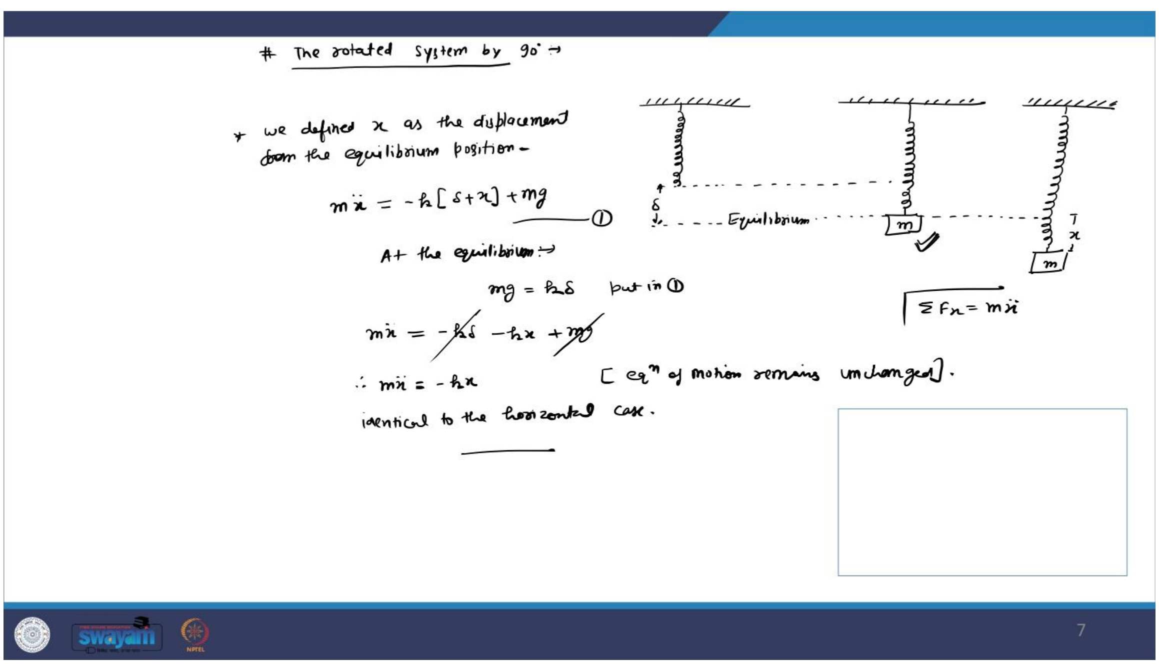
So, we have $x = x_0 cos\omega t + \frac{\dot{x_0}}{\omega} sin\omega t ------- (5)$. Now, note that we have taken $x = A\,cos\omega t + B\,sin\omega t$ as the solution of equation number 4 and we have seen that indeed this solution satisfied the equation number 4. We can also choose another solution of equation number 4, which is $\ddot{x} = -\omega^2 x$ and let us choose this solution $x = C\sin(\omega t + \phi)$. Let us find out what is $\dot{x}$ in this case. So, $\dot{x} = C\omega\cos(\omega t + \phi)$ and $\ddot{x} = -C\,\omega^2\sin(\omega t + \phi)$ and this I can write down as $\ddot{x} = -\omega^2 x$. So, this implies that $C\sin(\omega t + \phi)$ satisfies the equation $\ddot{x} = -\omega^2 x$. So, therefore, that is also a valid solution. Here, the constant are C and ϕ and their values can again be find out from the initial condition, and remember the initial condition we have at $t = 0$, we have $x = x_0$ and $\dot{x} = \dot{x_0}$. So, let us put this above in the equation. So, at $t = 0$, we have $\omega t = 0$. So, therefore, I have $x_0 = C\,sin\phi - - - - - (a)$ and we have $\dot{x} = \dot{x_0}$. So, herein let us put $\omega t = 0$. So, we have $C\omega\,cos\phi$ or $\frac{\dot{x_0}}{\omega} = C\,cos\phi - - - - - (b)$. So, we have two equations and we have two variable that we want to find out ϕ and C. So, for that let us square and add them, a and b. So, we have $x_0^2 + \frac{\dot{x_0}^2}{\omega^2} = C^2\sin^2\phi + C^2\cos^2\phi$. C^2 I can take outside and $\sin^2\phi + \cos^2\phi = 1$. So, therefore, we have this equation. Therefore, $C = \sqrt{x_0^2 + \frac{\dot{x_0}^2}{\omega^2}}$. Now, remember earlier we have find out the values of A and B. So, $A = x_0$ and $B = \frac{\dot{x_0}}{\omega}$. So, therefore, I can write down the relation between C and A and B. So, you can see that $C = \sqrt{A^2 + B^2}$. Now, to find out the value of ϕ, so let us divide A and B. So, we have $tan\phi = \frac{x_0}{\dot{x_0}}\omega$. And again, you can see that ϕ is also related with A and B because $A = x_0$ and $B = \frac{\dot{x_0}}{\omega}$. So, therefore, $\phi = \tan^{-1}\frac{A}{B}$. And let us put the values of C and ϕ above in the solution. So, we have $x = \sqrt{x_0^2 + \frac{\dot{x_0}^2}{\omega^2}}\sin\left(\omega t + \tan^{-1}\left(\frac{x_0\omega}{\dot{x_0}}\right)\right) - - - - - - (6)$.



So, we have find out two solution of the simple harmonic motion. One is given by equation number 5 and the another one is given by equation number 6. Since they represent the solution of the same simple harmonic motion, therefore the solution has to be the same. So, let me write down that equation number 5 and 6, they represent the same time dependent motion. Let me just summarize the result. So, we have the equation of motion of simple harmonic motion which was $\ddot{x} = -\omega^2 x$, wherein this $\omega = \sqrt{\frac{k}{m}}$. One of the solution was $x = A\, cos\omega t + B\, sin\omega t$. And the another solution we have taken $x = C \sin{(\omega t + \phi)}$. Now, let us plot the motion. So, on the horizontal axis, now we have time t. Initially, at $t = 0$, we have $x = x_0$. Now, the maximum value of x can be when, you know, sin is maximum and its maximum value is 1. So, therefore, x can be C at max. And since it is a sin function, so, therefore, the motion will be something like this. Now, one complete oscillation is called the time period T. And let us find out this time period T. So, we have $x = C \sin{(\omega t + \phi)}$. Let us increase t by $\frac{2\pi}{\omega}$ and see what happens. So, we have $x = C \sin{(\omega \left(t + \frac{2\pi}{\omega}\right) + \phi)}$. So, this I can write down as $x = C \sin(\omega t + \phi + 2\pi)$ and $\sin(2\pi + \theta) = sin\theta$. So, we have $x = C \sin{(\omega t + \phi)}$. So, this is the same value that we have earlier. So, that means if I increase t by $t + \frac{2\pi}{\omega}$, we get the same value. So, this implies that $\frac{2\pi}{\omega}$ is nothing but the time period. So, therefore, $T = \frac{2\pi}{\omega}$ and remember $\omega = \sqrt{\frac{k}{m}}$. So, therefore, this becomes $T = 2\pi\sqrt{\frac{m}{k}}$.



So, total energy will be the kinetic energy plus the potential energy, $E = T + V$ $-----$ (7). Now, we already know the potential energy of the harmonic oscillator V. $V = \frac{1}{2}kx^2$ because $V_0 = 0$. So, we have $\frac{1}{2}kx^2$ and the kinetic energy $T = \frac{1}{2}mv^2$. So, $v = \dot{x}$. Now, we have to find out what is $\dot{x}$. So, remember the equation of motion. So, equation number 4, we have x double dot equal to minus omega square x. So, we have $\ddot{x} = -\omega^2 x$. This was the equation of motion. Let us multiply this equation of motion by $2\dot{x}$. So, $\frac{d^2x}{dt^2} \times 2\frac{dx}{dt} = -\omega^2 x. 2\frac{dx}{dt}$. Now, let us integrate this with respect to t. So, we have $\int \frac{d^2x}{dt^2}\, 2\frac{dx}{dt}\, dt = \int -\omega^2 x. 2\frac{dx}{dt}\, dt$. And this integral is nothing but $\left(\frac{dx}{dt}\right)^2 = -\omega^2 x^2 + A$. So, therefore, we have that. Now, let us find out A from the initial condition, So, let's say we have this spring mass system and when the spring is stretched by an amount a, in that case the velocity becomes 0. So, let's say at $x = a$, $\frac{dx}{dt}$ which is the velocity that is equal to 0. So, we have $0 = -\omega^2 a^2 + A$ and that gives you $A = \omega^2 a^2$. Let us put it back. So, we have $\left(\frac{dx}{dt}\right)^2 = -\omega^2 x^2 + \omega^2 a^2$. And this is the velocity square. So, $\dot{x}^2$, this I can put it over here to find out the kinetic energy. So, $T = \frac{1}{2}m\omega^2(a^2 - x^2)$. And what is $m\omega^2$, remember $\omega = \sqrt{\frac{k}{m}}$. So, let me just mention that $\omega = \sqrt{\frac{k}{m}}$. So, therefore, $\omega^2 m = k$. So, this I can write down as $T = \frac{1}{2}k(a^2 - x^2)$. And let us put the kinetic energy and the potential energy in equation number 7. So, $E = \frac{1}{2}k(a^2 - x^2) + \frac{1}{2}kx^2$. This will get cancelled with that and we have $E = \frac{1}{2}ka^2$. So, therefore, the total energy of the harmonic oscillator is independent of x. Let us plot the kinetic energy, potential energy and total energy as a function of x.



So, we have x and the maximum value of x is $+a$ or $-a$, $\frac{1}{2}kx^2$ is a parabola and $a^2 - x^2$ will be the inverted parabola and the total energy which is the sum of both that will be constant. So, this is the potential energy V. This is the kinetic energy T and this is total energy E.

So, you can see that since the friction etc, are not involved therefore, the total energy remains conserved and when the mass m moves then the potential energy get converted into the kinetic energy and the kinetic energy gets converted into the potential energy. Now, let us look at the same system, okay, but let's say we rotate this configuration by 90°, okay. So, let us look at the rotated system by 90°. So, now the configuration that we have is following. We have a mass which is hanged by a spring. And let's say when you hang the mass, then we have the equilibrium. So, this implies that if the mass is not there, then the natural length of the spring will be smaller and let's say this length is δ. Now, you stretch this spring so that we set up the motion and let's say from the equilibrium we stretch it by an amount x. So, herein we have defined x as the displacement from the equilibrium position, okay. Now, let us look at the equation of motion of this system. So, the equation of motion will be $\sum F_x = m\ddot{x}$. So, we have $m\ddot{x}$ equal to let us look at the forces that are acting on the spring. So, that will be $m\ddot{x} = -k(\delta + x) + mg - - - -(1)$. Now, in the equilibrium case, the forces will be balanced. So, $mg = k\delta$. Let us put it above. So, we have $m\ddot{x} = -k\delta - kx + k\delta$. Therefore, we have $m\ddot{x} = -kx$ and this equation of motion is identical to the horizontal case that we have studied. There is no difference when you study the spring-mass system in the horizontal configuration or in the vertical configuration. So, the equation of motion remains unchanged.

With this, let me stop here. See you in the next class. Thank you.