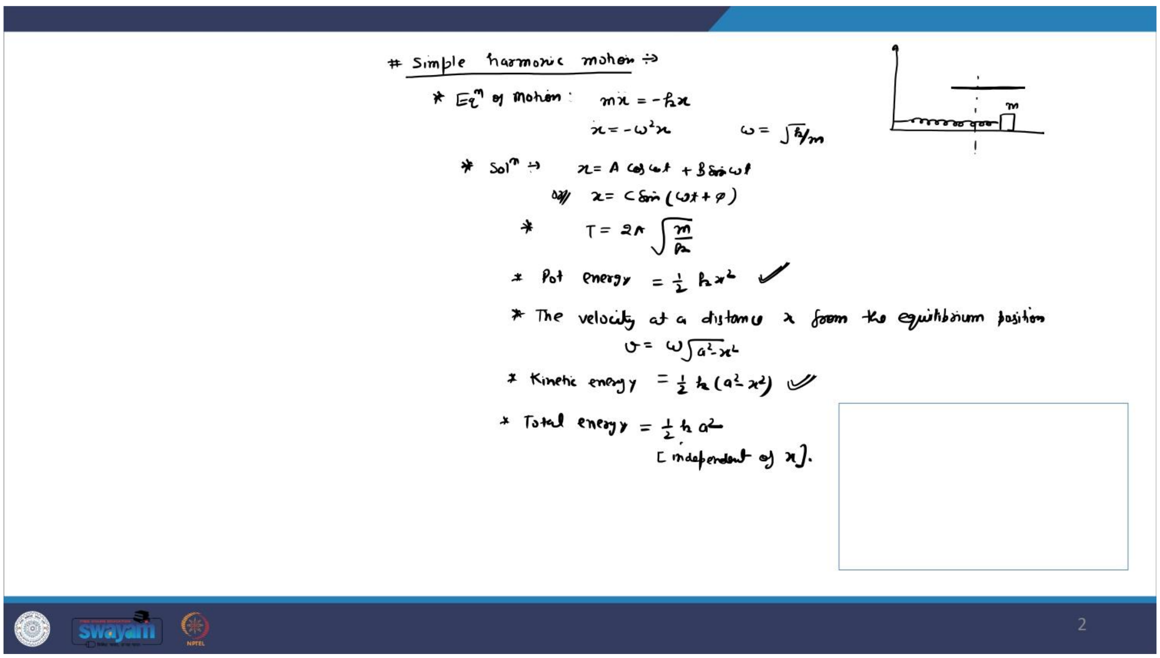
# MECHANICS

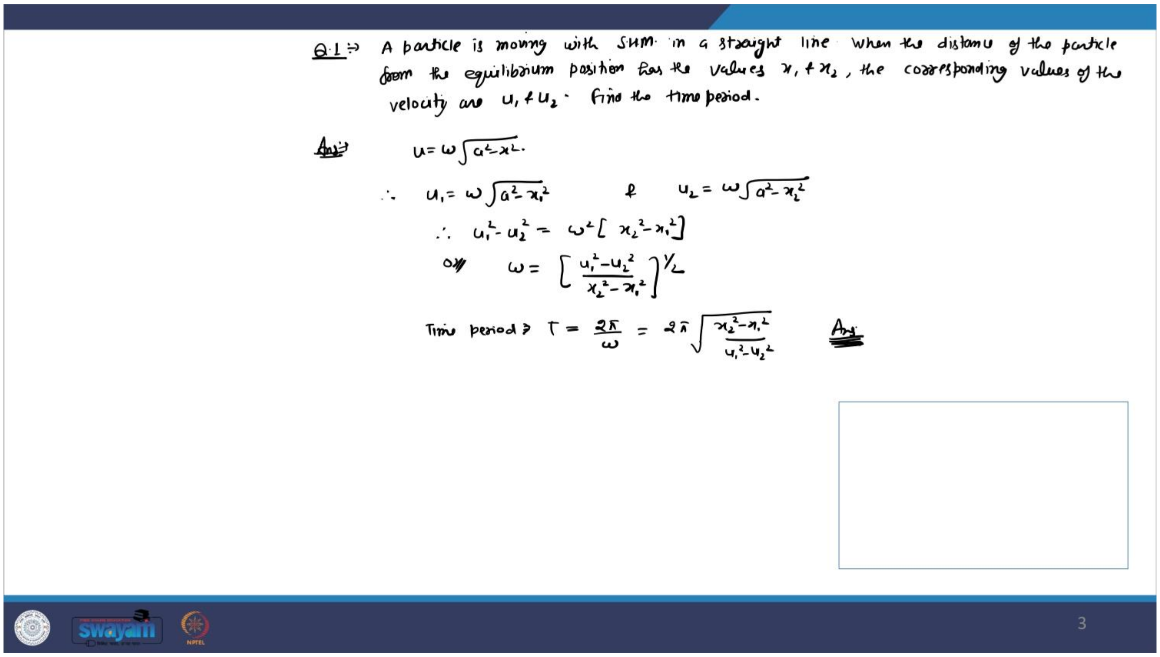
**Prof. Anjani Kumar Tiwari**
**Department of Physics**
**Indian Institute of Technology, Roorkee**

## Lecture 50
## Simple harmonic motion: examples

Hello everyone, welcome to the lecture again. In the last class, we discussed about the simple harmonic motion. Today, we are going to solve couple of examples, but before that let me summarize the result of the previous class.

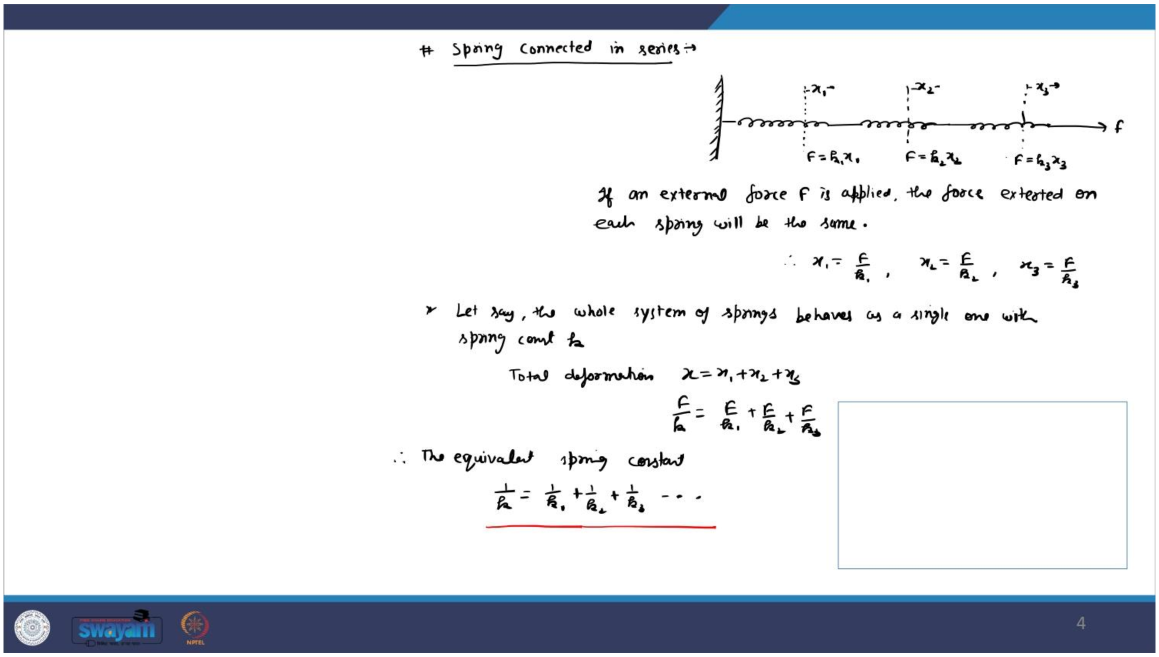



So, simple harmonic motion. In general, we have a mass which is attached to a spring and to set up the motion, you displace it by applying a force $F$ and then the mass $m$ makes an oscillatory motion about the equilibrium position. So, the equation of motion of the mass $m$ was $m\ddot{x} = -kx$, then we define a quantity $\omega = \sqrt{k/m}$, then the equation becomes $\ddot{x} = -\omega^2 x$. We saw that this equation has the solution and the solution can be $x = Acos\omega t + Bsin\omega t$. The solution can also be written as $x = Csin(\omega t + \phi)$ and here $A$ and $B$ and $C$ and $\phi$ they are related. The time period of this motion $T = 2\pi\sqrt{\frac{m}{k}}$ and the potential energy was $\frac{1}{2}kx^2$. Now, we also saw that the velocity at a distance $x$ from the equilibrium position is $v = \omega\sqrt{a^2 - x^2}$ and therefore, the kinetic energy was $\frac{1}{2}k(a^2 - x^2)$. Now, since the potential energy is this and the kinetic energy is that, therefore, the total energy was $\frac{1}{2}ka^2$ and this was independent of $x$.

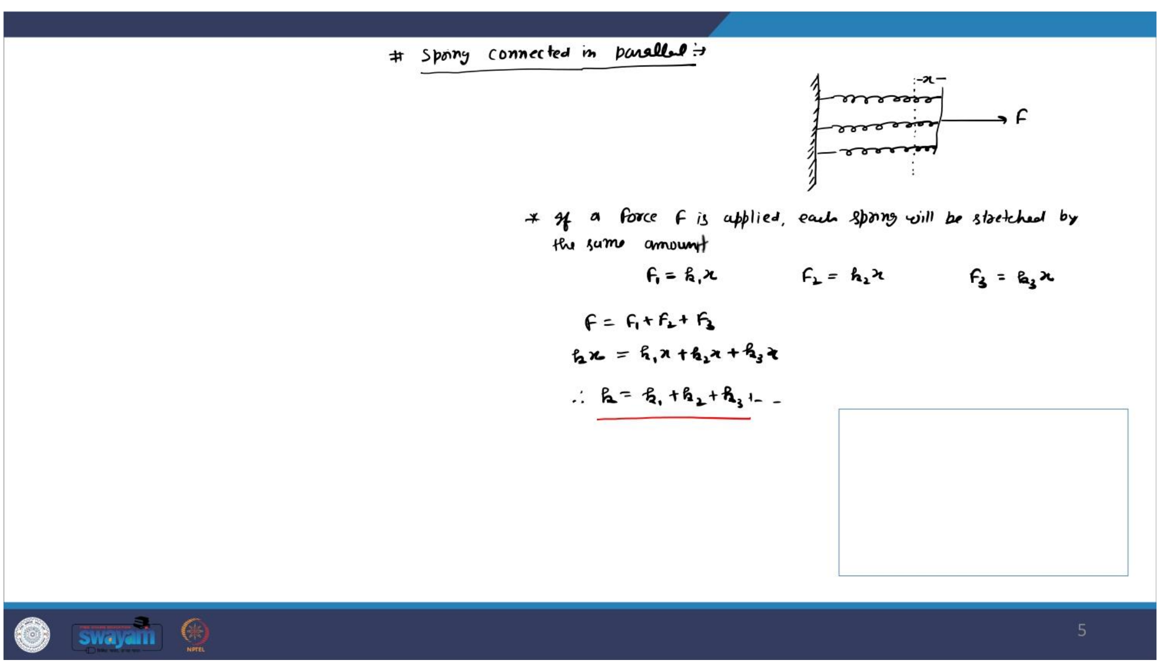



Now, let us look at one example. A particle is moving with simple harmonic motion in a straight line when the distance of the particle from the equilibrium position has the values $x_1$ and $x_2$, the corresponding value of the velocity are $u_1$ and $u_2$. Find the time period. So, we already saw that when a particle is at a distance $x$ from the equilibrium, in that case, the velocity $u = \omega\sqrt{a^2 - x^2}$. Herein, it is given that when the particle is at a distance $x_1$, its velocity is $u_1$. So, therefore, $u_1 = \omega\sqrt{a^2 - x_1^2}$ and here $a$ is the maximum distance that the particle can go. And similarly, you have $u_2 = \omega\sqrt{a^2 - x_2^2}$. So, therefore, from here, we can find out the value of $\omega$. For that, let us do $u_1^2 - u_2^2 = \omega^2(x_2^2 - x_1^2)$ or the value of $\omega = \left(\frac{u_1^2-u_2^2}{x_2^2-x_1^2}\right)^{1/2}$ and we already know the time period in terms of $\omega$. Time period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{x_2^2-x_1^2}{u_1^2-u_2^2}}$.

Now, let us discuss the springs which are connected in series and parallel. So, the first one is the spring connected in series. So, let's say we have the following situation. We have a spring, then another spring, and then another spring, and a force $F$ is applied. So, when the force $F$ is applied, all the springs are going to experience an extension. Let's say the first spring experience an extension of $x_1$, second one $x_2$, and the third one $x_3$. So, as I said, if an external force $F$ is applied, then the force exerted on each spring will be the

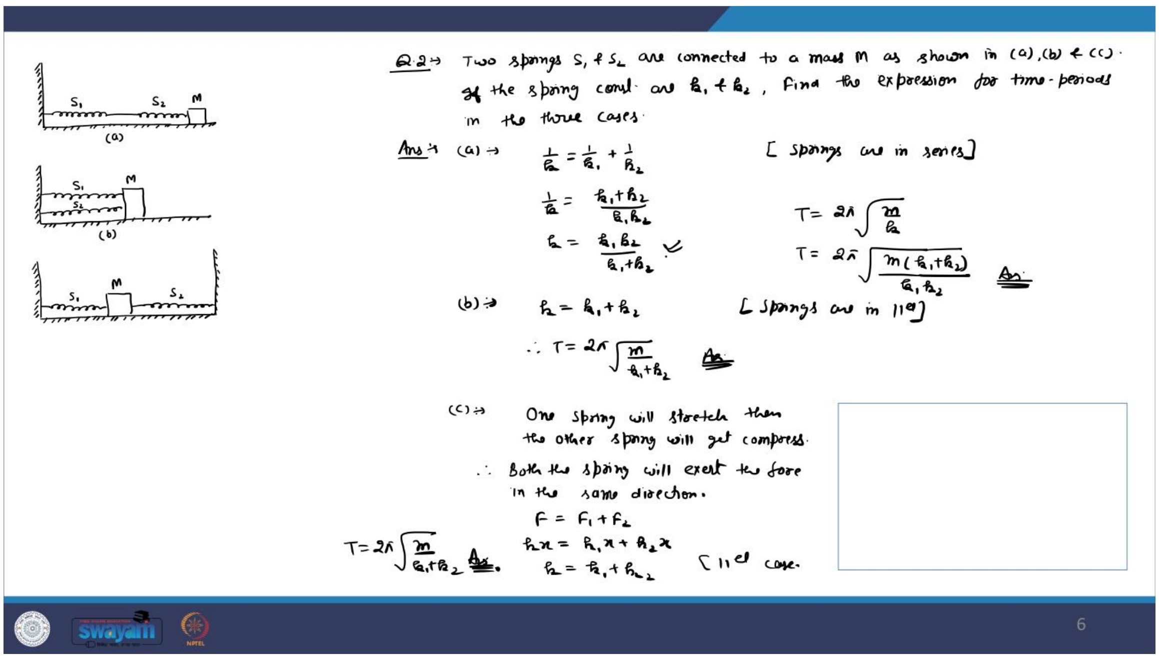



same. So, here the force $F$ is acting, so that $F$ will give you an extension, so that will be determined by $k_1 x_1$. Similarly, $F = k_1 x_2$ and here $F = k_3 x_3$, where $k_1, k_2$ and $k_3$ are the spring constant of spring 1, 2 and 3, okay. So, from here, you can see that $x_1 = \frac{F}{k_1}$ and $x_2 = \frac{F}{k_2}$ and $x_3 = \frac{F}{k_3}$ and so on. Now, let's say the whole spring system behaves as a single one with let's say a spring constant $k$. So, in that case, so, you can see from the figure that the total deformation will be $x = x_1 + x_2 + x_3$ . Now, since the whole system we said is behaving as a single spring, so therefore, $x = \frac{F}{k}$ and $\frac{F}{k} = \frac{F}{k_1} + \frac{F}{k_2} + \frac{F}{k_3}$. Now, $F$ will get

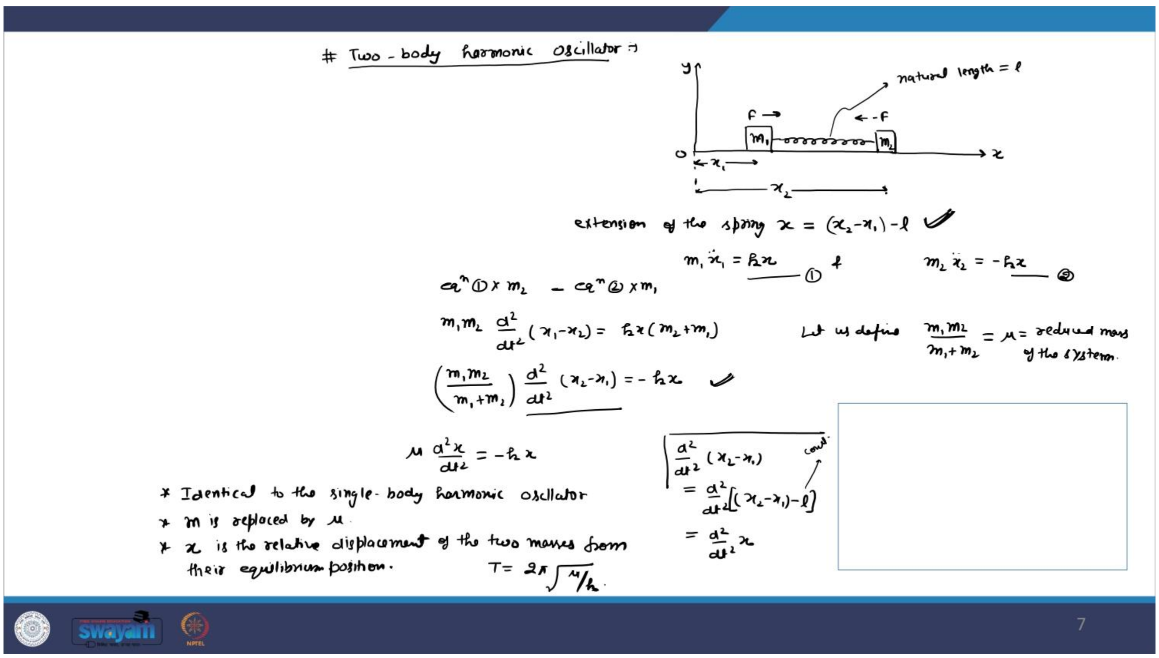


cancelled. Therefore, the equivalent spring constant will be $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \cdots$. So, this is the case wherein the springs are connected in series.

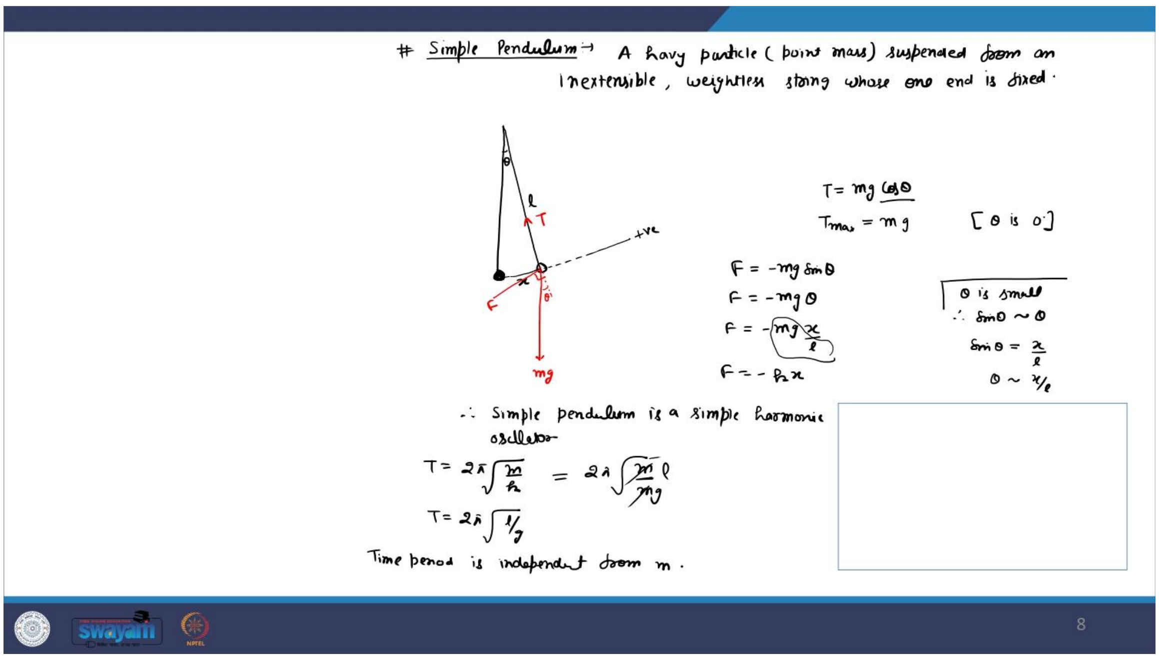
Now, let us see the case where spring are connected in parallel. So, here in the situation is like that, you have, let's say, three springs and on these springs, the force $F$ is applied, okay. So, if we apply a force $F$ then each spring will be stretched by the same amount. And let's say that amount is $x$. So, because of this force $F$, the spring get extended by $x$. So, therefore, for the first spring, the force $F_1 = k_1 x$ and for the second spring, $F_2 = k_2 x$ and for the third spring, it will be $F_3 = k_3 x$. And total force $F = F_1 + F_2 + F_3$. So, $F = kx$, wherein again we have said that let's say the whole system behaves like a single spring of spring constant $k$, then $F = kx$ and this will be $kx = k_1 x + k_2 x + k_3 x$ and from here we get $k = k_1 + k_2 + k_3 + \cdots$ if there are other springs. So, this is the effective spring constant when the springs are connected in parallel.




Now, let us look at another problem statement. Two springs $S_1$ and $S_2$ are connected to a mass $M$ as shown in $a$, $b$ and $c$, if the spin constant are $k_1$ and $k_2$, find the expression for time period in the three cases. In the first case, you can see that the springs are connected in series. So, therefore, the spring constant $k$ will be $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$ because the spring are in series. So, therefore, $\frac{1}{k} = (k_1 + k_2)/(k_1 k_2)$ and therefore, $k = \frac{(k_1 k_2)}{k_1 + k_2}$ and we know that the time period $T = 2\pi\sqrt{\frac{m}{k}}$. So, therefore, $T = 2\pi\sqrt{\frac{m(k_1+k_2)}{k_1 k_2}}$. In the second case, the springs are connected in parallel. So, therefore, $k = k_1 + k_2$ because the springs are in parallel. So, therefore, $T = 2\pi\sqrt{\frac{m}{k_1+k_2}}$. Now, let us look at $c$. So, here if one spring will get stretched, then the other spring will get compressed. So, you can clearly see that if one spring will stretch, then the other spring will get compressed. Therefore, both the spring will exert the force in the same direction. And the total force $F = F_1 + F_2$ wherein $F_1$ is the force by the first spring and $F_2$ is the force by the second spring. So, $F = kx$. So, this is the effective $kx = k_1x + k_2x$ and therefore, $k = k_1 + k_2$. So, this is same as the parallel case. Therefore, $T = 2\pi\sqrt{\frac{m}{k_1+k_2}}$.




Now, let us look at two-body harmonic oscillator. So, let's say I have two masses $m_1$ and $m_2$ and they are attached by a spring. To formulate the problem, let's say that the first mass is at a distance $x_1$ and the second mass is at a distance $x_2$ and the natural length of the spring is $l$. Now, to set up the motion, you stretch the spring. Therefore, let's say on the mass $m_1$, the force $F$ is acting in that direction, then the force on the second mass will act in this direction. So, the extension of the spring which is $x = (x_2 - x_1) - l$. Now, for the first mass, the equation of motion is $m_1\ddot{x_1} = kx - - - - - (1)$ and for the second mass $m_2\ddot{x_2} = -kx - - - - - (2)$. Because the forces are acting in opposite direction, okay ,then we can combine this equation. If we multiply equation number $1 \times m_2 = 2 \times m_1$ and subtract them, then we have $m_1m_2\frac{d^2}{dt^2}(x_1 - x_2) = kx(m_1 + m_2)$ . Or I can rewrite this as $\left(\frac{m_1m_2}{m_1+m_2}\right)\frac{d^2}{dt^2}(x_2 - x_1) = -kx$. Let us define this quantity $\left(\frac{m_1m_2}{m_1+m_2}\right) = \mu$ which is called the reduced mass of the system. It is called reduced mass because the value of $\mu$ will be smaller than either $m_1$ or $m_2$ and we have this quantity. $\frac{d^2}{dt^2}$. And from here, this is nothing but $x$. So, therefore, it can be written as $\frac{d^2}{dt^2}x$. So, therefore, the above equation becomes $\mu\frac{d^2x}{dt^2} = -kx$. So, this equation you can see that it is identical to the single harmonic oscillator. Let me write it down. This is identical to the single body harmonic oscillator. The only difference is your $m$ is getting replaced by $\mu$ and $x$ here is the relative displacement of the two masses from the equilibrium. Therefore, the time period or the time period of the oscillation $T$ will be $T = 2\pi\sqrt{\frac{\mu}{k}}$.




Now, let us discuss simple pendulum which is also an example of simple harmonic motion. So, in simple pendulum is you have a heavy mass point like mass which is suspended with an string and then the motion is set up by displacing it slightly. So, you have a heavy particle or the point mass suspended from an inextensible weightless string whose one end is fixed. So, let's say this is the string and you have the weight which is suspended through it and then the motion is set up by displacing it little bit, let's say by an amount $x$. Let's say the length of the string is $l$ and this angle is $\theta$. So, $\theta$ is small. Then the mass of the particle $mg$ or the weight of the particle is going to act downward. Since that angle is $\theta$, this angle will also be $\theta$. The tension $T$ is going to act upward and the restoring force is going to act which will be perpendicular to the strength. So, you can see from the figure that $T = mg\ cos\theta$ and the maximum value of $cos\theta$ can be 1. Therefore, $T_{max}$ or the maximum tension in the string will be $mg$ and this is the case when $\theta$ is 0°. So, that means when the particle is over here, then the tension in the string will be the maxima. Now, let's say this is the $+x$ direction. Now, the force $F - mg\ sin\theta$ and since $\theta$ is small, therefore, $sin\theta = \theta$ and you can see from the geometry that $sin\theta = \frac{x}{l}$. Therefore, $\theta = \frac{x}{l}$. So, I can write down $F = -mg\theta$ or $F = -\frac{mgx}{l}$. So, here this is in the form of $F = -kx$, where this $k = \frac{mg}{l}$. Therefore, the simple pendulum is a simple harmonic oscillator because it is in the form of $F = -kx$. Now, we know for the simple harmonic oscillator, the time period $T = 2\pi\sqrt{\frac{m}{k}}$. Now, let us put the value of $k$, $T = 2\pi\sqrt{m/(mg/l)}$. So, $m$ will get cancelled, you get $T = 2\pi\sqrt{\frac{l}{g}}$ which is very famous result. So, time period is independent from the mass of the particle.

With this, let me stop here. See you in the next class. Thank you.