MECHANICS

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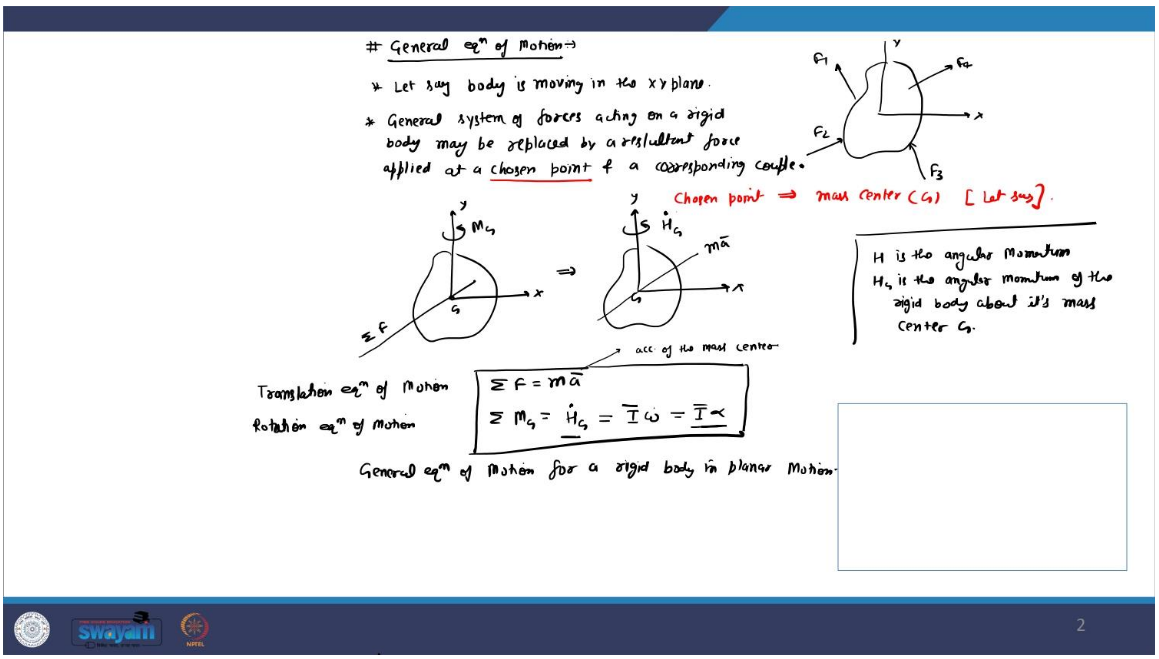
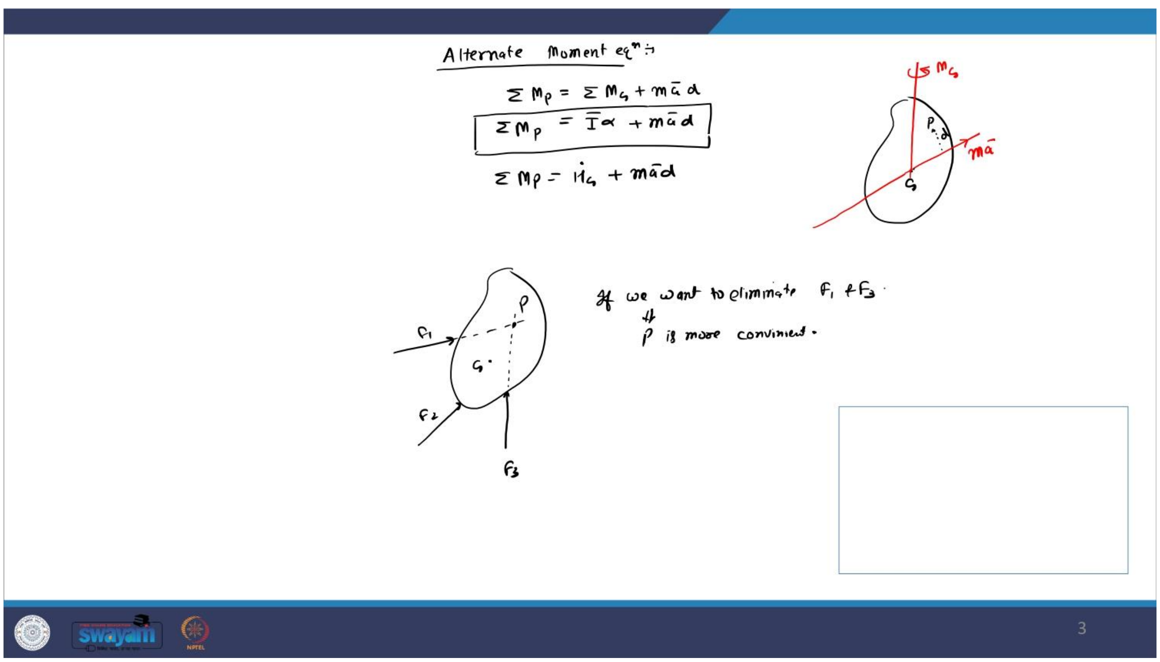
Lecture 54

Plane motion of a rigid body

Hello everyone, welcome to the lecture again. Today, we will discuss about the general equation of motion of a rigid body which is moving in a plane.

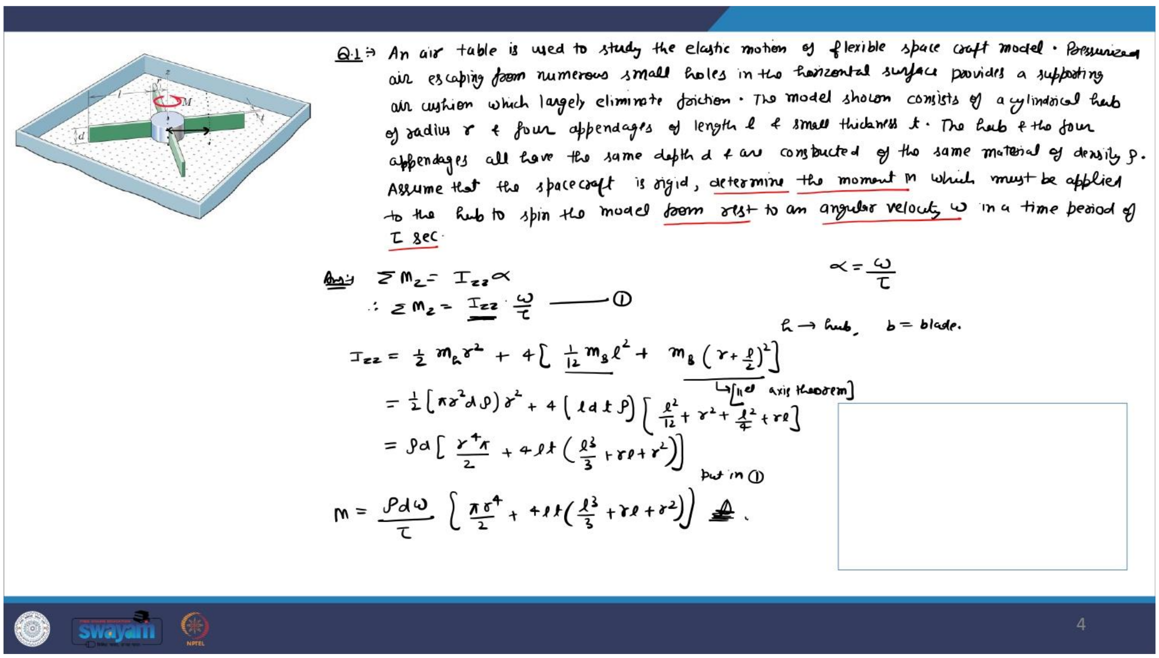


Let us say the forces on the rigid body is acting in the xy plane, and let us say there are many forces. So, F_1, F_2, F_3, F_4. So, as I said that let us say the body is moving in the xy plane because the forces are acting in the xy plane. Now, we already know that the general system of forces which are acting on the rigid body, they can be replaced by a resultant force and a corresponding couple. So, let me write this down. General system of forces acting on a rigid body may be replaced by a resultant force applied at a chosen point and a corresponding couple. Now, here we have a chosen point. This chosen point is up to me which point I would like to choose. Let us say that I choose the mass center as my point. So, here chosen point is let us say the mass center. Let me call it G. Now, the effect of these forces can be replaced by a resultant force which is passing through, let us say G. So, we have $\sum F$ and a couple about G. So, let us say M_G and we know the equation of motion. So, $\sum F = ma$, where a is the acceleration and the couple M is going to give me the rate of change of angular momentum. So, let us say I denote the angular momentum by H. So, it is passing through G. So, $\dot{H}_G$. So, here H is the angular momentum, and H_G is the angular momentum of the rigid body about its mass center. So, we have two equations, one is for the translation and one is for the rotation. So, translation equation of motion that is $\sum F = m\bar{a}$. $\bar{a}$ means the acceleration about the mass center. So, that means acceleration of the mass center and we have rotation equation of motion that is $\sum M_G = \dot{H}_G$ and as I said, $\dot{H}_G$ is the rate of change of angular momentum about its mass center. Now, we know that $\dot{H}_G = I\dot{\omega}$ and this I is again about the mass center and this can also be written as because $\dot{\omega} = \alpha$. So, therefore, that can be written as $\dot{H}_G = \bar{I}\alpha$. This is the general equation of motion for a rigid body in planar motion.



Now, note that to write down these equations of motion, we have taken or chosen the mass centre as our point, but we can also choose some arbitrary point on the rigid body. So, let us write down the alternate moment equation for that point. Alternate moment equation. So, we have this rigid body. Earlier, we have taken G. Now, let us take point P. Now, in the previous case, we saw that the equation of motion was $\sum F = ma$ and then there was the moment about the mass center, M_G. Now, let us say this point P is situated at a perpendicular distance of d from the line of action of ma, then $\sum M_P = \sum M_G + m\bar{a}d$.

Now, we know that $\sum M_G = \dot{H}_G = \bar{I}\alpha$. So, therefore, this equation can be written as $\sum M_P = \bar{I}\alpha + m\bar{a}d$ or we can write down $\sum M_P = \dot{H}_G + m\bar{a}d$. Now, let us see when the point P is more convenient than the mass centre point G. Let us say I have a rigid body and on this rigid body force F_1, F_2, F_3 are acting. So, we have our mass center G, let us say over here. Now, let me extend the line of action of F_1 and F_3. And at the intersection, let me take point P. So, if we have less information about F_1 and F_3, in that case, we can take the moment about P because then the effect of F_1 and F_3 will be 0. So, if you want to illuminate, F_1 and F_3, then the point P is more convenient. Now, based on these concepts, let us look at some of the problem.



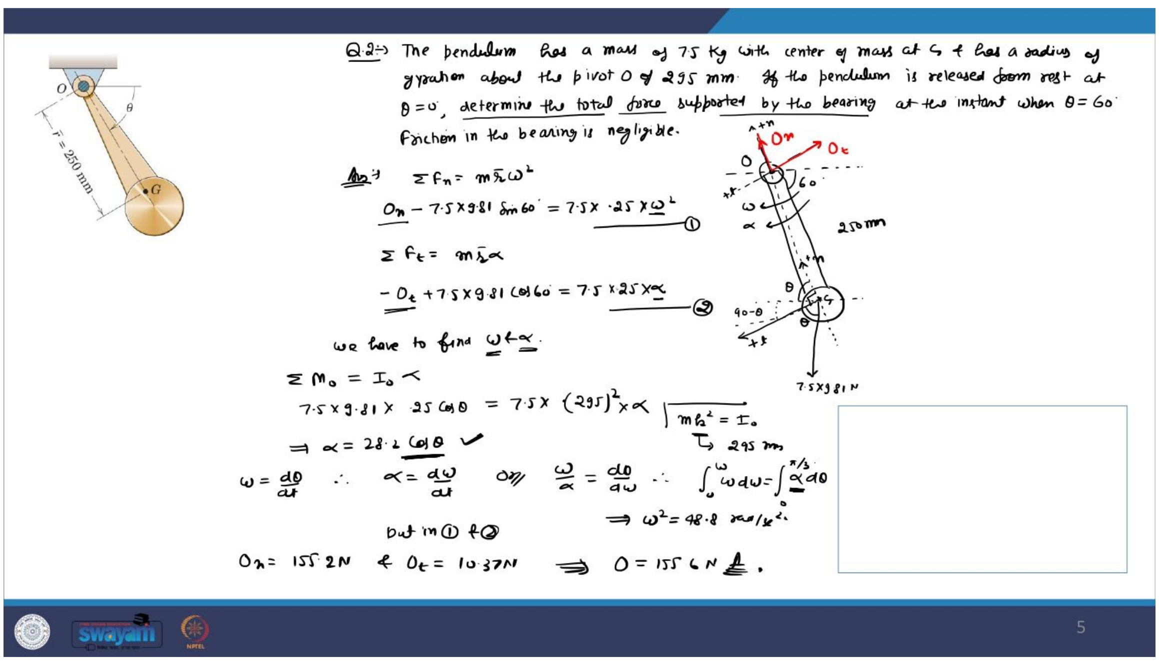
The first problem statement is following. An air table is used to study the elastic motion of flexible spacecraft model pressurized air escaping from numerous small holes in the horizontal surface provides a supporting air cushion which largely eliminate friction. The model shown consists of a cylindrical hub of radius r and four appendage of length l and small thickness t. The hub and the four appendages all have the same depth d and are constructed of the same material of density ρ. Assume that the spacecraft is rigid, determine the moment M which must be applied to the hub to spin the model from rest to an angular velocity ω in a time period of $\tau\ sec$. Now, in this question, we have been asked to determine the moment M which must be applied to the hub to spin the model from rest to an angular velocity omega in period tau. So, $\sum M_z = I_{zz}\alpha$ where α is the angular acceleration and α will be the angular velocity divided by time. So, it is τ. Therefore, $\sum M_z = I_{zz}\frac{\omega}{\tau} - - - - - - (1)$ and the only thing that we need to determine now is I_{zz}. So, let us say h for the hub and b for the blade. So, we have one hub and it is in the shape of the disc. So, therefore, the moment about the z -axis will be $I_{zz} = \frac{1}{2}m_h r^2 + 4\left(\frac{1}{12}m_b l^2 + m_b\left(r + \frac{l}{2}\right)^2\right)$.

Now, we can simplify this because the density is given. So,

$$= \frac{1}{2}(\pi r^2 d\rho)r^2 + 4(ldt\rho)\left(\frac{l^2}{12} + r^2 + \frac{l^2}{4} + rl\right)$$

$$= \rho d\left(\frac{r^4\pi}{2} + 4lt\left(\frac{l^3}{3} + rl + r^2\right)\right)$$

So, this is the moment of inertia about the z-axis. Now, let us put this in 1. So, we get $M = \frac{\rho d\omega}{\tau}\left(\frac{\pi r^4}{2} + 4lt\left(\frac{l^3}{3} + rl + r^2\right)\right)$. So, this is the moment that we have to apply to spin this model from rest to an angular velocity ω in time τ.



Now, let us look at another problem statement. The pendulum has a mass of 7.5 kg with centre of mass at G and has a radius of gyration about the pivot O of 295 mm. If the pendulum is released from rest at $\theta = 0°$, determine the total force supported by the bearing at the instant when $\theta = 60°$ and it is given that the friction in the bearing is negligible. In this question, we have been asked to determine the total force supported by the bearing at the instant when $\theta = 60°$. So, let us first identify all the forces that are acting on the pendulum and the pivot. So, we have this pendulum and we have the pivot at point O. So, first of all its mass is going to act downwards and it is given it is $7.5\ kg$. So, $7.5 \times 9.81\ N$ and it is making an angle θ which is $60°$. So, let me just write down this angle as θ. So, therefore, this angle will also be θ and let us take the direction along the pendulum as n and the tangential direction in which it is moving let us say that is t. So, we have $+n$ direction here and we have the t direction $+t$ direction. Now, since this angle is θ and this angle is $90°$, therefore, this angle will be $90 - \theta$ and therefore, this angle will be θ. Now, this pendulum is rotating about point O. So, let us say its angular velocity is ω and the angular acceleration is α. So, we can now write down the equation of motion in the n and the t direction. So, the force in the n direction will be, so note that about O, this is making a fixed axis rotation and in the fixed axis rotation, the force in the n direction, $\sum F_n = m\bar{r}\omega^2$. Let us identify the forces in the n direction. So, we have at O, we have the reaction forces. Let me mark them by red. Now, at the pivot, we have the reaction forces also. So, let us say we have from O in the t direction and from O, we have the force in the n direction. Now, let us balance all the forces. So, we have O_n, force in the n direction minus the projection of the weight along the n direction. So, that is

$O_n - 7.5 \times 9.81\ sin60 = 7.5 \times 0.25 \times \omega^2 ------(1)$.

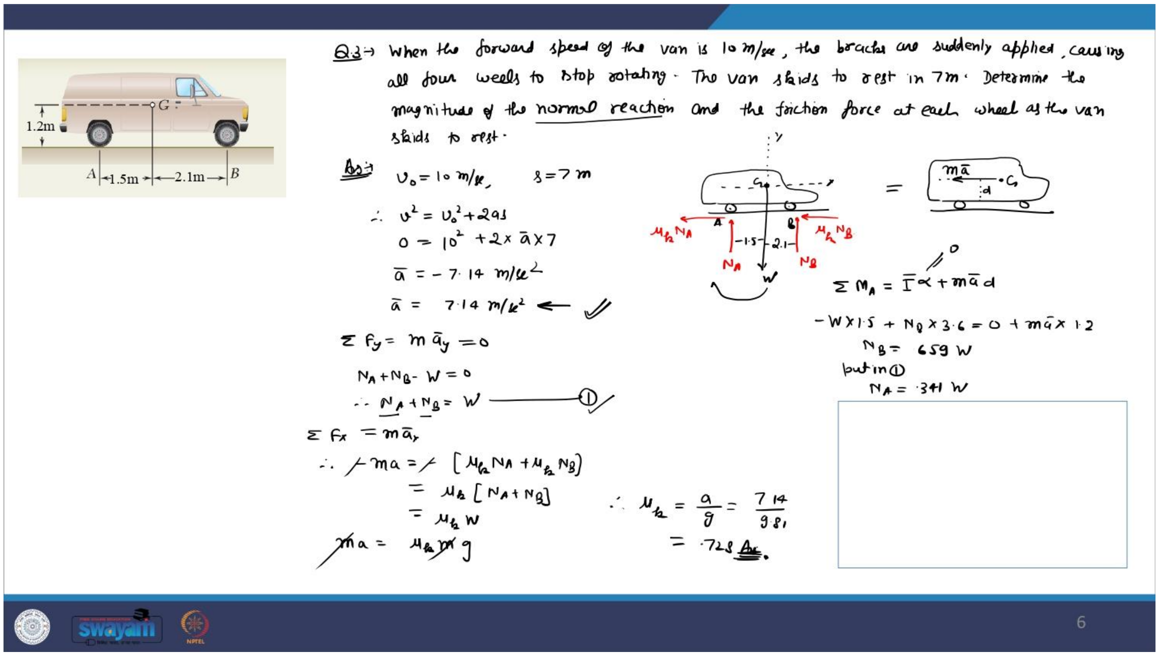
Similarly, let us balance the force along the t direction. So, we have the equation of motion $\sum F_t = m\bar{r}\alpha$. So, the forces are

$-O_t + 7.5 \times 9.81\ cos60 = 7.5 \times 0.25 \times \alpha ------(2)$.

Now, note that the support reaction at O, I can find out if I know the value of ω and α. Therefore, first we have to find out what is ω and α. Now, α I can find out by taking the moment about O. So, we know that $\sum M_O = I_O\alpha$ where I_O is the moment of inertia about O. So, $7.5 \times 9.81 \times 0.25\ cos\theta = 7.5 \times (0.295)^2 \times \alpha$ and this gives me $\alpha = 28.2\ cos\theta$ and if you put $\theta = 60°$, we get the instantaneous value of α. Now, let us find out ω. We know that $\omega = \frac{d\theta}{dt}$. Therefore, $\alpha = \frac{d\omega}{dt}$ or $\frac{\omega}{\alpha} = \frac{d\theta}{d\omega}$ and therefore, $\omega d\omega = \alpha d\theta$. Now, the value of θ changes from 0 to $\pi/3$ and we know the functional form of α as a function of θ, it is this. So, therefore, we get $\omega^2 = 48.8\ rad/\sec^2$ and we can put this in equation number 1 and 2 . So, we get $O_n = 155.2\ N$ and $O_t = 10.37\ N$ and therefore, $O = \sqrt{O_n^2 + O_t^2} = 155.6\ N$.

Now, let us look at one more problem. Problem statement is when the forward speed of the vane is $10\ m/sec$, the brakes are suddenly applied causing all four wheels to stop rotating. The vane skids to rest in $7\ m$. Determine the magnitude of the normal reaction and the friction force at each wheel as the vane skids to rest. So, let us look at the free body diagram. So, we have this vane. It has the center of mass. So, its weight is going to act downward. And at point A, we will have a reaction force. So, let us say N_A. Similarly, at B, we have

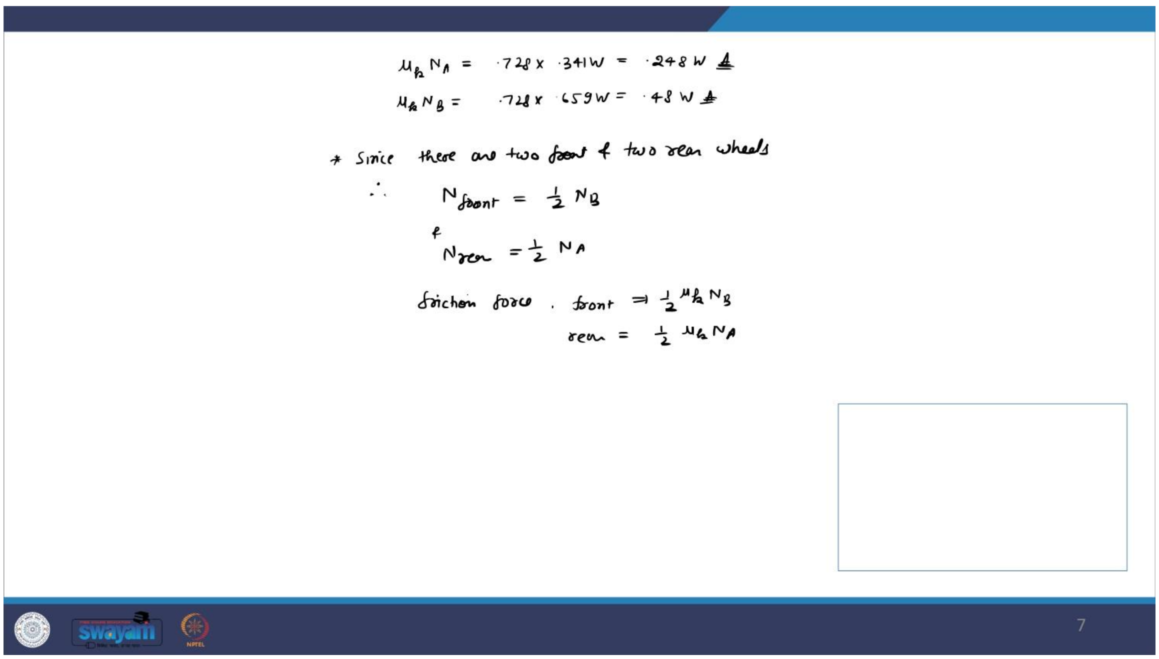
N_B. And the frictional forces are going to oppose the motion. So, here we have the frictional force at A is $\mu_k N_A$, where μ_k is the kinetic friction coefficient. And similarly, here we have $\mu_B N_B$. Now, because of all these forces, the van is eventually going to stop and going to experience a deacceleration force.



So, we have G and the force will be ma. So, it is given that the initial velocity is $10\frac{m}{sec}$d it travels a distance of $7\ m$ before it comes to the rest. So, we can use the deacceleration value $v^2 = v_0^2 + 2as$. Final velocity is 0. Initially, it is moving with a velocity of $10\frac{m}{sec}$o, $0 = 10^2 + 2 \times a \times 7$. So, that gives you the acceleration $a = -7.14\frac{m}{\sec^2}$or the acceleration is $7.14\frac{m}{\sec^2}$ in the opposite direction of the velocity. Now, let us look at the forces in the y direction. So, we have in the y direction, there is no acceleration. So, we have $\sum F_y = ma_y$, but since there is no acceleration, so that will be equal to 0. Therefore, the forces in the y direction should balance. So, we have force in the y direction $N_A + N_B - W = 0$. Therefore, $N_A + N_B = W ----- (1)$. Now, forces in the x direction $\sum F_x = ma_x$. So, therefore, $-ma$ equal to let us identify the force in the x direction they are only the frictional forces

$$-ma = -(\mu_k N_A + \mu_k N_B)$$
$$= \mu_k (N_A + N_B)$$
$$= \mu_k W$$
$$ma = \mu_k mg$$

m will get cancelled. And the value of a we already know. So, therefore, $\mu_k = \frac{a}{g} = \frac{7.14}{9.81} = 0.728$. Now, we are also asked to find out the normal reaction forces and for that we have equation number 1. But in equation number 1, there are two unknown N_A and N_B. Therefore, we need one more equation. And for that, we can use the moment equation. For that, we have to choose a point. So, the more convenient point in this problem is either point A or point B instead of point G because in that case, one of the reaction forces will go away. So, let me choose point A and then take the moment about that. So, the equation of motion is $\sum M_A = \bar{I}\alpha + m\bar{a}d$. So, let us take the moment about A. So, the tendency of the force W will be to rotate this in the clockwise direction. So, therefore, $-W \times 1.5 + N_B \times 3.6 = 0 + m\bar{a} \times 1.2$. Now, this equation gives me $N_B = 0.659\, W$. We can put this in 1. We will get $N_A = 0.341\, W$.



Therefore, the frictional forces which are $\mu_k N_A = 0.728 \times 0.341\, W = 0.248\, W$ and $\mu_k N_B = .728 \times 0.659\, W = 0.48\, W$. And note that there are two front wheels and two rear wheels in the van. So, therefore, the N_{front} or the reaction force will be the half of what we have determined. Let me just write down, since there are two front and two rear wheels, Therefore, $N_{front} = \frac{1}{2} N_B$ and $N_{rear} = \frac{1}{2} N_A$. Similarly, the friction force at the front will be $\frac{1}{2}\mu_k N_B$ and at the rear, it will be $\frac{1}{2}\mu_k N_A$.

With this, let me stop here. See you in the next class. Thank you.