MECHANICS
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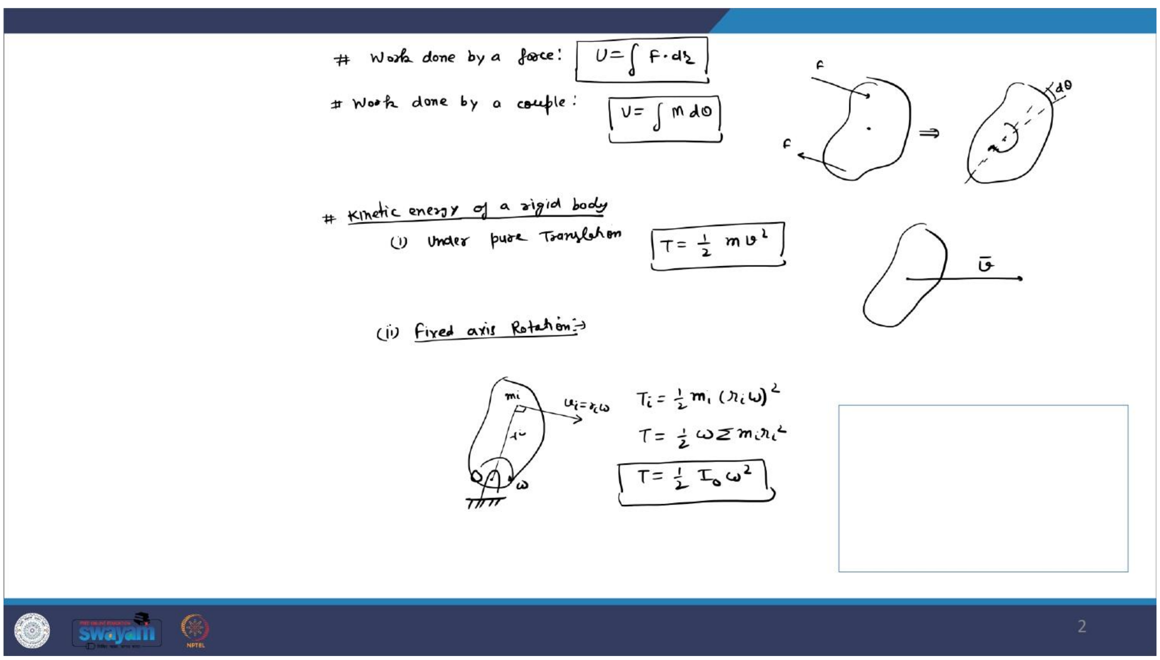
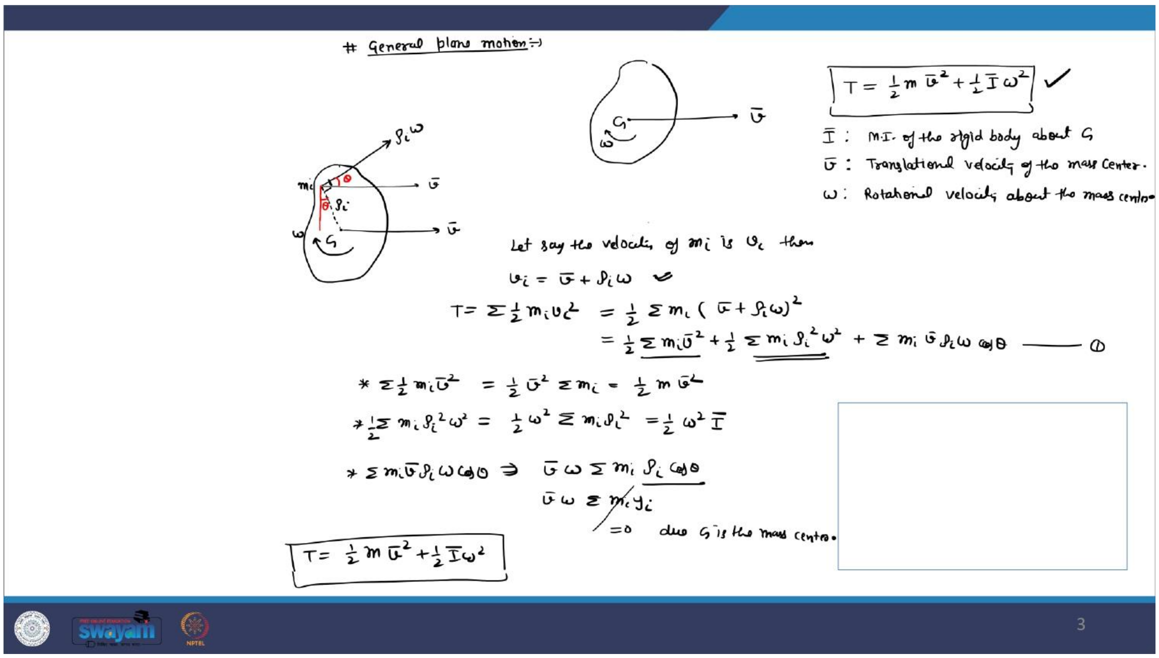
Lecture 55
Plane motion of a rigid body: work energy equation

Hello everyone, welcome to the lecture again. In the last lecture, we discussed about the equation of motion of a rigid body which is moving in a plane. We saw that the equation of motion can be analyzed by considering the pure translation of a point on the rigid body and the fixed axis rotation about that point. Today, we are going to discuss the work energy equation and for that we need some of the basic concepts that we already know.



Let me mention them here. So, we know that the work done by a force is $U = \int F.dr$, where dr is the infinitesimal displacement and F is the force. We also know that the work done by a couple is $U = \int Md\theta$. So, the situation is suppose I have a rigid body and on this rigid body the couple is acting. So, we have two equal opposite forces F and $-F$ that will give you the couple and because of that this rigid body is let us say rotating and it makes let us say rotate by an angle $d\theta$ then $U = \int Md\theta$. Now, we also know that the kinetic energy of a rigid body let us say under pure translation is equal to $T = \frac{1}{2}mv^2$. So, we have a rigid body and this rigid body is just translating with velocity v. Now, if the body is making a fixed axis rotation. So, we have this rigid body and let us say this rigid body is rotating about a point O with a velocity or the angular velocity ω. Then, the kinetic energy of the i^{th} particle will be $T_i = \frac{1}{2}m_i(r_i\omega)^2$, wherein this distance is r_i, mass is m_i and its velocity $v_i = r_i\omega$. Now, this ω is constant on the body because the whole body is rotating with a fixed angular velocity. Therefore, total $T = \frac{1}{2}\omega^2\sum m_i r_i^2$. And $\sum m_i r_i^2$ is the moment of inertia of this body about point O. So, $T = \frac{1}{2}I_o\omega^2$.



Now, let us discuss the kinetic energy of a general plane motion. So, as we already discussed that the general plane motion can be analyzed by considering a point on the rigid body and then translation of that point plus the fixed axis rotation about that point. Let us say we choose point G which is the center of mass of the rigid body. So, we have this rigid body. Let us say its center of mass is moving with a velocity $\bar{v}$ and the body is rotating about this G with an angular velocity ω, then the kinetic energy $T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$. Here, $\bar{I}$ is the moment of inertia of this rigid body about G. So, I is the moment of inertia of the rigid body about G or the mass centre and $\bar{v}$ is the translational velocity of the mass centre or G and ω is the rotational velocity about the mass centre. Now, let me show you that the kinetic energy of the rigid body will indeed be $\frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$. For that, let us consider a i^{th} particle on the rigid body. Let us say its mass is m_i. And as I said, this rigid body is moving with a translational velocity of $\bar{v}$. Let us say, the position of the mass m_i is ρ_i. Then the mass m_i will have a translational velocity of $\bar{v}$ plus a rotational velocity about G. So, it will be $r\omega$. So, $r = \rho_i \times \omega$. And that will be perpendicular to the line that joins m_i to G. Now, from the geometry, you can see that if this angle is θ, then if I draw the perpendicular to $\bar{v}$, then that angle will also be θ. Let us say the velocity of mass m_i is v_i, then this v_i will be the sum of the translation velocity plus the rotational velocity. So, that will be $v = \bar{v} + \rho_i\omega$. Now, the kinetic energy $T = \frac{1}{2}m_i v_i^2$. Let us put the value of v_i from here. So, we will have $T = \frac{1}{2}\sum m_i\,(\bar{v} + \rho_i\omega)^2$. And we can expand this bracket term. So, it will be

$$T = \frac{1}{2}\sum m_i\bar{v}^2 + \frac{1}{2}\sum m_i\rho_i\omega^2 + \sum m_i\bar{v}\rho_i\omega cos\theta - - - - - (1)$$

where θ is the angle between $\bar{v}$ and $\rho_i\omega$. Let us look at it term by term. So, the first term is $\frac{1}{2}\sum m_i\bar{v}^2$ and $\bar{v}$ is the translation velocity of the rigid body and also of all the particle. So, therefore, $\bar{v}$ is independent of the summation. We can take it outside. So, I have $\frac{1}{2}\sum m_i\bar{v}^2$ and $\sum m_i$ is just m. So, we have

$$\star \sum \frac{1}{2}m_i\bar{v}^2 = \frac{1}{2}\bar{v}^2\sum m_i = \frac{1}{2}m\bar{v}^2$$

Now, let us look at the second term. The second term is $\frac{1}{2}\sum m_i\rho_i\omega^2$. Again, ω is constant.

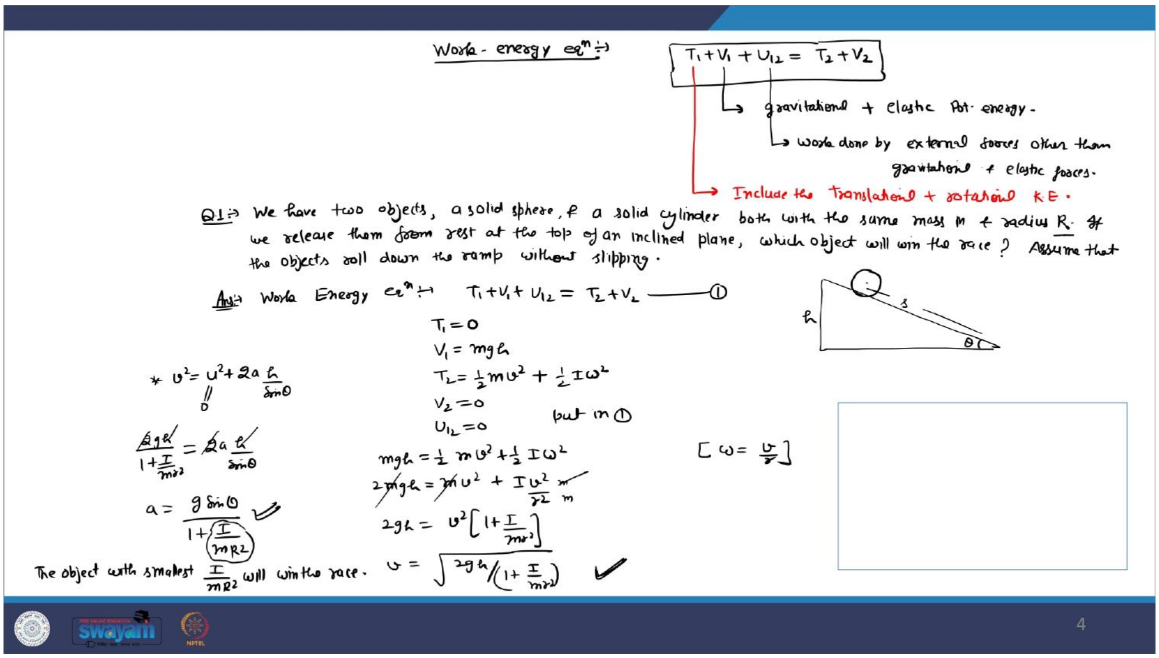
$$\star \frac{1}{2}\sum m_i\rho_i^2\omega^2 = \frac{1}{2}\omega^2\sum m_i\rho_i^2 = \frac{1}{2}\omega^2\bar{I}$$

Now, let us look at the last term that is $\sum m_i\bar{v}\rho_i\omega cos\theta$. So, this I can write down as $\bar{v}$ because that is constant. ω is also constant. $\rho_i cos\theta$ is the projection of ρ_i along the y direction. So, let me call it y_i. Now, this y_i is measured from G, which is the mass center and because of the definition of the mass center, $\sum m_i x_i$ or $\sum m_i y_i = 0$. Why? Because G is the mass center.

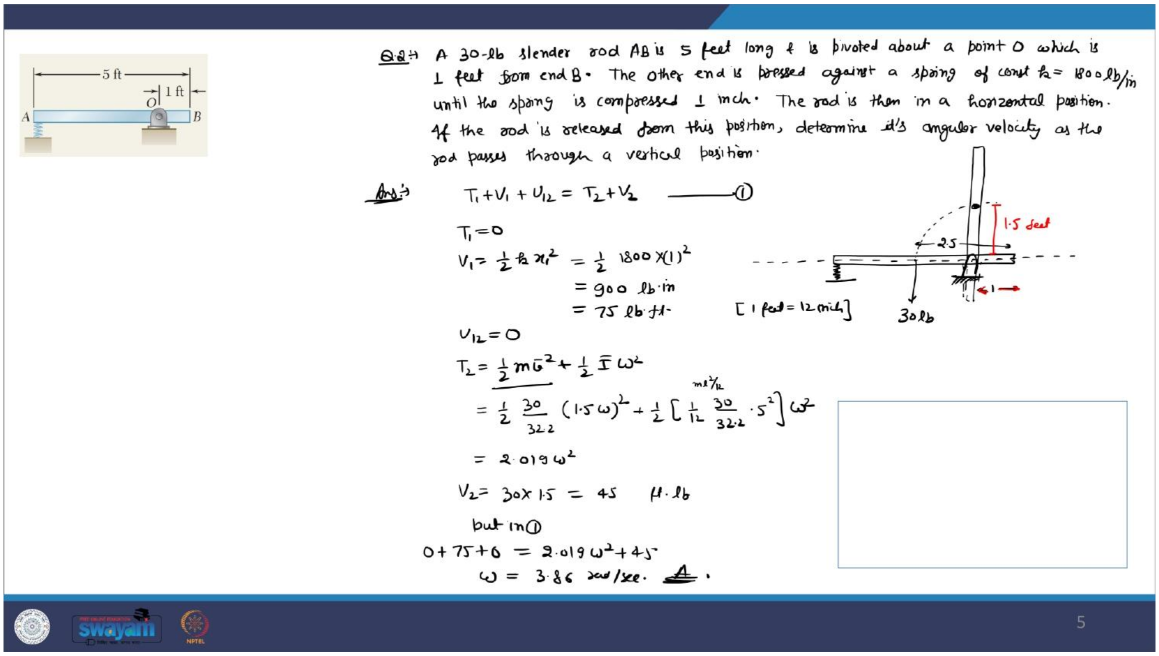
$$\star \sum m_i\bar{v}\rho_i\omega cos\theta = \bar{v}\omega\sum m_i\rho_i cos\theta = \bar{v}\omega\sum m_i y_i = 0$$

So, if we put this in 1, then we have $T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$.

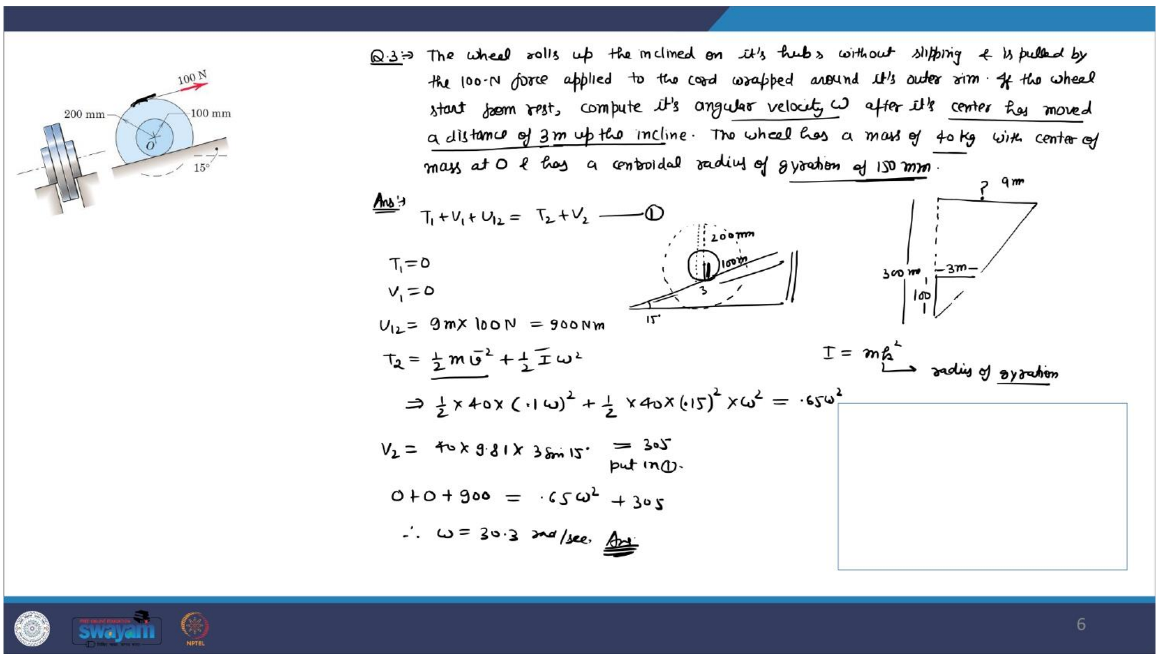
Now, let me write down the work energy equation for the rigid body which is moving in a plane. So, we already know the work energy equation for the particle and let us follow the same convention that we have followed earlier. So, we had $T_1 + V_1 + U_{12} = T_2 + V_2$. Here, T_1 is the initial kinetic energy, V_1 is the potential energy wherein we consider the gravitational potential energy and the elastic potential energy. So, this has gravitational plus elastic potential energy and U_{12} is the work done by external forces. Of course, other than the gravitational and you know elastic forces and you have to note that the term T_1 and of course, T_2 it will include the translational plus rotational kinetic energy. So, based on this, now let us look at a couple of problem. The first problem is the racing spheres. So, let me write down the statement. We have two object a solid sphere and a solid cylinder both with the same mass m and radius r. If we release them from rest at the top of an inclined plane which object will win the race and assume that the object roll down the ramp without slipping. So, we have the following situation we have this ramp to formulate the problem let us say this height is h and this angle is theta. So, we have our object here. Now, to analyze this problem we can use the work energy equation the whole thing is moving in a plane, which is $T_1 + V_1 + U_{12} = T_2 + V_2 ----(1)$. So, initially the object is at rest. Therefore, T_1 will be 0. The initial kinetic energy will be 0. V_1 is the potential energy.



So, it has only the gravitational potential energy that is equal to mgh. Now, the final kinetic energy $T_2 = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$. And finally, $V_2 = 0$, $U_{12} = 0$. So, let us put this in equation number 1. So, we have $mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$ and ω we can write down in terms of v because $\omega = \frac{v}{r}$. So, therefore, I have $2mgh = mv^2 + \frac{I\,v^2}{r^2}$ and let me multiply by m and divide by m, so that m will get cancelled and we have $2gh = v^2\left(1 + \frac{I}{mr^2}\right)$ or $v = \frac{\sqrt{2gh}}{\left(1+\frac{I}{mr^2}\right)}$. So, this is the velocity of the object when it covers a vertical distance of h. Now, let us find out the acceleration of the object for that we can use $v^2 = u^2 + 2as$ and from the geometry $s = \frac{h}{sin\theta}$. So, let me call this as s. So, initial velocity u was 0. Let us put v^2 from here. So, we have $\frac{2gh}{1+\frac{I}{mr^2}} = 2a\frac{h}{sin\theta}$. Now, 2 and h will get cancelled. So, we have $a = \frac{gsin\theta}{1+\frac{I}{mr^2}}$ and the radius of the object is given as r. So, let me write down r^2. This is the acceleration of the object. Now, this tells you that the object with the smallest $\frac{I}{mr^2}$ will win the race. So, the object with smallest $\frac{I}{mr^2}$ will win the race.



Now, let us look at another problem statement. A $30-lb$ slender rod AB is 5 $feet$ long and is pivoted about a point O which is 1 $feet$ from end B. The other end is pressed against a spring of constant $k = 1800\ lb/in$ until the spring is compressed 1 $inch$. The rod is then in a horizontal position. Now, if the rod is released from this position, determine its angular velocity as the rod passes through a vertical position. So, initially the rod is like this. Its weight is 30 lb, which is going to act at the center of the mass. So, this is 1 $feet$ and the whole length is 5 $feet$. So, therefore, this will be 2.5 $feet$. And so, this was the initial situation and when the rod is released, then it will goes into the vertical position and we have to find out its angular velocity in that case. So, let us use the work energy equation. So, we have $T_1 + V_1 + U_{12} = T_2 + V_2 - - - - (1)$. Initially, the rod was at rest. So, $T_1 = 0$. Now, let us look at V_1. Let me fix our reference point over here. So, there is no gravitational potential energy. We only have the spring energy. So, that will be $\frac{1}{2}kx_1^2$. So, that will be $\frac{1}{2}1800 \times (1)^2 = 900\ lb\ in$. Since all other parameters are given in feet, so let us write down this in feet also. We know that $1\ feet = 12\ inch$. So, therefore, this will be 75 $lb\ ft$. Now, U_{12}, there is no work done by, you know, external forces. So, that will be 0. Now, let us look at T_2, the kinetic energy. So, $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$. Now, let us find out $\frac{1}{2}mv^2$. So, that will be $\frac{1}{2}m$ is weight divided by the gravitational constant and the weight is 30 lb divided by g. g is 32.2 feet per second square. So, this is my mass into v. $v = r\omega$. Now, let us look at r. So, note that this length is 1, this is 2.5, therefore, from here to here, it will be 1.5 $feet$. So, that is our r. So, r is 1.5 into ω square plus half I, moment of inertia of the rod about the centre of mass will be $\frac{ml^2}{12}$ where l is the length of the rod. So, we have $T_2 = \frac{1}{2}\frac{30}{32.2}\,(1.5\omega)^2 + \frac{1}{2}\left(\frac{1}{12}\frac{30}{32.2}\,5^2\right)\omega^2$. Now, this can be simplified and it comes out to be $2.019\,\omega^2$. Now, let us look at V_2. V_2 is the potential energy in that situation. So, that will be you know there is no spring force now. So, it is only the gravitational potential energy. So, that will be mgh. So, mg is 30 and h is 1.5. So, that is 45 $ft - lb$. So, let us put this in equation number 1. So, we have $0 + 75 + 0 = 2.019\omega^2 + 45$ and that gives you $\omega = 3.86\,rad/sec$.



Now, let us look at one more problem. Problem statement is the wheel rolls up the inclined on its hubs without slipping and is pulled by the 100 N force applied to the chord wrapped around its outer rim. If the wheel start from rest, computes its angular velocity ω after its center has moved a distance of 3 m up the incline. The wheel has a mass of 40 kg with centre of mass at O and has a centroidal radius of gyration of 150 mm. So, in this question, we have a wheel which has a hub and this wheel moves up on the inclined plane which makes an angle of 15° from the horizontal. So, we have this hub and then this hub is connected to a wheel. The radius of the hub is 100 mm and the radius of the wheel is 200 mm. Now, it is given that the center move a distance of 3 m up the incline. So, when the center move a distance of 3 m, in that case, let us see how much the outer rim will move and that we can find out using very simple geometry because this distance is 100 mm. So, we have 100 mm and in that case, the center moves a distance of 3 m and the outer rim is at a distance of 300 mm. So, now, let us say it is 300 mm. In that case, how much the outer rim will move? So, how much is this? And you can find out from the very simple geometry that if this is 100, then this one is $3\ m$. So, therefore, when this is $300\ mm$, then that will be $9\ m$. Now, to find out the angular velocity, let us use the work energy theorem. So, we have $T_1 + V_1 + U_{12} = T_2 + V_2 ----(1)$. Now, initially the whole thing was at rest. So, therefore, T_1 was 0 and let us put our reference point at the bottom. So, therefore, initial potential energy was also 0. Let us look at the work done on the wheel. So, U_{12} will be the distance multiplied by the force. So, the distance was $9\ m$ into the force. Note that the force was applied here. Therefore, we have calculated how much distance this point has travelled. So, that was $9 \times 100 = 900\ N$. Now, let us look at the final kinetic energy. So, that will be $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$. Now, $\frac{1}{2} \times 40 \times (0.1\ \omega^2) + \frac{1}{2} \times 40 \times (0.15)^2 \times \omega^2 = 0.65\omega^2$. Now, let us look at V_2. V_2 will be the potential energy. So, that will be mgh. So, m is 40, g is 9.81. So, it travel a distance of $3\ m$ on $15°$ inclined. So, therefore, $h = 3sin15°$. Because this is 3. So, therefore, that distance will be $3sin15°$. Let us put all this in equation 1 . So, we have $0 + 0 + 900 = 0.65\omega^2 + 305$. So, from here we get $\omega = 30.3\ rad/sec$.

With this let me stop here. See you in the next class. Thank you.