MECHANICS

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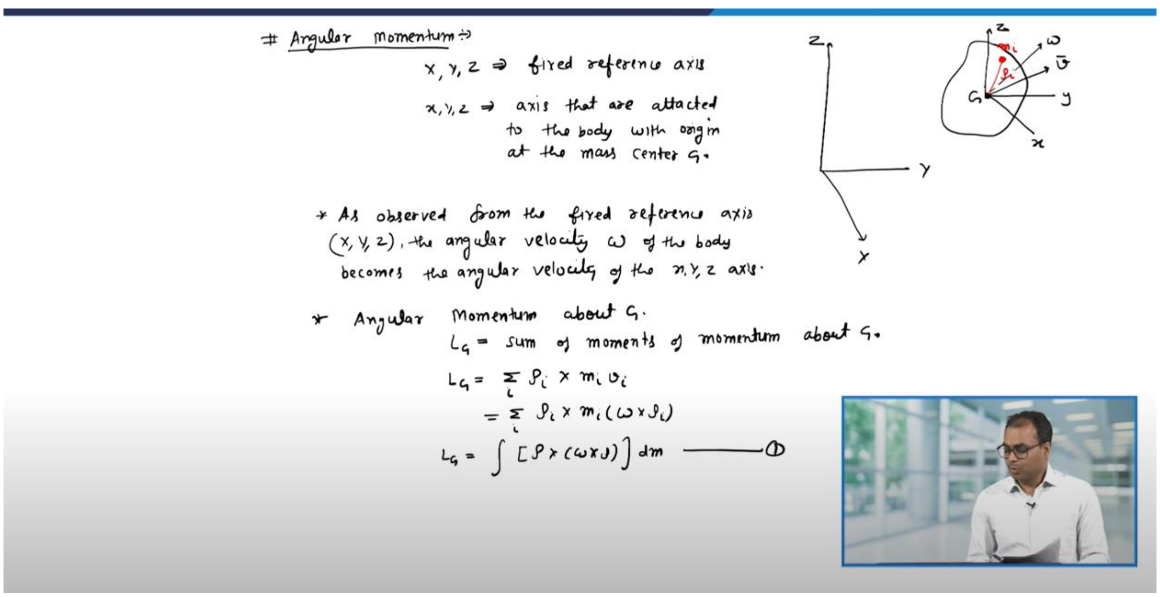
Lecture: 57

Three-dimensional dynamics of rigid bodies: angular momentum

Hello everyone, welcome to the lecture again. In the context of the dynamics of rigid bodies, in the last few lectures, we have discussed the fixed axis rotation, general plane motion, work energy equation and impulse momentum equation.



Today, we are going to discuss the angular momentum of the rigid body. Let us say XYZ are some fixed reference axis and then we have a rigid body and with this rigid body, we attach a xyz .

Okay, so $X, Y, and\ Z$, these are the fixed reference axes, and note that this rigid body is making a general plane motion. $x, y, and\ z$ Are the axes that are attached to the body and let us say their origin is at the mass center of the rigid body? So, this is G . Now, if we observe from the $X, Y, and\ Z$ reference frame and let us say this rigid body has an angular velocity ω , in that case, the xyz axis will also rotate with the same angular velocity ω because these axes are fixed with the rigid body.

So, let me mention this point that as observed from the fixed reference axis that means XYZ , the angular velocity ω of the body it becomes the angular velocity of the $x, y, and\ z-axis$ because the $x, y, and\ z-axis$ are fixed with the body. So, we are saying that let us say this rigid body is having an angular velocity ω and this also has a linear velocity let us say v . So, what is the angular momentum about the mass center G ?

Angular momentum about mass center G will be $L_G = sum\ of\ moments\ of\ momentum\ about\ G$, okay. And to find out the sum of the moment of momentum, let us say I have, let us say i of particles, its mass is m_i and let us say its distance from G is ρ_i, okay. So, L_G will be, so there are n particles, let us say, so $L_G = \sum_i \rho_i \times m_i v_i$. this is the moment multiplied by the momentum $m_i v_i$, and since $v = \omega \times r = \omega \times \rho_i$. So, therefore, this can be written as $L_G = \sum_i \rho_i \times m_i(\omega \times \rho_i)$. Note that ω is independent of i because the body is rotating about G. So, therefore, L_G will be integral.

So, I can write down summation into integral $L_G = \int [\rho \times (\omega \times \rho)]dm$. Let us call it equation number (1). Now, suppose instead of rotating about G, let us say the body is making a fixed axis rotation. So, in the same scenario, I have this rigid body, and let us say this rigid body is making a fixed axis rotation about O; then again, we have $X, Y, and\ Z - axis$. And then we have $x, y, and\ z$, which are fixed with the body.

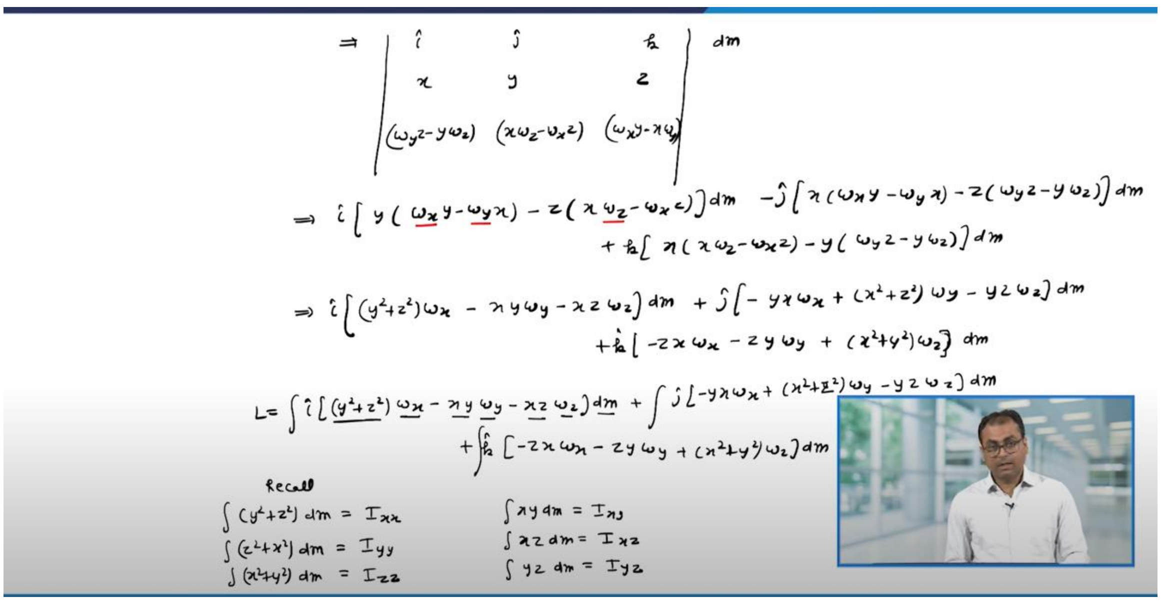
So, when the body rotates, the $x, y, and\ z - axis$ will also rotate. So, let us say this is the $x - axis, y - axis, and\ z - axis$. Let us say the mass of the i^{th} particle is n_i, and its distance from O is r_i. So, as I said, $x, y, and\ z - axis$ they are attached to the body.

Therefore, both the body and $x, y, and\ z - axis$ they have angular velocity ω . Now, in this case, what is the angular momentum about O? Well, again, it is a moment of momentum. So, $L_O = \sum_i r_i \times m_i v_i$ and $v = \omega \times r$. So, I can write down $L_O = \sum_i r_i \times m_i(\omega_i \times r_i)$. And of course, this I can write down in the integral form.

So, this is $L_O = \int [r \times (\omega \times r)]dm.$ Let us call it equation number (2), and you can see that equation number (2) and equation number (1), they have the identical form. So, in equation number (1), we calculated the angular momentum about the mass center G, and in equation number (2), we calculated the angular momentum about a fixed point O. So, equations number (1) and (2) have identical forms. Okay. And I can also write down this equation as you know $L_O\ or\ L_G = \int\ dL$.

So, in this case, this $dL = r \times (\omega \times r)dm$, okay. Now, to find out what is this quantity? Let us write down component by component. So $r \times (\omega \times r)dm$. We are interested in finding out this quantity so that we can find out what is L_O or L_G. So, this I can write down as $r = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ in the Cartesian coordinate. So, this is $(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}) \times$. then we have ω . ω Also has xyz component.

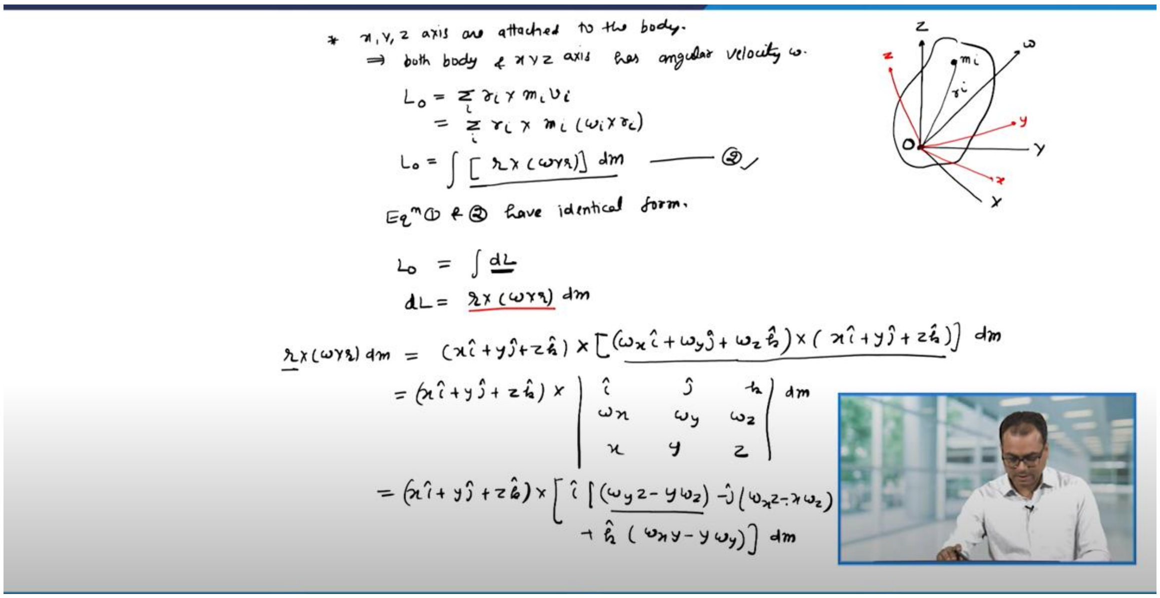
So, $(\omega_x\,\hat{\imath} + \omega_y\,\hat{\jmath} + \omega_z\,\hat{k}) \times$. And cross product with again r. $r = (x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k})$, and then we have dm . So, first, let us calculate this cross product, and that is equal to $(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k})$ cross product $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$, ω_x, ω_y, ω_z, *and* x, y, z, and then we have to multiply it by dm . So, this becomes $(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k})$ cross product. So, we have $\hat{\imath}(\omega_y z - y\omega_z) - \hat{\jmath}\,(\omega_x z - x\omega_z) + \hat{k}(\omega_x y - x\omega_y)$ and then you have to multiply by dm .



Now, again, this is a cross product of two vectors. So, $a \times b$ again, we can write down in the matrix form and we can evaluate its value. So, this becomes $\hat{\imath}$, $\hat{\jmath}$, *and* $\hat{k}$ and we have x, y, z, and then we have coefficients of $\hat{\imath}$, $\hat{\jmath}$, *and* $\hat{k}$. So, the first one is $(\omega_y z - y\omega_z)$, and then the coefficient of y is $-(\omega_x z - x\omega_z)$. So, this I can write down as $(x\omega_z - \omega_x z)$ and then the coefficient of k which is $(\omega_x y - x\omega_y)$, and then we have dm . So, let us calculate this also. So, this becomes $\hat{\imath}[y(\omega_x y - \omega_y x) - z(x\omega_z - \omega_x z)]dm - \hat{\jmath}[x(\omega_x y - \omega_y x) - z(\omega_y z - y\omega_z)]dm + \hat{k}[x(x\omega_z - \omega_x z) - y(\omega_y z - y\omega_z)]dm$.

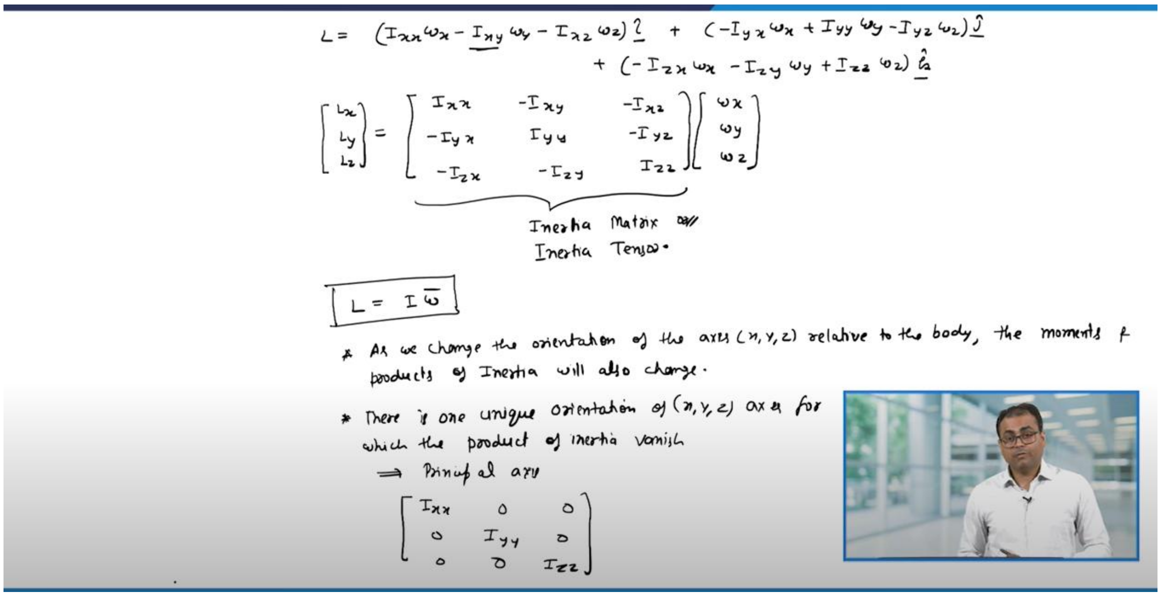
Now, we can collect the coefficient of ω_x *and* ω_y *and* ω_z inside $\hat{\imath}$, $\hat{\jmath}$, *and* $\hat{k}$. So, here I have ω_x, ω_y *and* ω_z. Let us collect their coefficient. So, this I can write down as i . We have $\hat{\imath}\left[(y^2+z^2)\omega_x - xy\omega_y - xz\omega_z\right]dm + \hat{\jmath}\left[-yx\omega_x + (x^2+z^2)\omega_y - yz\omega_z\right]dm + \hat{k}\left[-zx\omega_x - zy\omega_y + (x^2+y^2)\omega_z\right]dm$. So, now we have the value of $r \times (\omega \times r)$ and we can put that value in equation number (2) or equation number(1).

So, we have to just take the integral of that. So, L will be $\int \hat{\imath}\left[(y^2+z^2)\omega_x - xy\omega_y - xz\omega_z\right] dm$ plus $\int \hat{\jmath}\left[-yx\omega_x + (x^2+z^2)\omega_y - yz\omega_z\right] dm$ plus $\int \hat{k}\left[-zx\omega_x - zy\omega_y + (x^2+y^2)\omega_z\right] dm$.



Now, note that this ω_x, ω_y and ω_z, they are independent of the i^{th} particle because the body is making a fixed axis rotation. So, therefore, they can be taken outside the integral. Now, another important point is look at this quantity $(y^2+z^2)dm$. Similarly, $xydm$ and $xzdm$. So, you have to recall that that $\int (y^2+z^2)dm$ is the moment of inertia about the x-axis.

So, this is I_{xx}. Similarly, $\int(z^2+x^2)dm$ is the moment of inertia about the y-axis, and $\int(x^2+y^2)dm$ is the moment of inertia about the z-axis. Similarly, $\int\ xydm$ is the product of inertia about the x, y-axis, and $\int\ xzdm$ is the product of inertia about $x,\ z-axis$ and $\int\ yzdm$ is the product of inertia about the $y,\ z\ -$axis . So, let us put that in the above equation. Keeping in mind that $\omega_x,\ \omega_y\ and\ \omega_z$ are independent of the integral.

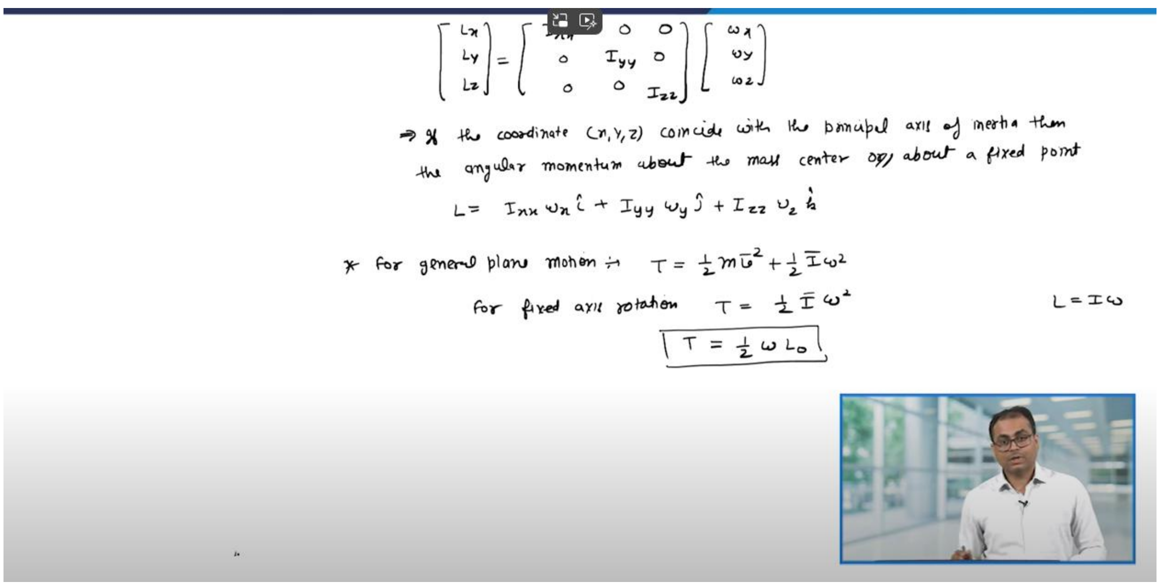


So, we have L equal to $\left(I_{xx}\omega_x-I_{xy}\omega_y-I_{xz}\omega_z\right)\hat{\imath}$. This is the first term because this becomes I_{xx}. So, we have $I_{xx}\omega_x$ minus $\int\ I_{xy}\ dm$ becomes I_{xy}. So, that is the second term and then you have ω_y similarly, $I_{xz}\omega_z$ plus, $\left(-I_{yx}\omega_x+I_{yy}\omega_y-I_{yz}\omega_z\right)j$ plus $\left(-I_{zx}\omega_x-I_{zy}\omega_y+I_{zz}\omega_z\right)\hat{k}$.

So, these are the component of the angular momentum along the $i-axis$, along the $j-axis$ and along the $k-axis$ or unit vectors. This I can very nicely write down in the matrix form. So, $L_x,\ L_y,\ and\ L_z$, it can be written as $I_{xx},\ -I_{xy},\ -I_{xz},\ -I_{yx},\ I_{yy},\ -I_{yz},\ -I_{zx},\ -I_{zy},\ I_{zz}$. and $\omega_x,\ \omega_y\ and\ \omega_z$. And note that this matrix over here is what? This is the inertia matrix that we introduced when we discussed the moment of inertia and product of inertia part.

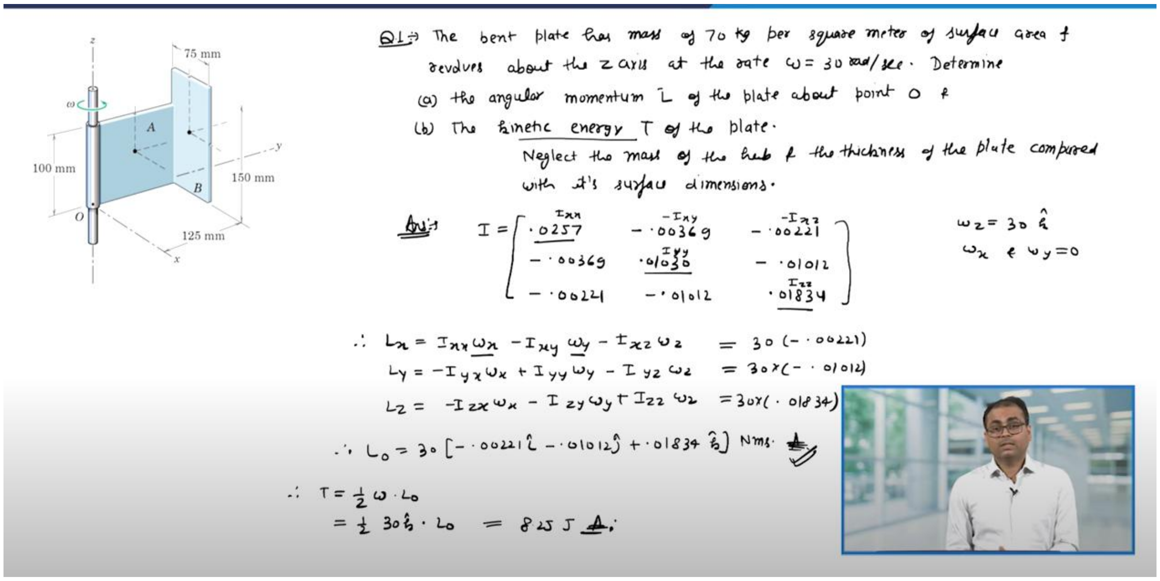
So, this is the inertia matrix or you also call it inertia tensor. Okay. And therefore, I can write down this in more compact form, $L=I\omega$, okay. Here, this I is the inertia tensor. Now, you have to note that as we change the orientation of the $axis\ x,\ y,\ z,$ then the moment and product of inertia also changes because we have fixed them with the rigid body. So, relative to the body, then the moments and products of inertia will also change.

And we have also discussed that you will always find unique orientation of the body or of the $x,\ y,\ z-axis$ for which the product of inertia will vanish and this is called the principal axis. So, this also we know that there is one unique orientation of $x,\ y,\ z-axis$ for which the product of inertia vanishes, okay? And this is called the principal axis, okay? So, in that case, the inertia tensor will be $I_{xx},\ I_{yy},\ I_{zz}$ and all the product of inertia, they will be 0. Now, if that happens, in that case, the total angular momentum $L = I_{xx}\omega_x\hat{\imath} + I_{yy}\omega_y\,\hat{\jmath} + I_{zz}\omega_z\hat{k}$.



Therefore, in this case, your L_x , L_y and L_z, they become I_{xx} , I_{yy} , I_{zz} and the product of inertias are 0 into $\omega_x,\ \omega_y$ and ω_z. And this tells you that if the coordinates x, y, and z, they coincide with the principal axis of inertia, then the angular momentum about the mass center G or about a fixed point O will be L equal to, so let us multiply that. We have $L = I_{xx}\omega_x\hat{\imath} + I_{yy}\omega_y\,\hat{\jmath} + I_{zz}\omega_z\hat{k}$.

Now, we have also shown that for general plane motion, and this is something that we proved that the kinetic energy $T = \frac{1}{2}m\overline{v}^2\ + \frac{1}{2}\overline{I}\,\omega^2$, where this $\overline{I}$ is the moment of inertia about the mass center, and $\overline{v}$ is the linear velocity of the mass center. Now, for fixed axis rotation, there is no $\overline{v}$. So, $\frac{1}{2}m\overline{v}^2$ term will not be there. So, therefore, $T = \frac{1}{2}\overline{I}\ \omega^2$ and just now we have seen that $L = I\omega$.



Therefore, I can write down kinetic energy$T = \frac{1}{2}\omega L_o$. With this introduction, now let us look at an example and the problem statement is following. The Bent plate has mass of 70 kg per square meter of surface area and revolves about the z-axis at the rate $\omega = 30\ \frac{rad}{sec}$, determine (a), the angular momentum L, of the plate about point O. And (b) the kinetic energy T Of the plate. And it is given that neglect the mass of the hub and the thickness of the plate compared with its surface dimensions. So, in this question, we have been asked to find out the angular momentum L of the plate about point O. So, about point O, this is making a fixed axis rotation, and first, we have to find out what is the inertia matrix. So, the inertia matrix I am not calculating because it is something that we have done earlier.

Let me write down the final answer for the moment of inertia matrix for this geometry, the dimensions argument and I come out to be $0.0257,\ -0.00369,\ -0.00221,\ -0.00369,\ 0.0130,\ -0.01012,\ -0.00221,\ -0.01012\ and\ 0.01834$. So, note that this is the moment of inertia about the $x,\ y\ and\ z-axis$ and these are the rest of them are the product of inertia. Now, in this question, it is given that ω_z or the angular velocity about the z-axis is $30\ \frac{rad}{sec}$. And there is no rotation about $x\ and\ y-axis$. So, therefore, $\omega_x\ and\ \omega_y = 0$. Now, the angular momentum about the $x,\ y\ and\ z-axis$, we have already found out.

So, this is the angular momentum about the x-axis, this one is the angular momentum about the y-axis and this one is the angular momentum about the z-axis. So, let me just write it down. The angular momentum about the x-axis is $L_x = I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z$, $L_y = -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z$ and $L_z = -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z$. Okay. Now, the values of $\omega_z,\ \omega_x\ and\ \omega_y$ is known and also the moment of inertia and the product of inertia.

So, we can put the values. So, $L_x = 30(-0.00221)$ because the $\omega_x\, and\ \omega_y = 0$. So, this term will not be there. That term will not be there.

We have only $I_{xz}\ and\ \omega_z$. And I_{xz} is 0.00221. So, let me just write down this was your I_{xx}. This was I_{yy}. This was I_{zz} and this was $-I_{xy}$, $-I_{xz}$ and so on, okay. So, similarly, $L_y = 30 \times (-0.01012)$, and $L_z = 30 \times (0.01834)$. Therefore, the angular momentum about point O $L_o = 30\left[-0.00221\ \hat{\imath}\ - 0.01012\,\hat{\jmath} + 0.01834\ \hat{k}\right] Nms$. Now, let us find out the kinetic energy T of the plate. So, since this plate is making a fixed axis rotation, so therefore, kinetic energy $T\ = \frac{1}{2}\omega\ L_o$ and L_o, we have found out. ω Is also known. This is 30 $\hat{k}$.

So, it will be $T\ = \frac{1}{2} 30\hat{k}. L_o$ and L_o is here. So, this comes out to be 8.250. With this, let me stop here. See you in the next class. Thank you.