Micro and Smart Systems
Prof. G.K. Ananthasuresh
Department of Mechanical Engineering
Indian Institute of Science – Bangalore

Lecture - 16
Microdevice Suspensions: Lumped Modeling

Hello today as part of the micro and smart systems course we are going to look at some mechanical aspects. Mechanical aspects are very important for micro systems because there are lot of elements that have to move we are talking about solids that have to move and that analysis requires mechanical treatment.

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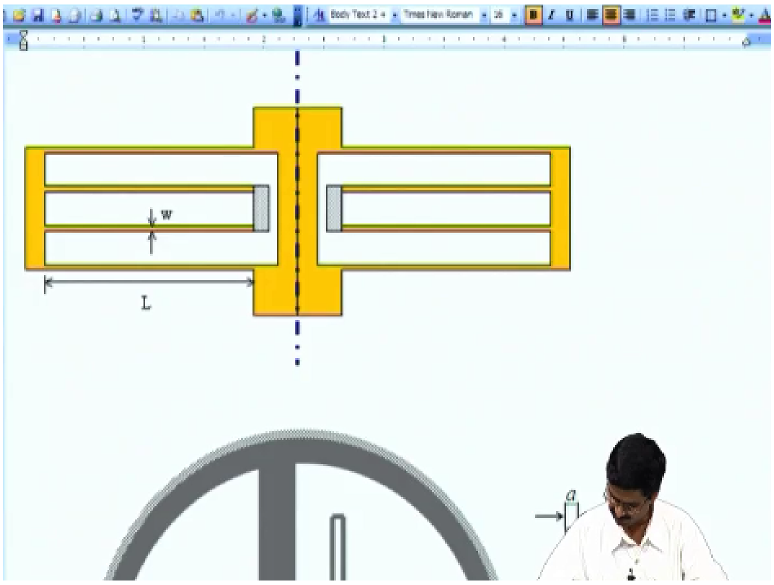

What we have on the screen that we are going to see a suspension. A suspension is called a suspension because the middle I shaped alloy beam that you see here is suspended by a bunch of beams 4 on the left slide, 4 on the right side. These beams suspend this mass so that it can move only in the vertical direction like this where the axis is shown here, but it will have very high stiffness to move in the perpendicular direction in this plane.

And also it will have very high resistance to rotation about the axis which is perpendicular to the screen. So it is able to move freely in this direction, but very stiff to move in this direction as well as for rotate. It is a special kind of suspension meant for mini micro electromechanical system devices. One of the prominent ones is a comb drive which is where a shuttle mass will be going back and forth like the one that is shown here with electrostatic force.

Another one is to use it as an accelerometer as you all know senses acceleration. So to sense an acceleration you need to have a mass which due to the inertial force caused by the acceleration will experience a force, but if there is no spring all the suspension will just move away and that is why the suspension is necessary. In fact, the deformation of the beams that you see here.

These 4 beams on one side and 4 on other side if you measure the displacement of the mass which is determined by the stiffness of the spring that is these beams then we can calibrate for acceleration. That means that we have to lump this whole device as 1 spring mass system. Let us look at that kind of lumped modeling today. So we saw there a proof mass which had 4 beams on one side, 4 beams on the other side and those all of them can be made to look like just one spring.

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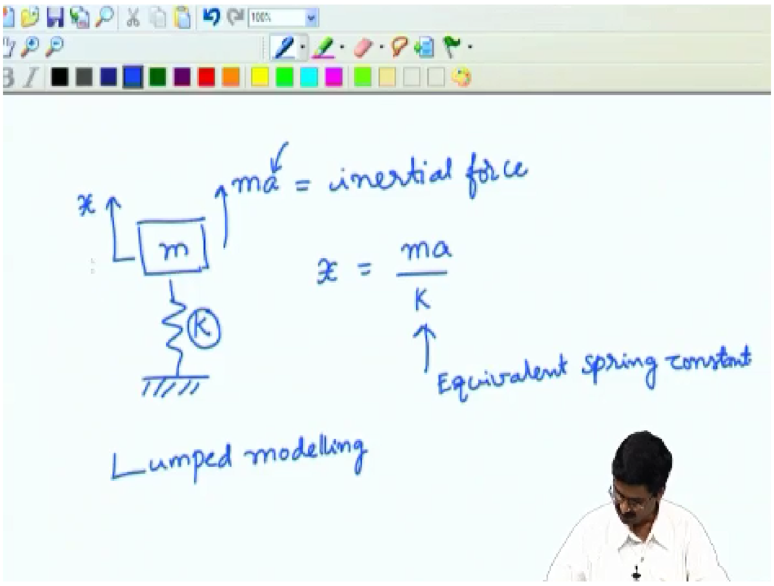

So this is the mass let say this is M which is a mass of central I shaped pattern that we saw on the screen. And then we want to have one spring constant which represents the effect of all of the beams that we saw over here. Let us look at the beam one more time. So we have this is one beam we have 4 such beams on this side and 4 such beams here and there is a mass here. So this mass is lumped into one single mass which is fine because there is only one mass here and of course we neglect the mass of all this springs.

And these spring beams that beams which act like springs. So all these beams we would like to get it to just one spring of spring constant K. If we are able to do that what we can get is if there is inertial force on this which M times A that is the inertial force due to an acceleration A which is what we want to measure. In order to measure that we need to see how much displacement this mass would experience.

Let us call the displacement X that X will be given by the force which is inertial force which is M times A/spring constant K. How do we get this equivalent spring constant? This is equivalent because it is equivalent to the effect of all these 8 beams in the structure equivalent spring constant. We need to determine this based on the dimension of the beam. Let us look at a few more examples before we discussed this lumped modeling.

We call this lumped modeling which is a very important concept in any field because we do not want to deal with all the details of the geometry that exist in this suspension, but get it all to just one quantity spring constant, equivalent spring constant.

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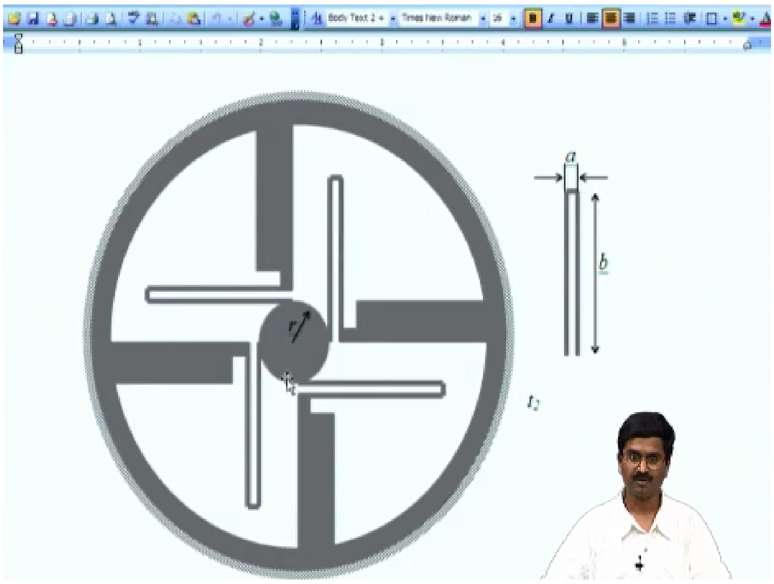

To look at other examples let switch back to this. Here we have a different device. This is a platform what you see at the center that is the central disk here can move up and down in the direction perpendicular to the screen. And that is enabled by these 4 U shaped folded beams. There are 4 of these each one of them looks like this that is a length A for this B here there is a width for this folded beam, 3 or 4 of them which are all attached to an outer ring which is fixed.

This hatching shows that it is fixed to a reference frame and now the central disk can move up and down and act like a Z stage. Here also we would like to represent this with one mass and a spring constant equivalent spring constant corresponded to all these 4 folded beams.

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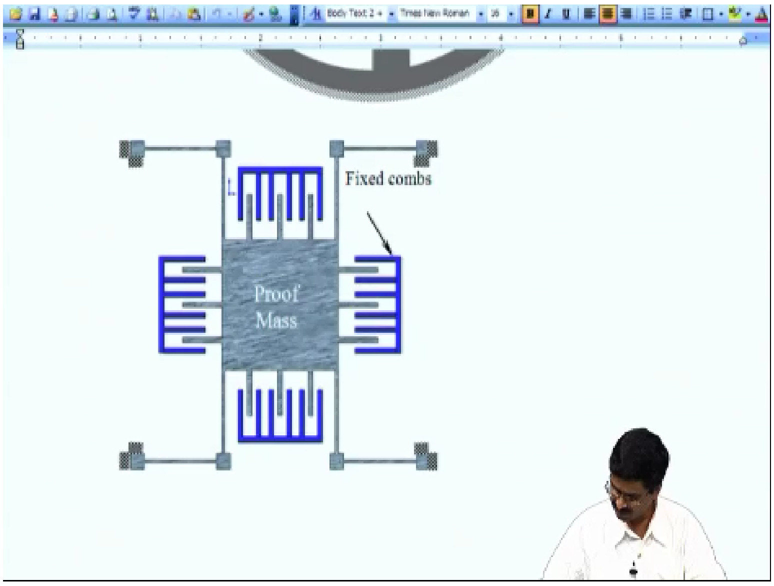

Let us look at one more example. Here the central mass which is called the proof mass we wanted to move in the X direction that is this direction as well as Y direction. So here we have a little coordinate system shown there this is the X direction, Y direction with complete decoupling meaning that when there is a force acting in the X direction Y direction is not affected.

And likewise when there is a force acting in the Y direction X direction is not affected. So here the mass will have then two springs.

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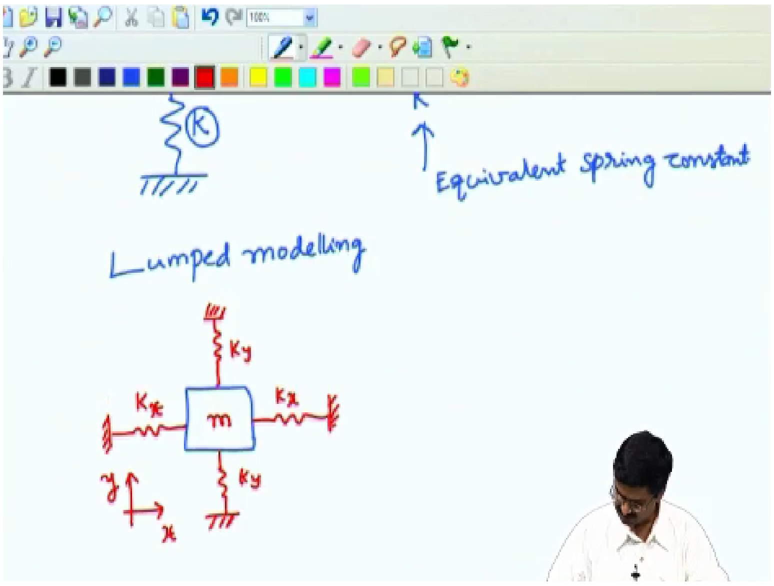

So if I go back to the lumped modeling. So far this one we can draw a lumped modeling with a mass at the center and the springs letting it move in the X direction with some stiffness. If the beams were not there, suspension is not there it will just fly away when there is a force in the X direction. Likewise, we will have the springs in the Y direction also. So we have X direction and Y direction and we have the central mass here.

So this Kx here and Ky here. In this particular example the Kx spring constant equivalent spring constant the X direction happens to be = Ky because of the symmetry that is there in this particular device. So how do you get this Kx and Ky?

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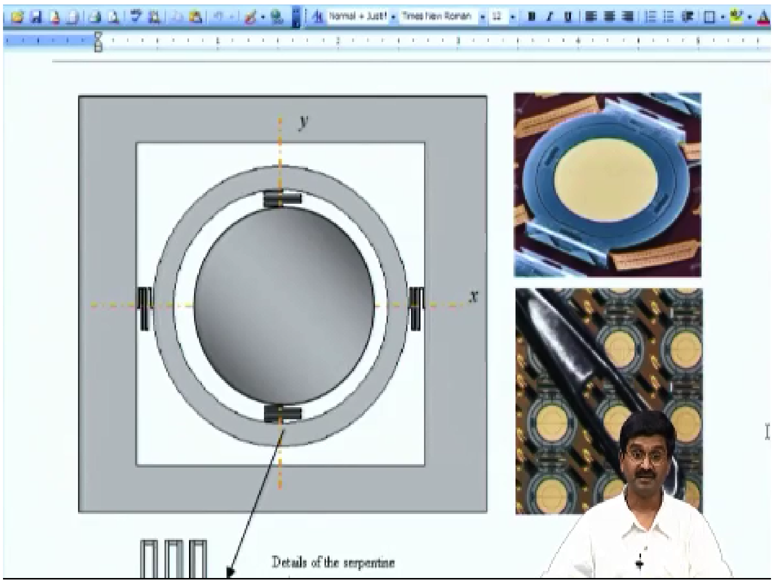

Let us look at one more device. Here it is a micromirror array where this mirror that you see this is the eye of the needle you can see on the wafer how many of them you can make and this particular mirror diameter of this is about 150 to 200 microns and that is why you can pack a lot of them on a single wafer. This mirror can tilt about X axis as well as Y axis.

We can tilt it like this, tilt it like that and for that there are again the suspensions. We have the circular mirror at the center there is a suspension here which enable this mirror to rotate about this Y axis and this one is connected to annular ring which is suspended with another pair of springs which enable this annular ring to rotate about the X axis. So this mirror can rotate about X axis because of the annular ring and rotate about Y axis because of this other spring.

So how do you get the rotational stiffness of this about X axis and Y axis.

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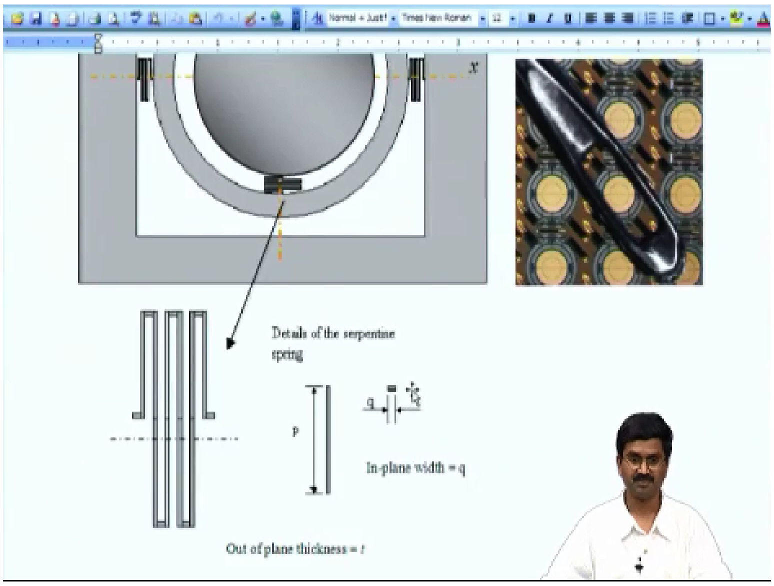

So if we look at the details of this beam suspension it is going to look like this. We have here if I look at this spring it is starting one end corresponds to this end here. Another end corresponds to this and here is our axis. And it can twist about this axis here to make this one rotate about that axis. How do you get an equivalent spring constant or something like this? So looking at these 4 examples let us look at the lumped modeling of these beams.

Going back to our first examples this one where let us look at the spring constant of this first in the X axis that is horizontal axis. We have several beams we want to look at the stiffness of all of these beams in the X axis that is horizontal on the length of the beams. Let us look at just one of them first. If we look at just one we have axial deformation happening in these beams.

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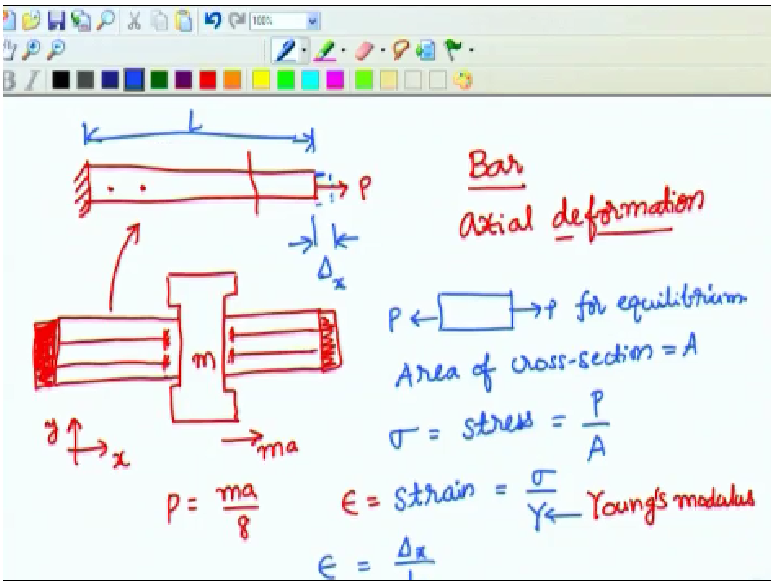

If I show one of those beams here which is connected and fixed to one reference frame and then here we want to see what happens which is a force in the X direction. We have this what we call a proof mass. It can be of any shape here it happens to be that way. We will just show these beams. This is somewhat a rigid motion and then we have these beams which is fixed here and fixed here.

This part is not fixed it just happens to be rigid. So that means like mass here this is completely rigid, but not fixed to anything and these are the 4 beams that move. So this again is connected here. So this goes like this and this one and similarly we have it on the right hand side. So it is fixed here, fixed here and there is a rigid mass over there connecting them. Now if there were to be let us call this X axis.

Let us call this Y axis if there were to be inertial force on this mass M and there is Ma. We would like to know what the stiffness is in that direction. So we should look at each of these beams. Let us say one of these beams I have shown like that this force Ma would act here. Let call that force P and this force it seems there are 4 beams here and 4 beams here and they are experiencing axial force due to this Ma.

This P we can say will be 1/8 of total inertial force Ma because there are 4 of them shearing this force and they are all in parallel here because there force is sheared. This force Ma is sheared by each of them equally. Now I would like to know how much it deflects. If I look at this problem, we want to think about what effect of force on this beam in the axial direction. A beam that experiences force in the axial direction is called a bar.

A bar moves only in the axial direction. So it undergoes axial displacement or deformation because it is not a rigid body displacement, but actually it elongates if I take two points on this after deformation these two points would have moved to 2 different new points where the distance would have changed. Distance between them would have changed that is not rigid body motion and it is called a deformation.

Deformation causes stress and strain in this bar. If I look at any point here if I take a cut and look at what internal force exist on that that has to = P because this element this segment has to be in equilibrium. So if I take the broken one that I take this we have this force P acting on it and I have cut here. So this should have a force P to keep it in equilibrium. This is for static equilibrium that is a force.

If we assume that the area of cross section of this bar. Area of cross section=A then we define a quantity sigma which is called the stress in this axial stress in the normal direction to the surface of cross section here it is called a normal stress that is given by the force in it divided by area of cross section. Now we can define another quantity which is very important in mechanics which is strain and that is given by the stress divided by Young's Modules.

This Y here is Young's Modules which arises because of the assumed linear relationship between the stress and strain. So strain we will use a symbol epsilon and stress and strain related by the Young's Modules. Now what is strain? The strain is defined as change in length that is how much deformation this undergoes let us call it delta X not X. Let us call that delta sub X that is the change in length in the X direction/original length of the bar.

So if I take this length of this bar as L that is strain.

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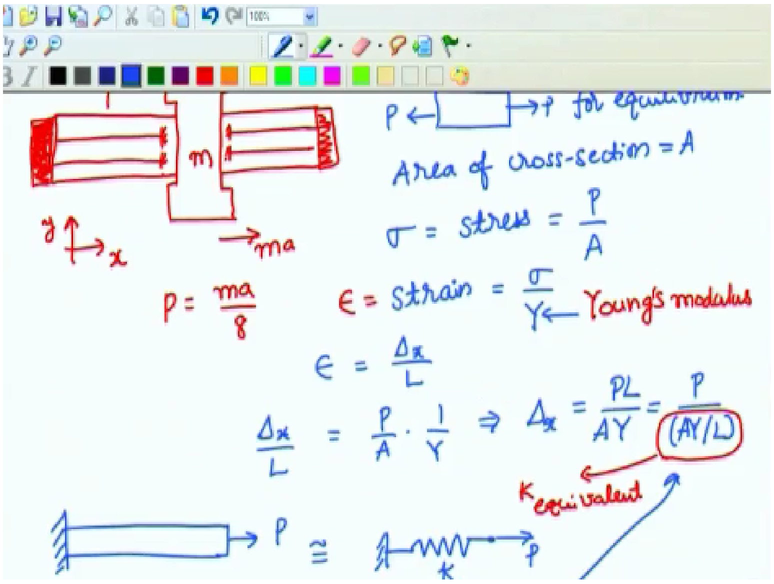

Now knowing the relation between the stress and strain we can now write. So we have strain which is delta X/L= stress which is P/A and then 1 over Y. This tells us that the delta X the deflection of the bar = PL force multiplied by length of the bar/area of cross section and Young's Modulus. Now if we look at this expression and then see that we can separate it out in this fashion P/AY over L.

Now this quantity can be seen as the equivalent spring constant that means that the bar that we have which is experiencing a force P can be written as a spring linear spring with a spring constant K which is = as we have shown AY/L area of cross section times Young's Modules/length of the bar. Once we know this we go back here. So we have this K=AY/L and we have 8 of those in the suspension.

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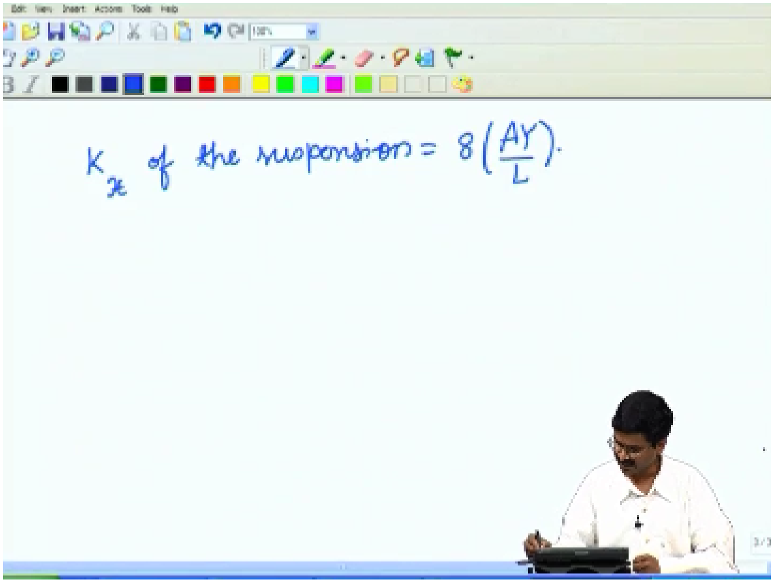

So we can write the total stiffness of this in the X direction of the suspension. This is KX of the suspension as 8 times the spring constant we just derive which was A*Y/L that gives the Kx. Going back to our figure we have this suspension and this spring constant the X direction which is this Kx we have derived expression. Now if there is a force in the Y direction on it which is again let say MAy

We could have called it MAx here. So if we call it MAx this force will be MAx that is inertial force in the X direction. Now if there is a force in the Y direction these beams now will start bending. We said that we call something a bar if it experiences only axial force whereas if there is a force in the Y direction these things experience a transverse displacement or transverse deformation. We need to determine Ky for it.

Before we proceed let us look at one more aspect of this. We had a bar several of them they all shear the force applied and that is why we call them springs in parallel.

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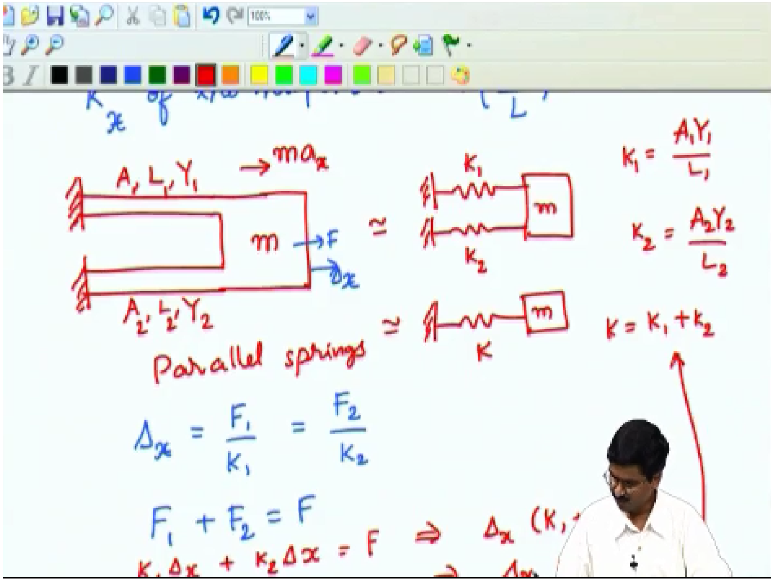

That means that if I have a bar here and another bar here connected to let say a proof mass a very big mass which is fixed here which is fixed here based on what we just did in the X direction if there were to be a inertial force M times Ax. So this is M times acceleration in the X direction that will be M times Ax. Now we have two let say they both have area of cross section A length L and made of material which has Young's Modulus Y and same thing for this AL and Y.

Then if we get spring constant as we just did get for one bar and take another bar which has a same spring constant you can just add because these are springs in parallel that is this is equivalent to the lumped model where we have one spring and another spring both are fixed here and we have a mass M. So if this spring constant is K let us say this as let make these things different that is A1, this is A2 L1, L2, Y1, Y2.

So we will have 2 different spring constant here K1 and K2 where K1 is A1 Y1/L1 and K2 is A2 Y2/L2. This further can be reduced to just one spring because they are spring in parallel with an equivalent spring which is K and that K is going to be the sum of this individual springs in parallel. This is the case of parallel springs. How many were they are we simply add because they all shear the force and they have the same displacement.

How does this K=K1+K2 come about? We know that the deflection if we denote the displacement or deflection of this by as we did delta X that delta X is the F1 the force in the bar 1 F1/K1 and it is also the same in the other one F2/K2 because they both are attached to the same mass. The mass moves by delta X they both have to move by the same amount

F1/K1, F2/K2, but we know that the force F1+F2 = total force acting on this mass because the springs in parallel shear the force and this is what gives us these relationships.

So we write F1 will be taking from here it will be K1 times delta X+K2 times delta X=F. This gives us that delta X times K1+K2=F or delta X is F/K1+K2. So that is what we get as the total spring constant. What if these are in series as opposed to being in parallel.

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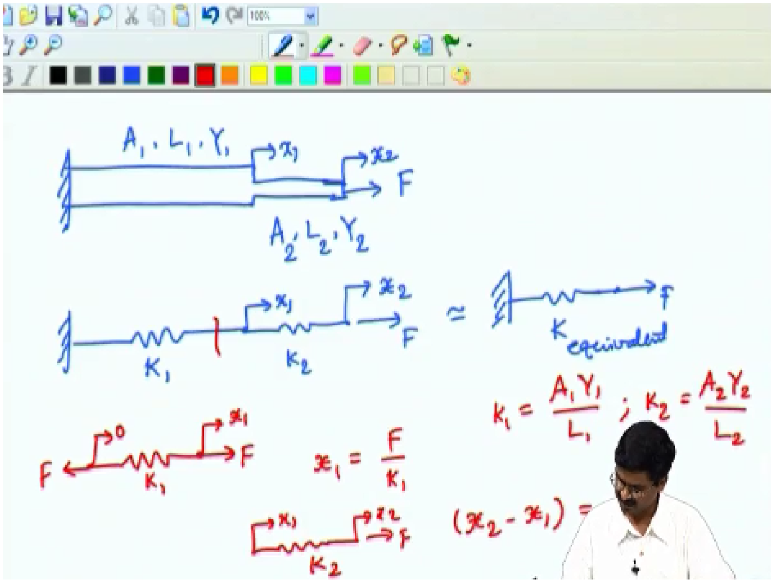

Let us take that example of a bar where we draw this bar of certain cross section. Let us say there is a change in cross like this which is fixed here and we have the (()) (23:37) cross section let us call this A1, L1 and Young's Modulus Y1. For this part the area of cross section is A2 length is L2 and modulus Y2. Y2 can be the same, but we have taken general case where they are different.

Now this can be represented as spring 1 and then spring 2. If there is a force F is acting on it. We will need to get an equivalent spring constant for this example. So let us say this point that is this point moves by X2 that is this is moves by X2 and this moves by X1 then we have to see what will be the equivalent spring constant here. What we like to get is these two we want to get to one spring where there is a force F we want to get this K equivalent to K1 and K2.

So here what we notice is whether we take the first spring or the second spring they both experience the same force because if I want to take a cut over here and draw this spring. It has to experience the force F here and reaction force F there and this displacement we know is 0 because it is fixed and this displacement is X1. So we can write that this is K1 which is K1 we will write one more time.

It will be A1 Y1/L1 and knowing the displacement or elongation of this will be X1-0 or just X1 we will get X1=F/K1. Likewise, if we take this portion that spring let us draw it. So this is moving by an amount X2. This is moving by an amount X1 the difference of that will be the elongation in the spring and we have a force the same force F acting on it. So we can write X2-X1 which is the deflection of the spring elongation of the spring.

If X2>X1 it is elongation. If X2<X1 it will be compression or contraction of the spring that can be written as the force/spring constant K2 of this and K2 will have the same form, but area of cross section 2 A2 Y2/L2. If we have this our aim is to get the K equivalent here.

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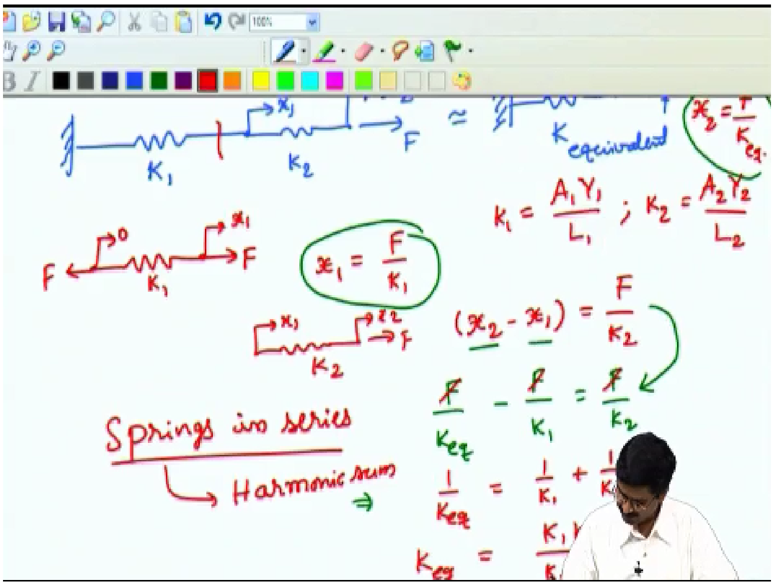

Looking at these two expressions and the equivalent one we will have another expression which will be the X2 that we see here this point will move by an amount X2 just as here. We can write X2=F/K equivalent. I will just write it as K (()) (26:52). So if I now take these expressions that one and that one and put it into here. So we have X2-X1/ X2 is F/K equivalent-X1 which is F/K1=F/K2. So we have got this by substituting for X2 and X1 from here and here.

Now we see the same forces everywhere. So we can write by taking this other side this term. First we can cancel F throughout because the same force acts there on all of them. So now we can take it the other side. So we get 1 over K equivalent=1 over K1 this goes other side becomes a + sign+1 over K2. Now for springs in series these are springs in series for the spring series we have to take harmonic sum that is the reciprocal of K is equivalent is=sum of the reciprocals and that is why we can write K equivalent =K1 K2/K1+K2 the harmonic sum of the two springs constant.

So we have to remember that when we take several beams that are in a suspension we have to identify whether individual spring corresponded to the beams are in series or in parallel and accordingly do this summation. This applies to whether it is axial spring that is a spring that models in a lumped form the axial deformation of a elastic elements such as beam or transverse it does not matter.

When you identify them as lumped elements the spring in series in parallel is a useful concept. Now going back to our example looking at this suspension we determine the equivalent in the X direction that is X direction here. We would like to do now for the Y direction if there is an acceleration and inertial force MAy there. Now we have to see how these beams deform.

So for that we have to look at these beams and then see how it deforms.

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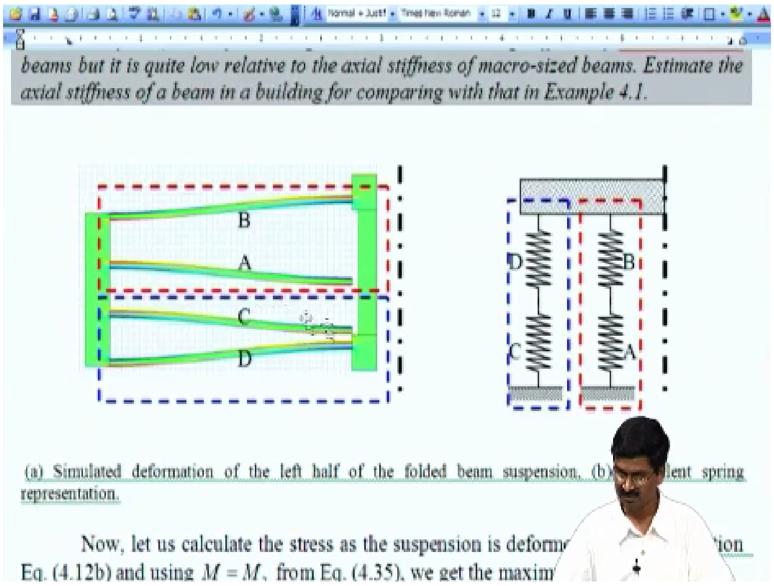

Just to give you an idea as how it looks when it deforms let us look at one result of a finite element analysis about which you will learn later in this course as well as you can look at standard textbooks and parental meta analysis or look at a course content that describes this method. What this method does is given any arbitrary shaped structure when you want to analyze its deformation you have to solve a differential equation.

And it is a numerical method parental meta analysis numerical method which gives you solution of differential equation. Here one of the results is shown where we see this beams which are bend. We had 4 beams we are looking at the left side of course there are 4 beams on the right side as well and this mass has moved and each of these beams has deformed. So this deformation if you see if I look at this beam.

Beam A is fixed at one end the other end it is able to move in X direction, but if you see the slope here is 0. The slope here is 0 because it is fixed the slope of this beam and here also it is 0 because it is attached to a rigid element that is movable in the vertical direction, but does not allowed to rotate. So this kind of a beam is called a fixed guided beam which we will consider now to see how we can determine the deflection of that one beam.

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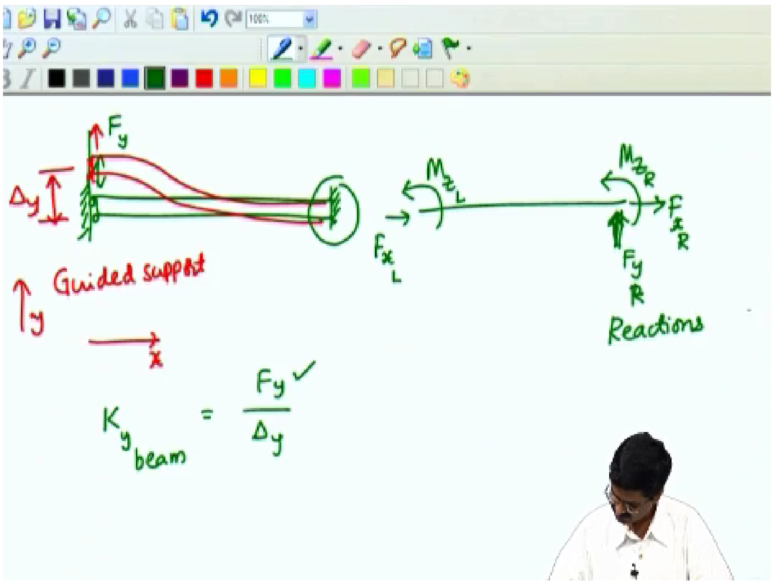

Let us look at the deflection of that one beam. So I draw that beam which was fixed at this end. It was guided that is what this is called it was guided here so that it can move up and down when there is force on it. It is fixed here so how would it deform? So we saw that it was deforming because it can move in that direction, but it need to have a 0 slope. So it starts from a 0 slope here that ends up having 0 slope that is how it is going to deform there are this (()) (32:00) imagine them this is called a guided support.

So we would like to know if there is a force acting on this in this direction how much would it move if I say this is my delta Y because I remember that we have our X axis in this direction and Y axis in this direction. So there is a force here due to that acceleration of Y direction we would like to know how much this delta Y is for that we need to analyze this beams.

For that we need to know first the reaction forces that one would feel at this end as well as at this end. If I take this beam let us take the un-deformed shape. We have fixed this beam completely here so that means that you cannot move in the upward direction. So since it is restrained from moving in the upward direction there must be a force holding it which we call reaction force. So we are going to have a reaction force here FY on this.

Likewise, it is not allowed to move in the X direction. So it must have a reaction force in the X direction Fx or this end. We are looking at this end so it is fixed in the X direction there is a force reaction force. Likewise, it is not allowed to rotate there. So there must be a movement about that Z axis. Z axis here is perpendicular to the screen so there will be a moment reaction moment.

So these are reactions because you are not allowing it to move it will develop a reaction. So at the other end what will be that? It is able to move in the Y direction so there is no reaction in the Y direction at this end. Whereas there will be reaction in the X direction here. We should call this really indicate the left side. So this is the left side, this is the right side, let side there is a right side, this is also right side.

So it is not allowed to rotate here so there must be a reaction here FZ on the left side (()) (34:31). So we have to know what reactions exist once we know what reactions exist then we can draw the bending moment diagram for this. We can compute bending moment. Once we know bending moment we can determine how much it deflects and from the deflection we can get the tip delta Y and use that to get our equivalent spring constant.

So if there is a force acting let us say the force here is Fy in the Y direction then the spring constant of this beam in the transverse direction that is Y direction of this beam will be that force FY/delta Y. So we know this force because we can look at the beams that are there. So if we go back to the diagram where we wrote this suspension. We have 1, 2, 3, 4 beams here, 4 beams here if we analyze that they are all identical they will have the same spring constant in the Y direction or X direction.

But you need to analyze whether their series are parallel. So once we know that we know the total force how they shear that so we can take this MAy and take that portion back to our model here and put that as Fy. Once you know Fy if I can somehow get this delta Y then I can determine KY that is the goal here. How do we get this Ky that means that how do you get this quantity delta Y.

So before that as we just said we need to know what reactions act and we need to determine those these 2 quantities these 3 quantities. Once we know them we can draw the bending moment diagram or we can compute bending moment. So bending moment is critical to determine deflections. So the M that we have the reaction moment.

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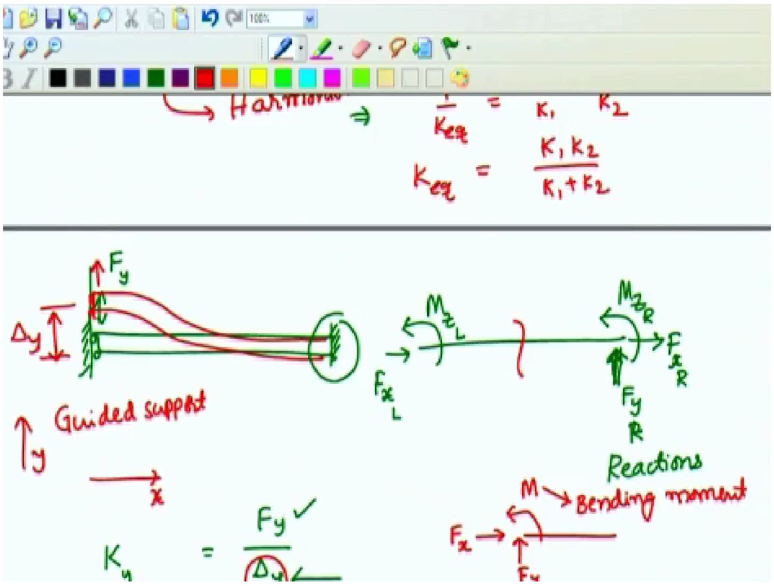

But if I take a section of this anywhere I can again show the vertical reaction and horizontal internal force (()) (36:55) because it is internal and there will be a moment. Notice that all of these we are writing in the positive direction. These are positive X direction, positive Y direction and positive direction for moment either reaction moment or bending moment is counter clockwise.

So in fact let us call this only M rather than Mz that is a bending moment we call it bending moment because the effect of this is to bend the beams bending moment. Once you know the bending moment we can determine the deflection.

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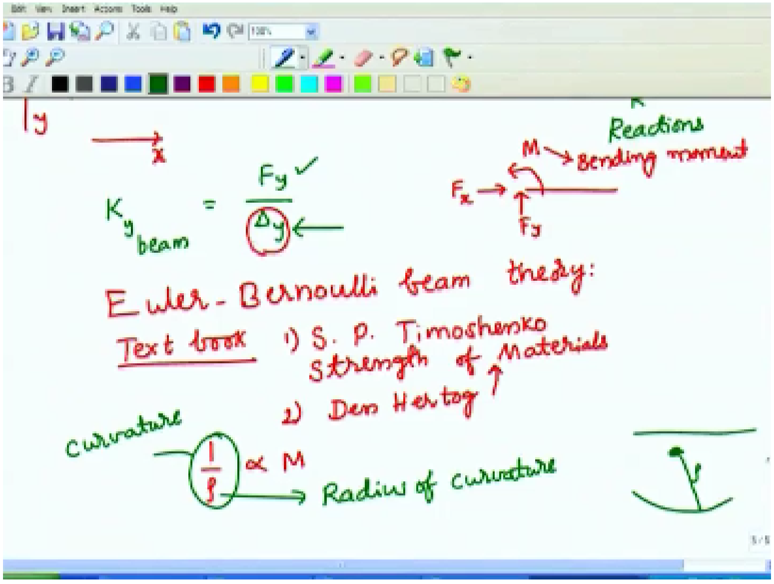

Because we know from beam theory this is we need to know Euler–Bernoulli Beam Theory which is usually discussed in a strength of material or mechanics of materials course which I am sure you can find in many standard text books. One of the good text books to look at the strength of material is by SP Timoshenko authors name and this is title the strength of materials.

There is another good book this is one book another book is by Den Hertog and this also has a same title. There are many other books in strength of material if you read them you will know what the Euler–Bernoulli Beam Theory is and once we are able to get this bending moment we can get the deformation because we know that the curvature of the beam as it bends is proportional to the bending moment the more the bending moment the more the curvature.

So rho here is the radius of curvature. Radius of curvature determines in fact the reciprocal it means the curvature because if radius of curvature or a straight line is infinity so its curvature is 0 because if I take a straight line its radius is somewhere very far an infinity. So its inverse means that curvature is 0. On the other hand, if I have a circular arch which will have a center here then this becomes the radius of curvature.

So we have this here that is a rho. So it shows how much is bending and that is called the curvature. So 1 over rho this quantity rho is radius or curvature. This is curvature. So curvature is proportion to moment. If we are able to determine the reaction moments here and then taking a section everywhere we can compute this bending moment and once we do that here we get this relationship.

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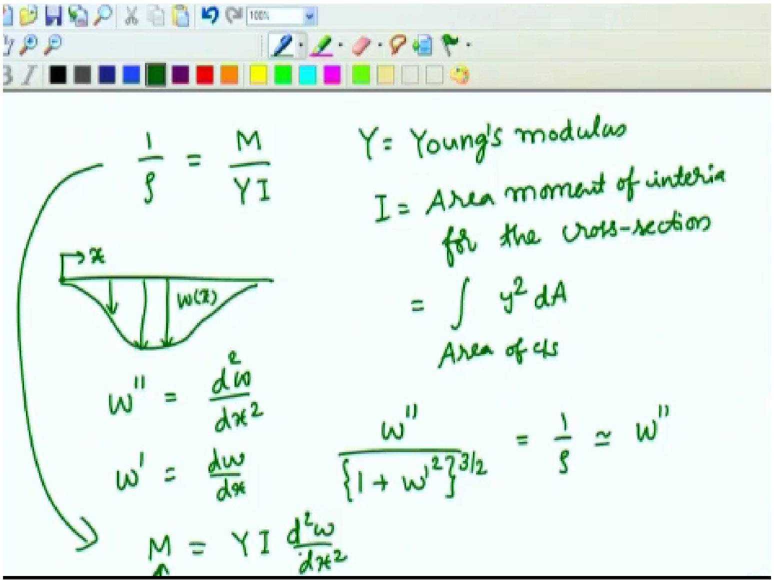

And we also know that I over rho that constant proportionate is here is M/EI. Based on this well let us use YI where Y again is Young's Modulus and I is area moment of inertia for the cross section. So the cross section plays an important role in how much the beam is going to bend because of the bending moment. So we need to have that come here as I which is given by integrating over the area of cross section a quantity Y square DA that is the area moment of inertia.

And material Young's Modulus is there and we have the radius of curvature and radius of curvature is given by if we denote the deformation of a beam. Let say beam deforms like this if I denote as WX where from the starting end of the beam we measure X WX is W at different points. If I take second derivative let us write as W double prime that is D square W/DX square where the first derivative would be simply DW/DX.

Then we can write this curvature as W double prime/1+W prime square raise to 3 over 2 that =1 over rho. This comes from geometry. You substitute that over there then you get a relation between W double prime and this M that we have. What we do in small displacement is to neglect this quantity. Look at this. This is 1+W prime square with a beam bending is very small.

I have shown here exaggerated fashion, but if it takes small displacement W prime will be very small W prime square will be much smaller 1+something that is very small. We can approximate it to one. So we can say this is =just simply W double prime that is this that is=M. So we get from here M is =Yi times W double prime that is D square W/DX square. If we know bending moment we can integrate it twice to solve for W and the W here is what we want because if you recall what we want is this deflection.

If I know W of X of this beam everywhere it will be easy matter to know how much it is over there, but then we have to first determine the reaction moment, get the bending moment, go to this differential equation that is only for small displacement. Determine W of X everywhere and then substitute at one end delta Y which we need here to get our lumped modeling, but we can avoid all that by using an energy theorem which is called Castigliano's theorem.

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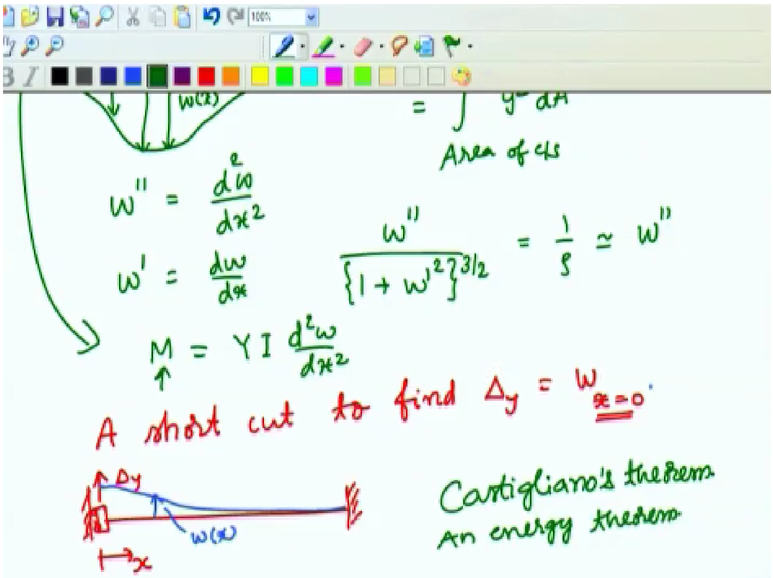

So a shortcut we want to find the shortcut to find delta Y which for us W at X=0 because we have a beam which is fixed at one end and guided at this other end. You want to determine this delta Y. So if my X is starting here at this point X=0 and what we are saying is when this deforms. Let us chose a different color it is going to go to that point we would like to know this is our W.

That is WX everywhere at X=0 we want to know delta Y and that is what we would like to have. A shortcut to find delta Y is given by a theorem known as Castigliano's theorem which is an energy theorem which are convenient for a certain calculation such as this one and energy involved here is a strain energy.

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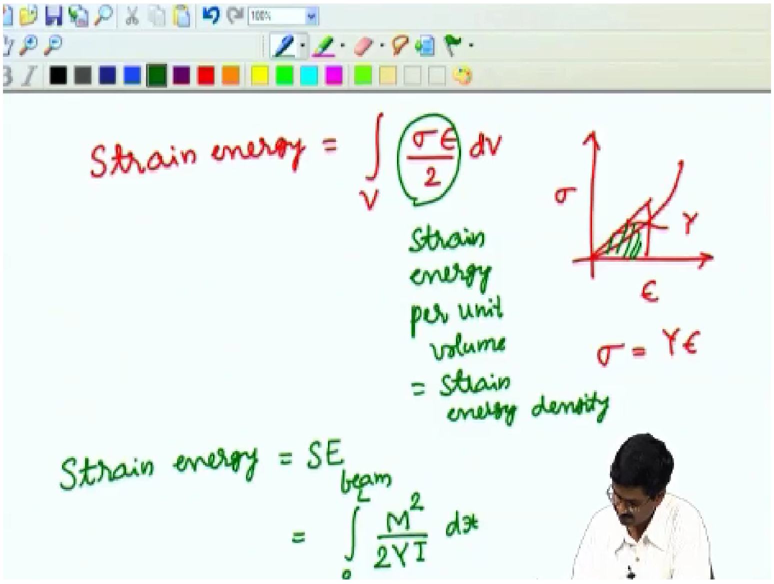

Let us write what this stain energy is? The strain energy is integral of sigma that we introduced earlier stress times strain half of it over the volume of the body. In other words, stress times strain if I were to take a graph where I have stress here strain here if I have a curve like this or a straight line in the case of linear things the slope of this is given by the Young's Modulus because we know that stress=Y times epsilon where Y is the Young's Modulus.

The area under this curve. This area is half sigma epsilon 2 and that is called the strain energy density. So this is strain energy per unit volume that is strain energy density. Strain energy density if I integrate it over the entire volume I will get the strain energy expression, but the case of beams this strain energy can be written the strain energy which will denote by two letters SE strain energy for beams is given by integration from 0 to L.

We have to reduce this volume to length after doing an integration area of cross section for the derivation of this we can look up a book on strength of materials it will be given by Young's Modulus which we have been using Y here. This is given by the bending moment square/2 times Young's Modulus times moment of inertia I DX. So if we have a beam we can determine bending moment that is absolutely necessary.

But what we are trying to avoid is fall in the differential equation using an energy method using Castigliano's theorem if I do this integration M square/2YI of this.

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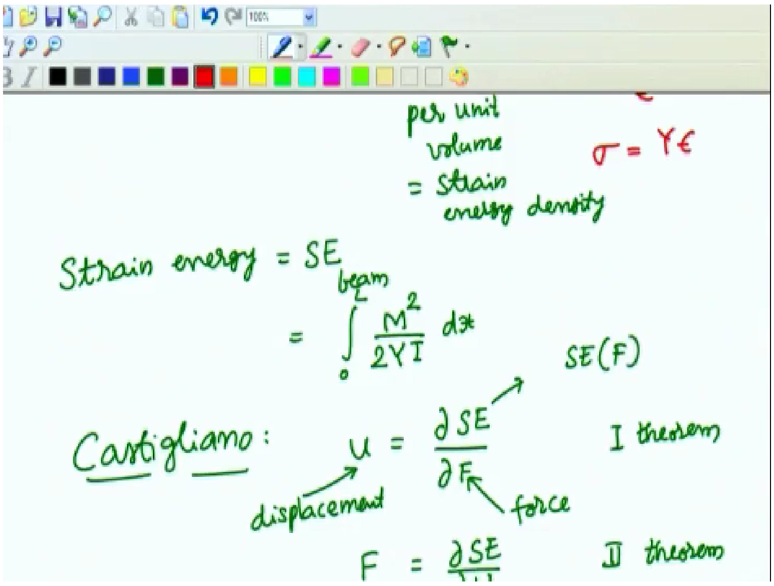

Then what Castigliano's theorem says that the deflection at a point. Let us say U at somewhere is=the derivative of the strain energy with respect to the force correspond to this one. So this is the displacement that we want at a point. At that point if there is a force then we write the strain energy. This strain energy we should write in terms of that force. In fact that will be true for this beam.

Because this beam we apply a force F and try to get the bending moment. So bending moment here will be in terms of that force. Y is a material property I is a geometric property of the beam. So we can have this whole expression of integration come in terms of this force F. Now if we take a partial derivative of that we get the displacement. So once we do this we can determine this U just at the point where we want without having to solve the differential equation.

So that is a shortcut using Castigliano's method. There is another Castigliano's theorem which says the force is given by the partial derivative of that with respect to displacement. So F and U are the conjugate quantities. At any point if I apply a force we cross some displacement. On the other hand, if we were to grab something that is causing the displacement then there will be force.

So force and displacement are conjugate related to each other you can have this thing. So this is called Castigliano's first theorem and this is Castigliano's second theorem. We use that Castigliano's theorem here to compute this deflection at the end without having to solve this differential equation. So here you have bending moment an expression you get and then you have to integrate it twice to get the deflection everywhere that is W of X.

And then determine delta Y W at X=0 instead you can use a Castigliano's theorem. This is one very powerful way to obtain lumped models of elastic beams such as the ones you find. Now going back to our motivating examples. So for this what you see for this beams we can use Castigliano's theorem to get the deflection at one point and for these beams as well and these beams or any other beams that you find you can easily get the lumped models.

Because Castigliano's theorem an energy method enables you to compute the deflection at one point without having to solve the differential equation. And once we have the deflection we can go back and use our lumped modeling that is the force/deflection that is this quantity we can get the spring constant K and see where the springs are in series are parallel we can get the total spring constant of that.

So let us look at the example of the suspension by 8 beams which is called a folded beam suspension which is what we see here. Now for each of these beams first we need to use Castigliano's theorem to get an equivalent spring constant in the vertical direction which is given by the way by this quantity the FL cube/12 YI or in other words K which is force/delta which will be Y T W cube. T is the depth of the beam W cube is the width of the beam raise to the power of 3/length of the beam raise of the power of 3.

So this is the spring constant in the vertical direction and we have to see that each of the beams in the suspension all of them have this K which is given by Y T W cube/L cube and how are they arranged in series in parallel. If you look at the 4 beams here we have label them A, B, C and D. We can notice that if you follow beam A from its fixed end to here and here you will notice that these two are in series.

Because this moves a little bit this rigid mass moves with it and then with respect to rigid mass this point moves more because this is folded back like this. So A, B are spring in series. They have the same force, but not the same displacement. So A and B are in series we can get if this is K, this is K both of them together will become 2K. Likewise the beams C and D are also in series. So we get KK these are in series.

So we get actually half of K as equivalent spring constant for these and half of K here and now this assembly this assembly the one put in blue dash line box and red dash line box are in parallel as you can see because they shear the same displacement. So this is K/2 overall it will be K and then this set of beams on the left slide and set of beams on the right side are in parallel.

The total thing here will be two times K. So we can get the total spring constant of this particular suspension. So we look at each beam and do this. One more thing that we need to keep in mind to end this discussion is what happens when you have beams of different boundary conditions. So this what you all see in the scrolling on the screen is basically the center material which you can find in the two books that I mentioned.

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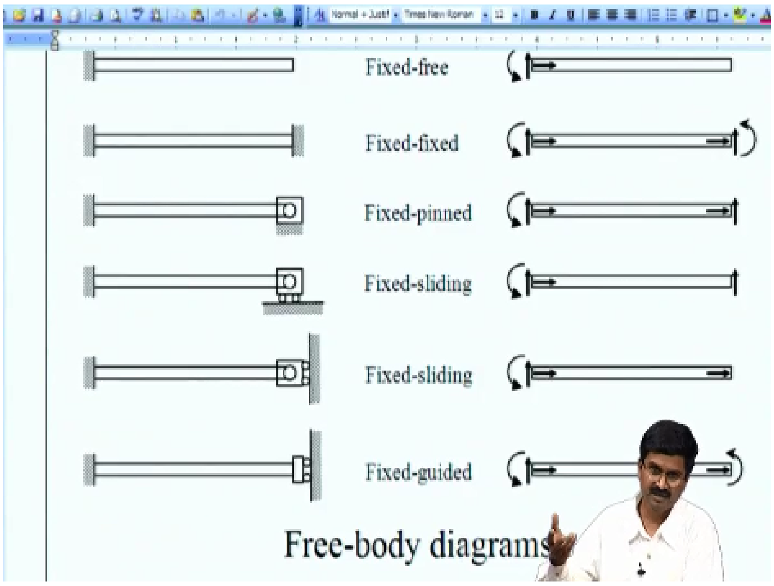

Now let us look at the beams of different boundary conditions. Left side we have shown all of them as fixed. Right side this is completely free so this will not have any reactions at this end at the right end because it is completely free that is called fixed free no reaction moments. So you need to get only these three and for a planar case you have 3 equilibrium X direction can be balance Y direction can be balanced and a rotation can be balanced.

So we can get the 3 equations these 3 quantities and you get the bending moment here and then apply Castigliano's theorem to get the equivalent spring constant of this in X direction, Y direction as well as the slope that is if I want to denote it by the torsional spring I could also do that. Now when I fixed in both sides then quantity like this and this is called a Statically Indeterminate Beam which means that we cannot determine the reaction here without knowing the deflection of the beam.

But that situation can be resolved by using Castigliano's theorem. Again you can refer to standard books on center material and if there were the pinned condition like this here then there will not be reaction moment and it if is fixed and sliding at this end it can freely rotate there is no moment it can freely move in X direction so there is no X reaction force only Y reaction force.

This concept of free by diagram that is if I take a situation we get a free body drawing the reaction forces is a very important element of center material. Once you get these then you can get the bending moment and apply Castigliano's theorem and determine the deflections and then lumped spring constant seeing springs in series are parallel you can get the total lumped spring constant for any of the thing that we saw we can get these lumped constant whether they are corresponding to the fixed rotation thing.

It cannot rotate there will be a torsional spring constant you can get the K of that as well as Kx and Ky for any system. So it is taught by the free body diagram concept go to beam analysis get bending moment, get the strain energy take partial derivative and get the deflection and force divided by deflection gives you the equivalent lumped spring constant. So with this we have discussed how suspension of micro system devices can be lumped into just a few springs in masses which makes it reduced degrees of freedom.

So we can easily solve. In the next lecture we will look at more types of deformations of elastic bodies and look at how we will analyze them. Thank you.