Micro and Smart Systems
Prof. G. K. Ananthasuresh
Department of Mechanical Engineering
Indian Institute of Science – Bangalore

Lecture - 22
Modelling of Coupled Electrostatic Microsystems: Part -2

Hello, as part of the micro and smart systems course, we will continue with the modeling. In the last lecture, we had discussed the electrostatic system modeling, where we had discussed how the elastic mechanics and electrostatics couple with each other leading to interesting phenomenon which is known as pull-in phenomenon. We also talked about the pull up phenomenon. Now today, we will continue the discussion and introduce what happens when we consider the dynamics of such a couple system.

So we will continue with the couple modeling but move from statics to dynamics, when we do that we also need to bring in the fluidic effects which is a very important aspect of micro systems we are going to cover just a little bit of that microfluidics aspect as it relates to model in the couple behavior that exists in electrostatic and elastic systems.

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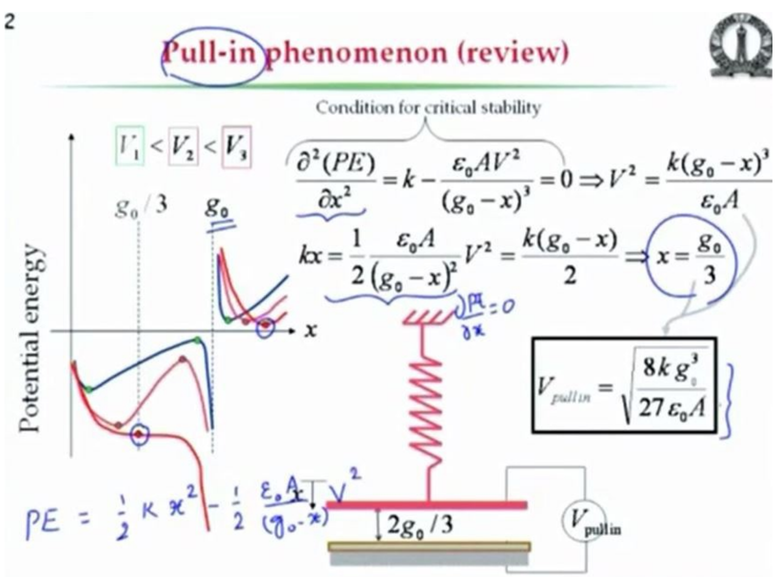

Let us just briefly review what we discussed in the last lecture. When we have the potential energy which we had written it as this potential energy, which is this, if I call it PE. It is half k x Square, where x is the displacement of the top plate that is moving in a parallel plate capacitor-

1/2 c V Square, where c is Epsilon 0 a/g0-x times V Square. So that is the electrostatic energy. We discussed why it should be negative sign and then we have the strain energy, spring energy, half k x Square, this is the potential energy.

If you plot it for a voltage V1 as it is shown here we get a curve like this. We have 3 solutions for this one dimensional approximate problem. There are 2 minima and a maximum all 3 are equilibrium solutions. The minima correspond to the stable equilibrium and maximum corresponds to unstable equilibrium.

So, we have a situation like that and as we increase the voltage let us say V2 then obviously this equilibrium solution of minimum has to increase because we get more displacement when we increase the voltage and the unstable one also moves towards this way, this dashed line here which is g0 over 3, one third of the gap that unstable equilibrium solution value decreases it comes like this.

So at some point, which we call the pull-in voltage the stable and unstable coil is into one, beyond that there is no more stable solution apart from this infeasible one. Because this is all after g0 that if one plate is going through the other plate that is not a realistic solution. So after that we do not have any realistic stable solution, which is we called pull-in phenomenon. So if you take the 2 plates when you increase the voltage it will move a little bit little bit, at some point it will suddenly jump up, jump down and pulling, that is what we call pull-in phenomenon.

So the equations you saw taking this potential energy we have to make the second derivative of potential energy = 0 because that is what we see here and then the first derivative being0, that is this equation comes from dou P E/dou x being = 0 and when you solve that we get that instability occurs at one third the gap between the 2 plates.

So, the formula for that pull-in voltage was derived to be what is shown here depends, of course on the spring constant k there is a stiffness of the structure. Remember that this k is simply a lumped parameter, indicator of the stiffness of the structure and g0 is the gap between the 2 electrodes of the parallel plate capacitor.

Epsilon is the permittivity if it is free medium air and A is the overlapping area of the plates. So here, if we have the situation like that when we apply a voltage that is pull-in voltage it just reaches the one third of the gap and then just pulls in if anything more than that, it just pulls in as it is shown here.

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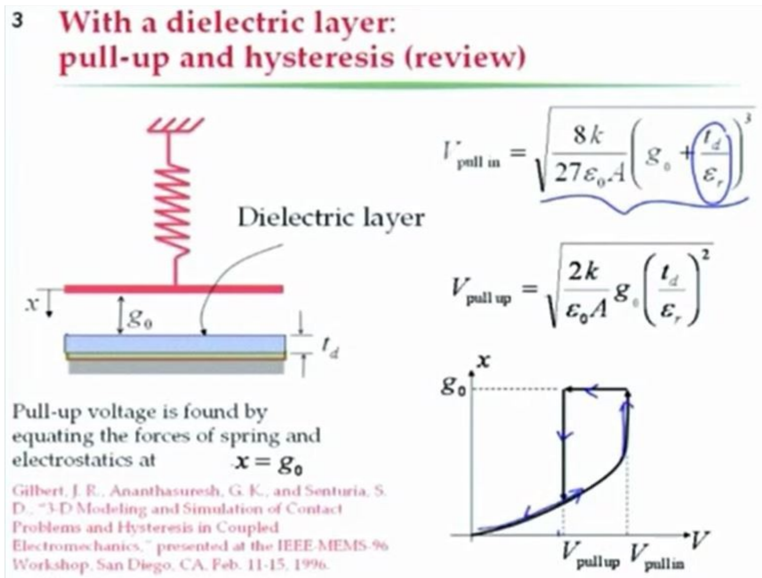

And then when you imagine that there is also a dielectric layer in between even after pull-in short circuit does not occur. So we do see that in that case the pull-in voltage is given by this formula. Essentially this is the new part, where the thickness of the dielectric/relative dielectric coefficient or the dielectric constant of that material Epsilon R that is effective thickness that adds to this g0 otherwise the formula is the same 8k effective gap that is g0+t d/Epsilon r cubed/27 Epsilon 0 A.

Whatever we had in the previous one without the dielectric, which is this. Effectively g becomes g0+dielectric thickness/dielectric constant. Now, we saw that it also leads to another phenomenon called pull up phenomenon. As you increase the voltage we go like this that some point just pulls in goes to the g0. That is this plate would reach this point and then now if we decrease the voltage it will not jump back up immediately that is because there is enough electrostatic force to keep it in the down State.

So it will remain like that until the voltage decreased is decreased to something called V pull up where it just pulls back up again. Here in the curve x value will decrease so the plate has gone down now it just stays there for some time and then pulls back up and then follows the equation. So it goes like this goes here, goes here then comes back.

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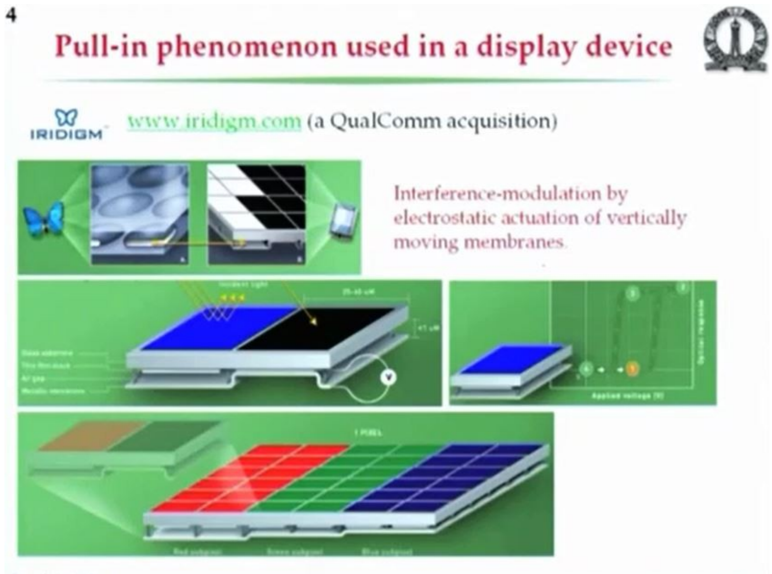

These are strange phenomenon and this has been used in devices and we have talked about this particular device.

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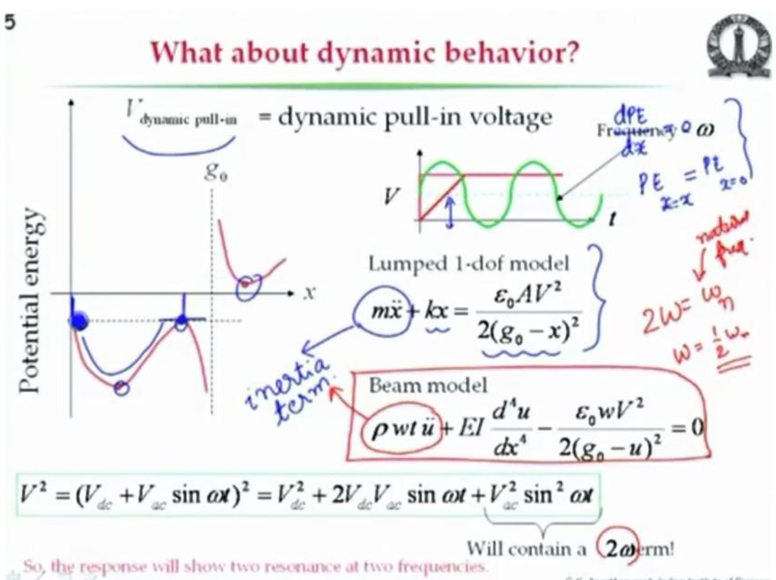

Now let us look at the dynamic behavior. Now what we consider is the static behavior, so here we have to apply the voltage very very slowly to characterize it as a static or a quasi-static phenomenon. But if your voltage is changed rapidly or applied suddenly then what happens? It turns out that there is another pull-in voltage that we can define which is called dynamic pull-in voltage.

So here again we are plotting the potential energy, we are showing the situation, where imagine that there is a ball, if there is a landscape like this when I leave it that it will come here and then it will go to the same height. For any reason, let us say for voltage larger than what whose energy shown here then if this were to be higher than this unstable point. Note that this stable point here, unstable point start moving towards each other, so if I were to leave a ball here you can imagine that it comes down and then it can go up to that level.

So, it will obviously go over this hill and still has some potential energy left, which converts to kinetic energy just go down the hill and again pull-in occurs. So here even before the minimum and maximum (()) (08:31) into 1 equilibrium solution not stable that will be kind of a transition after that it will not exist at all, in other words in your cubic equation 2 roots will become Complex conjugate by the third root remains real which is this, which is not of interest to us.

So, now we need to work out another formula for pull-in voltage, where the potential energy is initial 1 when $x = 0 =$ the potential energy at the unstable equilibrium that is one equation. Other is of course that this has to satisfy the equilibrium equation meaning $dPE/dx = 0$ there. If you do that then for a voltage that is more than dynamic pull-in voltage but less than the static pull-in voltage, pull-in will still occur that is one effect in the dynamic behavior.

You can work out the dynamic pull-in voltage by solving these 2 equations. One is $dPE/dx = 0$. Other is PE let us say you get some x here, PE evaluated at x, $x = x =$ PE at $x = 0$. That is we are saying potential energy here is same as here that $x = 0$ that is the starting point and $x =$ where $dPE/dx = 0$. Slope $= 0$.

These 2 equations you solve then you get both the value of x at which dynamic pull-in occurs as well as the voltage at which it occurs just as we had gotten that in the previous case, where let us just look at take a quick look at that we had one equation here and another equation here, we were able to solve for x and then for V. Similarly, we have 2 equations and 2 unknowns and we can solve for it.

Now if it is like this like a ramp and go they will be a transient and they will be accompanying effect for it. But imagine that this voltage here is actually a sinusoidal input something like that, this voltage goes from there is a bias and then it just goes like this. There is a DC component which is this much over that we have the AC component that goes up and down. Now then what happens? So we were to go to our 1d model. So far we had kx = the force, this is a mechanical force, this is the electrostatic force we equated those 2 and had got in the equation.

Now we have to add another term which is mx double dot, this is what we call inertia term, if we add that then we have to solve this ordinary differential equation in x second derivative in time, so it is second order system. Any dynamic system is a second order system when we consider in this particle form or rigid body form or other ways. Now when we solve it we get something but you know that in this case as you might recall the lecture that we had on the vibrations in general, a concept called resonance should come to your mind.

Resonance occurs when the frequency of the forcing function in this case, the voltage coincides with the frequency of the natural frequency of the system, when both are the same things will be in phase in a way that it will start exerting more and more displacement. Actually the force will be the same, just that the consequence of the amplitude of the system keeps on growing and we get a lot of displacement and that is what you would expect to happen.

But in this particular problem, this parallel plate as you increase the voltage of your frequency of your voltage resonance occurs much sooner. The reason for that you can understand by looking at let us erase these lines. If you look at that there is in (()) (13:29) there is a V square when you have V dc and V ac, so V dc+V ac Sin Omega t square when you have that we get V dc Square to V dc, V ac Sin Omega t and V ac Square Sin Square Omega t.

Because of Sin Square essentially there is a true Omega term, because you know how to write Cosine 2 Omega as Sin Square Omega. So, whenever you have Sin square Omega it means that there is a 2 Omega harmonic term so when this 2 Omega = natural frequency, again we should expect to have resonance occurring and that is why when this 2 Omega = Omega n, which is the natural frequency as we call natural frequency that is when the applied Omega is only half of natural frequency.

Already we will have resonance occurring in the system that is one thing we need to remember, because there is a V Square nonlinear term for the force. Now here is another equation that we should pay attention to because we had discussed apart from lumped modeling what if we take a distributed model such as a beam model. So here earlier we did not have this term, this is a new term, which is the inertia term. Inertia term is simply mx double dot in the case of a beam if u is the transverse deflection which is a function of x the axial distance that is u of x.

So, Rho w t will be mass per unit length and then u double dot will be like x double dot Equivalent d Square u/dt Square and then this other portion of the same thing, this is the beam equation, these are electrostatic force. And that is the equation that we need to solve and it will display the same behavior as this 1d model that we have shown, so that we will have static pull – in voltage, dynamic pull – in voltage and all other features.

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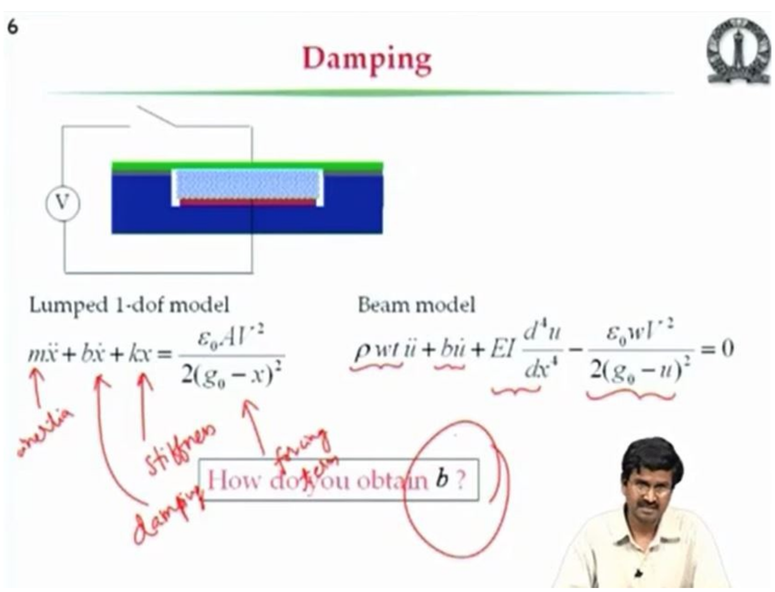

Now, when we talk about damping we have to deal with the fluids because microsystems necessarily involve fluids unless it is vacuum packaged. If I take an accelerometer and vacuum package it, there is no really fluidic effect there. Because only the solid moves but even under packaging there is nothing like a perfect vacuum packaging, there is nothing like a perfect vacuum so you do have some air and you have to consider the effect of that.

And that happens in many modes in micro systems, we will only consider a few standard modes and before that let us discuss the basic Fluid Mechanics that will enable us to think about problems where we can estimate so the damping term here, this is the inertia term let us write things down that we know already and this is stiffness term and this is our damping term. Thus in the 1d model mx double dot+bx dot+kx = forcing term that in this case is electrostatic.

In case of beam modeling, we have the same thing inertia term, damping term, stiffness term and the forcing term, that can be on the right hand side or minus on the left hand side. So, we get the dynamic equation. So, we have discussed how to get k, we have discussed how to get m also because when you want to get equivalent inertia we use the fact that kinetic energy of the deforming system should be same as the lumped models kinetic energy just like the strain energy of the lumped spring is same as strain energy of the elastic body.

So, similarly what concept gives us the ability to compute this b that is our main point of discussion? So we are not going to discuss fluid mechanics in at length and in fact that is not possible in a lecture and microsystems. So you have to refer to books or take a course or watch a course that discusses fluid mechanics in general.

Today we will discuss fluid mechanics as it pertains to microsystems very briefly. So we will have a slight review, I would not even say review it is just a few comments on fluid mechanics in general and try to derive the expressions for getting damping Coefficient that you can put it in your lumped modeling or 3 dimensional modeling or 2 dimensional distributed modeling.

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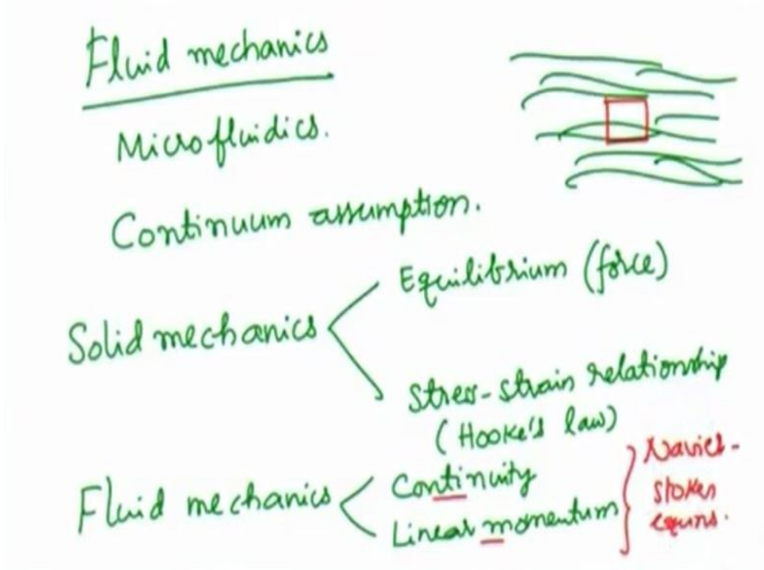

Let us say Fluid mechanics or microfluidics as this is called micro fluidics. It is basically the study of fluids at the micro scale, so you might ask what is different about fluid mechanics at micro scale as opposed to Fluid mechanics of the macro scale. We did not bring it up when we are discussing the modeling of solids whatever beam theory that applies to macro scale structures, we said that the same thing applies to micro scale structures as well. So however fluids different.

Fluids are different from solids in one fundamental way which is that they cannot take any shear force that is one thing. They are very different from solids we know but technically put that is the effect, other is they have viscosity. Viscosity is the difference that you will find between water and let us say honey.

So you can see that something like honey is very sticky if you put a pen or your finger into honey try to move it versus try to do the same thing in water you will find that moving your finger in water is very easy whereas in honey you find lot of resistance for your motion that is the viscous effect, that is a property of fluids that is very important.

Now if you look at fluids in general since they cannot take shear stress or in other words their shape changes when you apply some shear drastically. That causes a lot of differences in the way solids and fluids behave and when you come to micro scale the properties of fluid will change much more quickly than properties of solids change. Properties of solids also change when you go to micro scale but you have to go to very much smaller scale, as opposed to what happens at the fluids. Main thing is what we call continuum assumption.

Continuum assumption in order to the modeling with the differential equations and other thing that we have discussed, we have to worry about is continuum assumption meaning how continuously the species they can do particles or atoms or molecules how closely are they located to each other. That is if I take a chunk of material take a little piece out of it, I cut a little piece out of it, out of that little piece I take another little piece and I keep on doing it many many times.

When I do that can I end up with some species, how many other times I do it if that is true we call that material a continuum, say everything is continuous to each other. So that depends on the volume of object we take but when it comes to micro scale solid continuum assumption holds good much longer that is much longer when I say you can go to much smaller scales than fluid mechanics.

There are lot of reasons for it. We will just look at a couple of reasons today and to write the governing equations for solid mechanics, so let us just review that, solid mechanics we needed 2 things one is equilibrium and that is usually force equilibrium that we wanted and other is the stress strain relationship which is known as Hooke's law. That is all we need to formulate the equations for solid mechanics and there are a lot of details of course, everything follows from force equilibrium and stress strain relationship also known as constitutive relationship.

When it comes to fluid mechanics it is much more complicated but we can say that these 2 derives from 2 things one is continuity and other is linear momentum conservation. So if there is a fluid something. Now if I take small control volume how much enters, how much leaves. There is really no source within it. It has to be 0, whatever enters, whatever goes out the net flow will be 0.

You need to have that continuity plus a linear momentum because all around field particles you going to experience forces. So if you take that milliner moment correspond to that, that conservation of that, these 2 will lead to what are called Navier-Strokes equations. We certainly are not going to derive them or even state them. We will just look at a couple of flow patterns that we find in micro systems.

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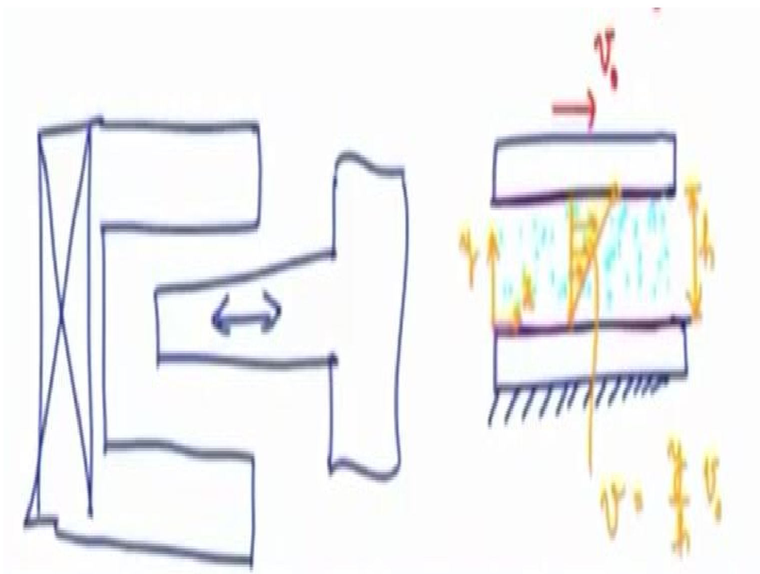

Let us take the first example of what happens in the case of comb drives. Let us say that this portion is fixed comb drives and then in between we have moving comb finger. So that means that this is going to go back and forth. So we had earlier assume that we have a parallel plate capacitor now there were the comb fingers if you have imagined actual I will put my fingers to show.

If you have fingers like this they will go in and out like this or they have actually a surface, so it will go like this. If I look at these 2 now one, I could say it is fixed other is moving like that so in between there is fluid. So if we have one plate at the bottom and another plate at the top, let us assume that this part is fixed and there is fluid in it showing like dots but there is a continuum fluid here and now if this were to be moving with some velocity V0.

Top plate is moving, there has to be 0, what will happen? So for that let us imagine that there is a water tank and you put a very thin glass plate that you are holding with our hand and you put it over it and try to move. We will try to move you will feel that there is some resistance to your motion as opposed to putting it in a solid and try to move. Then also it will be resistance either the plate will simply move over the solid or the friction is high enough you might move the solid also along with your plate. When it comes to fluid, the fluid that is over here on this edge will not move because it is in touch with something that is fixed.

Whereas things that is over this edge has to move with the same velocity V0. So we can show velocity profile here that is V0, this is 0 it turns out that if you apply the Navier-stokes equation here with assumptions of what happens in a one dimensional flow such as this you can derive that the second derivative is 0 or the second derivative of the velocity is 0 or the first derivative is linearly varying and that is what we are showing here.

So the last will be 0 here and V0 here that is same as the top plate in between little linearly vary like this. So we can write that as this velocity of the fluid anywhere, let us take this gap as h, velocity profile that we have written this V can be written as y/h*V0. So what is why here we say this is y axis this is x axis. When y = 0, V is obviously = 0 then y = h that is at the top place it has a velocity of V0 because h and h will cancel and become one so we get V = V0, so this is the velocity.

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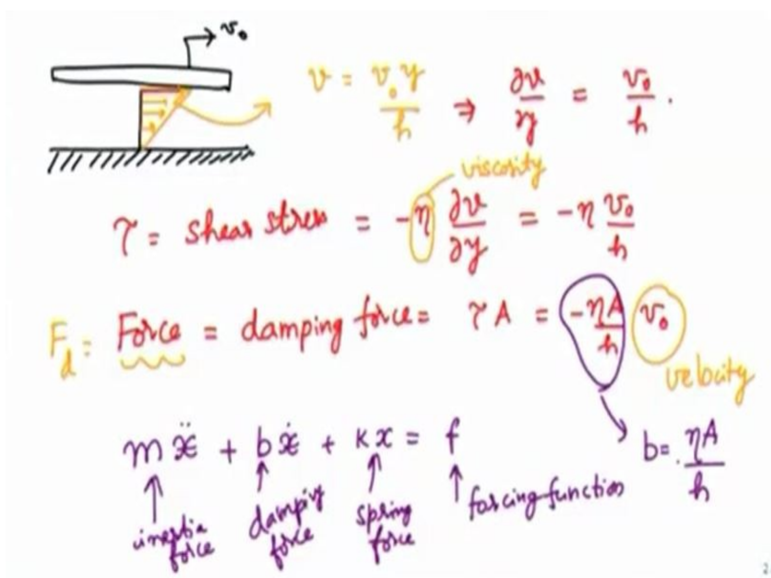

So in this flow where we have one plate fixed like this like what happens in comb fingers. In comb drive and other structures that are, there are accelerometers many of them. If you take an accelerometer there is a proof mass and the proof mass is going to move for the in plane acceleration and it experiences a fluid force. Because there is air underneath around the top it is moving and we have to consider that force.

The reason we need to look at fluid mechanics is that the damping is a result of the fluid force and there is dissipation. So we talked about viscosity that is moving your finger in thick honey, then you are actually dissipating some other force and as opposed to move in water, which is that viscous but they are also they do some dissipation. To capture the dissipation, we need to look at fluid mechanics.

The spring is only going to store energy and that is that energy strain energy and a mass inertia effects is also another kind of energy, which is a kinetic energy. Whereas a damping term leads to energy that is lost and that is what we are after to get that damping coefficient b, so have this plate fixed and the top one is moving.

We said is moving with velocity V0 and we wrote that velocity profile that is V0 there and then moving, other words V anywhere here = V0*y/h that is in this direction. Now we have to consider the force or we call it shear, because if this plate is there in a water type if you try to move the plate it will not be there, there will be some force that it feels. Now we define this shear stress, which is the force experienced per unit area of the plate that you are moving over water or any other liquid.

Here at micro scale even with the presence of air whether things move it experiences substantial shear due to the fluid beneath that shear stress is given by this relationship. This is like a constitutive law just like Hooke's law we had which relates stress and strain, here it relates the stress and something like a strain like quantity, that is how velocity varies in the Y direction dou V/dou y. So in this case dou V/dou y is-Eta V0/h. Look at that V is this, so dou V/dou y simply = V0/h. Now, when we want to get the force at the wall let us over here.

That is actually the damping force that acts on the top plate as you move there is a resistant that you feel if you have again you have to take try this experiment on a tank of water, take a plank a plastic plate or something or even a steel plate, steel dish and you put at that there try to move slowly versus try to move fast when you try to move fast there will be more force as opposed doing slowly.

So, let us say that we have a flat surface of water in a tank put your hand or even with the hand out also you can try when you move slowly it will be less force if you try to move fast it will try to drag more that is called drag. The same thing ships will experience if they are going in water or boats or anything else.

So that is apparent here in our relationship, when you say this shear stress is proportional to the velocity, so shear force is a damping force we have to multiply Tau/area of cross section. So when we do that, the substitute for this Tau, which is-Eta * area of cross section h times V0. We have not defined what this Eta is, that is a term that you have used already and that is called the viscosity.

There is a kinematic and dynamic viscosity, 2 phrases but we are not going to get into the details of that, because at micro scale one needs to actually modify the viscosity based on an assumption called slip flow velocity. Here we have seen that at the wall the velocity is 0 but that is not necessarily the case, in the case of the microfluidic channels and flows through microfluidic channel there will be a slip between the wall and the fluid particles out of there.

So this viscosity can you modify to account for those effects and when you go down to really small scales, then you have to treat each fluid particle as a separate entity and 2 molecular dynamic simulation, where you would just write Newton's laws for every particle and integrate and take the interactions among all these particles and do what is called molecular dynamics simulation or MDS. But for now let us say this particular flow, where we have damping force Tau A = this.

Now if you see this is force, let us call this FD damping force = something times velocity. So this is velocity, if you recall the expression that we had written for one dimensional model, where m times x double dot second derivative of the displacement, which is acceleration times mass is inertia force + we had b x dot that is sum b times velocity of the thing + k times displacement, which is a spring force = the forcing function.

This is the forcing function, the force that we have applied externally other thing and this is spring force once again to recall this is inertia force. Now, what is this? This is a damping force. Damping force is or at least viscous damping force such as the one that we are considering now is proportional to the velocity and the (()) (35:48) proportionality is b. So if you look at this, this is what is b.

And the sign tells you that when you are moving in one direction, the damping force will be there because the drag that is why it is called drag, it drags you back. If you move in this direction, there will be drag in airplane or a boat that goes in other direction. So, essentially we can say from here b = viscosity times area of cross section divided by the gap between the plates and that is how we need to take the b.

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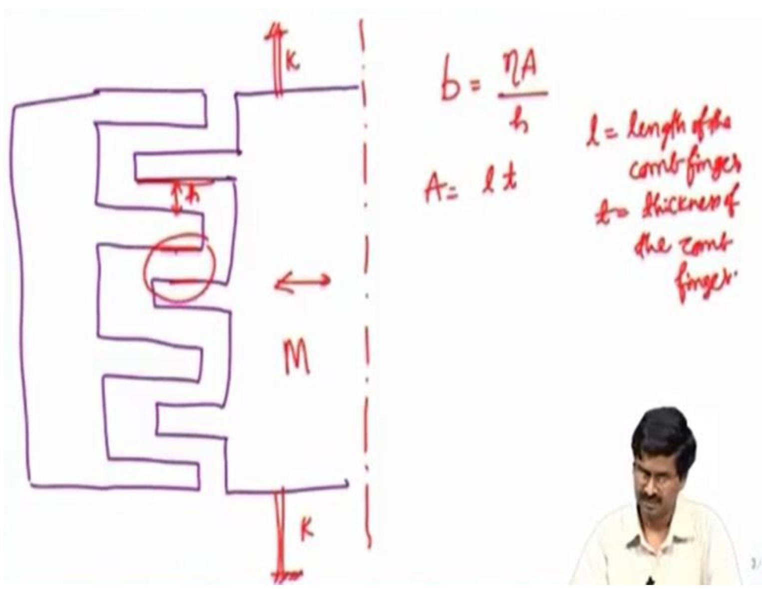

So if we have a comb drive let us say that I will show a number of combs now, number of comb fingers not really combs. So, this is let us say this is the fixed comb and there will be a moving comb. Let us say this is our proof mass and let us say this is axial symmetry. So, their similar thing is there mirror this side. Now this will have some suspension for it to move, suspension maybe a beam like this, which will and also another beam like maybe here which will make it move back and forth this way.

We can get k from these things. And of course mass is there like itself, it gives the mass for mx double dot we have that k times X we have, but the damping comes over here. So this is the fixed plate and this is the moving plate with some velocity for that we have derived b = Eta times Area/the gap. So the gap here now will be h and there will be area that will be length of the comb times the thickness of the comb. l is length of the comb finger and t is thickness, out of plane thickness of the comb finger.

If you do that you get the b and that is what you would use to do the Dynamics, that is one effect that we consider. This kind of flow that we just discussed that is one plate stationary, other plate moving such as this there is the name for it and that is Couette flow. Couette flow, under Couette flow which happens in accelerometers and comb drives and many other devices we can put an equivalent damping like a lumped constant. That is Eta viscosity times area of cross section divided by the gap between the fixed surface and the moving surface.

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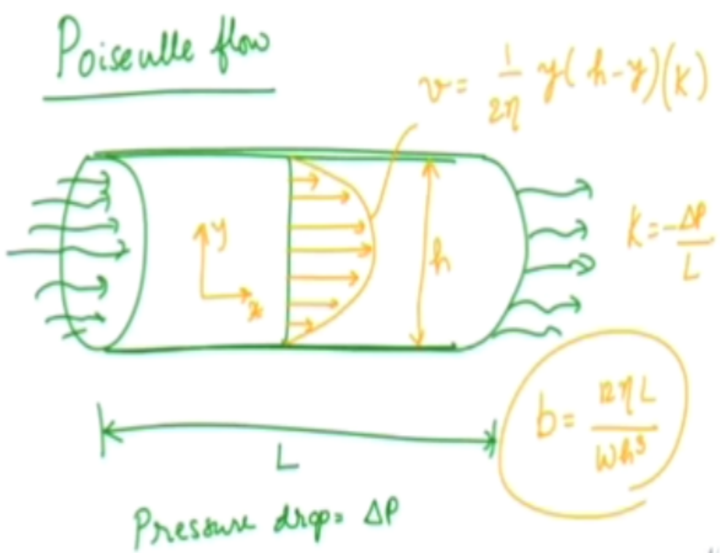

Now there are lot more things that we need to consider in when it comes to fluid mechanics but we will consider another flow, we considered what is called a Couette flow. Let us consider another flow which is called Poiseuille flow or pipe flow. So here imagine that we have a pipe microfluidics, no fluid will be going through it and coming out inside what happens and let us say there is a pressure drop across this.

There is a Delta P actually this is Delta L and pressure drop across Delta L is, let us say Delta P. This is actually L, why call it delta L, just call it L. Pressure drop cross across that, let us say delta P. Then what happens, from velocity here to here, so there also the walls of this tube that we are looking at, they will experience, this cylindrical surface if we take, they will experience some sheer stress as well. How does that, that causes dissipation, also causes sheer force on the body, how do we estimate that?

If we consider the Poiseulle flow condition, which you say, if you take a section, the velocity profile is parabolic. Again if you consider, if you look up any fluid mechanics book that tell you how the conditions of Poiseulle flow reduce the velocity profile to parabolic, which is given by this velocity V. It will be 1/2 eta times y, again this is our y axis, (h-y), where h is the hydraulic diameter of this flow in circular pipe, it is just h for other cross section equivalent hydraulic diameter, (h-y) times the pressure dropped that K is written as – delta P/L.

There is a pressure drop, so that is how we put in the equation, we get this. Now, if we take this velocity and then write the sheer stress like we did, that is viscosity eta times dou U or dou V, the velocity/dou X. That is this our x direction. Viscosity times the velocity gradient in the x direction, will give you the sheer stress, that is acting. And if you multiply that by the area that you take here, you get the sheer force.

If you work it all out, you would get the b here, so we are skipping a derivation, so that you can work it out for this case, following what we did earlier, that is previous example of create flow, if we do the same thing for Poiseulle flow, there we get this b, to be 12 eta L/wh cube, where w is the wall thickness into the paper. So if the pipe flow like this, if we take that that is your damping coefficient.

And that you can take into a formula in the lump modeling and to the dynamics. If you want to do the full analysis, like we do 3 dimensional analysis for the elastic solid body, here you have to use fluid dynamic equations, where we have to solve this so called Navier-Strokes equations and get the result. There are lot of fluid simulation software programs out there.

Any one of them can do it or solve this better to take a sample problem, small problem and try to code it ourselves. So we discussed 2 flows now, Quid flow and Poiseulle flow, where Quid flow will derive the damping coefficient to be eta times A/h, the gap between the fixed plate and the moving plate and Poiseulle flow, which goes in a pipe, as if things are going in terms of the pressure drop that occurs there. We derived an equivalent damp efficient, which is this.

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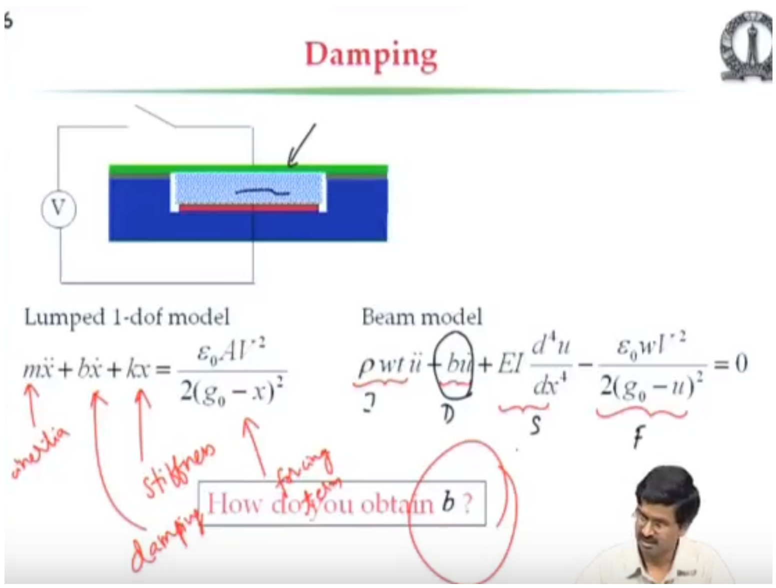

Now let us switch to another very interesting effect of the fluid that happens. First we said, if the plate is moving like it happens in accelerometer proof mass moving over the air film, then we have the Quid flow condition. But there is another case, where (()) (44:00) accelerometer, which can move up and down like this. There is a bottom substrate, other one goes up and down, then the air here get squeezed as the plate is moving.

It is squeezed and it can move out from the sides because you have the finites as a plate, when it moves like this, all the air that is here can escape from the sides. But there will be different effects, based on whether I am doing it slowly or fast. Imagine a cycle pump. When you try to push the pump slowly, you would not feel as much resistance from the fluid, as you would feel if you would have to pump fast.

The faster you do it, the more resistance to viscosity that you feel in the cycle pump. Similarly, here, when this stop play it at a beam over here, let us get that done here. This top plate here, when it moves slowly, you would receive much less damping force or fluid force, as of doing it very fast. We will analyze that. So what we want essentially is to obtain this b that occurs in this bu dot term.

It is the inertia term high is the damping term that is what we will discuss now. This is the stiffness term. This is a forcing function kind of electrostatics, which is there. So now, we will look at a problem that operates in 3 domains. There is elastic solid, the main and there is electrostatic energy domain, which is the forcing function here. And then we have the fluid that we are considering now here, that effect of damping to do that. So it is a coupled field problem of 3 things.

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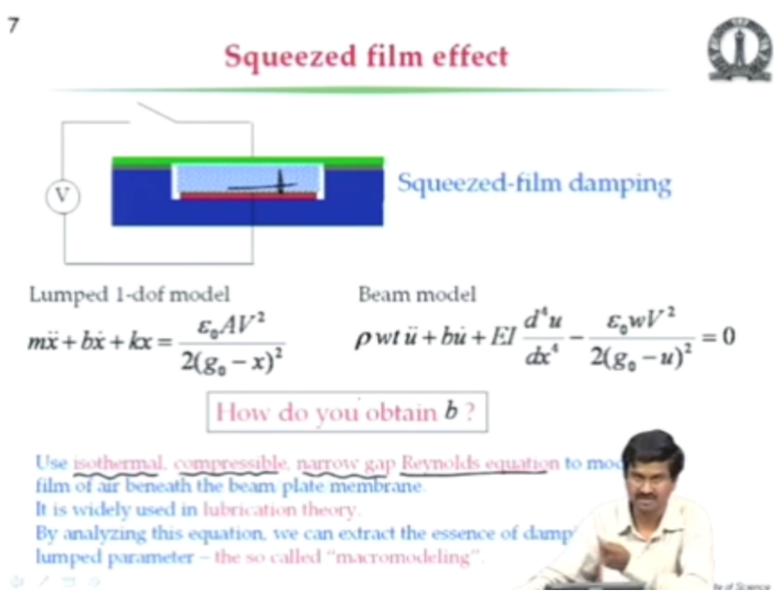

So in order to analyze this, we have to discuss this squeezed film effect, which is model using a special form of the fluid governing equation that is Navier-stokes equation called a Reynolds equation. In fact, there are lots of adjectives. It is isothermal; the temperature is not changing, isothermal. Compressible, because we are squeezing the air, it is a compressible. Narrow gap, so you assume that along this y direction, your pressure is not changing. Pressure is changing only here, but not in this direction over here.

And the film of air is very thin that is navigable already said and this particular equation actually comes from lubrication theory. So if you imagine journal bearing, where there is a fixed shaft and actually it is not even fixed, the reality it will rotate, the journal and the shaft there, in between there is a narrow gap and that narrow gap, if you put oil, that oil is getting squeezed. And that is where this Reynolds equation is very useful and very well developed.

There is lot of solutions worked out and now we are using that to model the damping due to the squeeze film effect and something else that happens, besides damping with this squeeze film effect.

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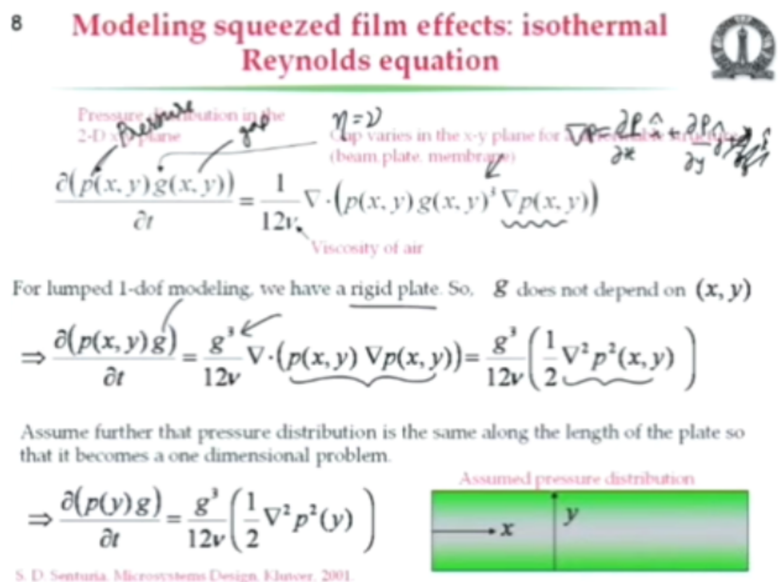

Now what is that Reynolds equation? So, here is the equation. It looks a bit intimidating at first sight. But we will simplify it, so we can extract the b from it. So here we have the pressure. We have the plate, there is x and y. So we assume that there is pressure in z direction that is not changing, only at x and y they are changing. So we have put p(x, y). And the gap is also is x, y. There is gap between the 2 plates.

When the beam moves just like this, the gap x, y will be the same everywhere. When the beam deforms like this, then there will be gap different. There will be dou pg/dou t, so it is a time varying, it is a dynamics equation. We have this nu for viscosity or where we have used eta. So whatever we had eta earlier, is same as this nu here. And this del dot, that is divergence, that is x, y, z components here, you will take dou of x component/dou x+dou of the y component/ dou z+dou of z component/dou z. That is what divergence means.

Then pg cube, notice that there is cube, just like we had cube in the electrostatics, in the pull in formula. And then we have the gradient of the pressure, that is del p = dou p/dou x in the ith direction. It is a vector gradient, z direction, that is x direction, y direction, dou p/dou z K, that is what this means. In x, y, is our final z term will not be there, static equation. It can be solved numerically.

But we will get simplified for one dimensional modeling, where we will say that it is a rigid plate that is moving in parapet approximation, we had a spring to account for the stiffness, the plate is rigid mass moving up and down. If it a rigid plate, then this g will not be dependent on x and y any more, we just have one gap for the entire plate. That is one simplification. And this dou/dou x, dou/dou y, we do here, that will not affect this g because g does not dependent on x and y, so we can take it that is shown here

And we can further write this p times del p, with little bit of vector manipulation, del square p square (x, y). And then you make one more assumption that this p varies only in this direction, but not in the x direction, because if it is very long plate, which is what to know will have either in a comb finger, we can say that nothing is changing so much in this direction, all that change in the y direction, where the squeezing is going to occur.

So if you consider that, then p depends only on the y direction, p(y)g dou y dou t of that, g cube/12 eta and we have this equation. This is still nonlinear because there is p square, del square, g cube and you know all of these things. And p and gap, both are occurring as a product form. There is lot of non-linearity.

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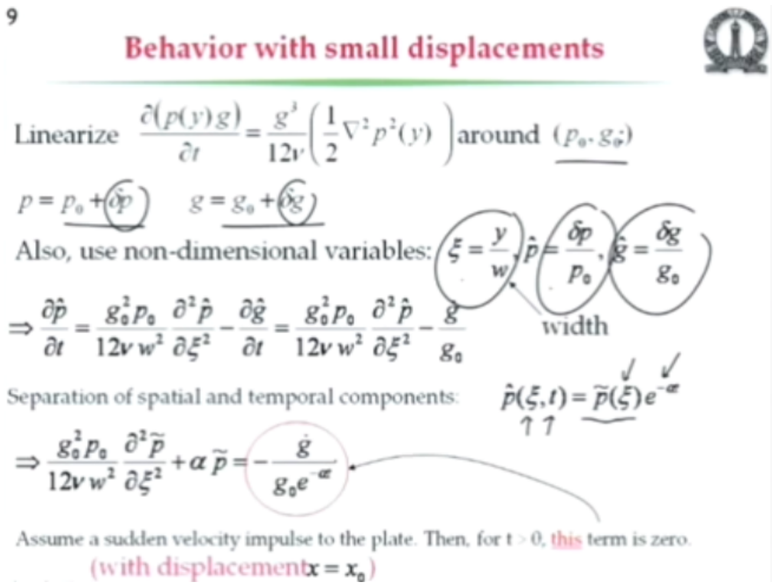

So you can linearize around a point p0 and g0 and substitute the linearized part. So we will get an equation in del p and del g, small pressure (()) (50:47), small gap (()) (50:50). That is you imagine the plate to be slightly vibrating like this. That is you have a small voltage, alternating voltage component, so because that plate is vibrating like this. One is fixed, other is vibrating like that.

So if you take this equation, introduce some non-dimensional variables such as these for pressure position, pressure and in gap. Position is y/w, width of the plate. This is pressure atmosphere at the outside and original gap g0. Then you get an equation, then you can solve by using separation of variables, there is something that depends on the position, something depends on the time because pressure is good depending on the position and time.

We can separate it out, which is standard technique for solving differential equations. Then this one can be solved this way by using an impulse excitation on the problem and try to get the solution. Again you should work out the details of this. Because deriving this takes a long time, but we have just shown the steps. If we look at the steps carefully and it is actually worked out in a couple of books, so the solution of this is given like this.

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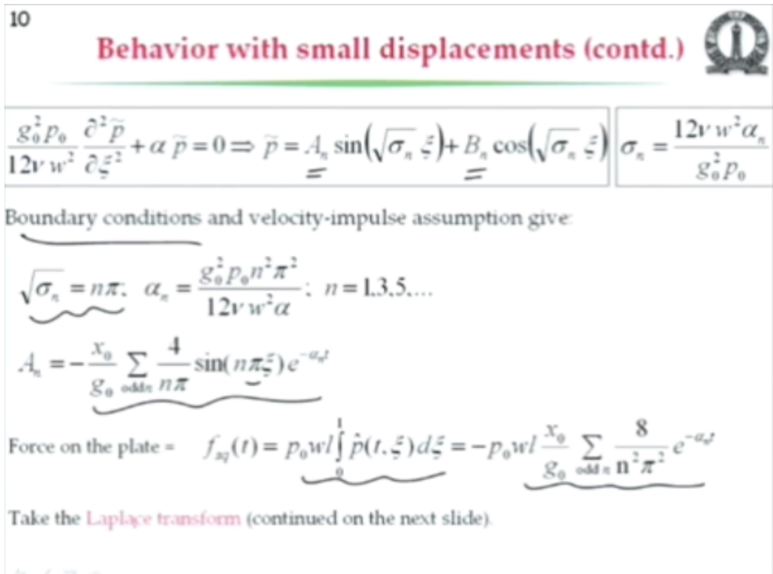

That solution is shown here, for that equation we showed in the last slide. Then we have some unknown constants. You can determine it from the boundary conditions. Then you will get a series solution such as this, only odd n because even n will cancel out. There is something called sigma n, which is also used in these equations, which is shown as n pi, which occurs in the series solution as well.

Now we can get a forced term from this because it is the pressure multiplied by area, we will get the force. That is what we are doing here the integration. So you get the force term, you take Laplace transform of this and you will see why we take that.

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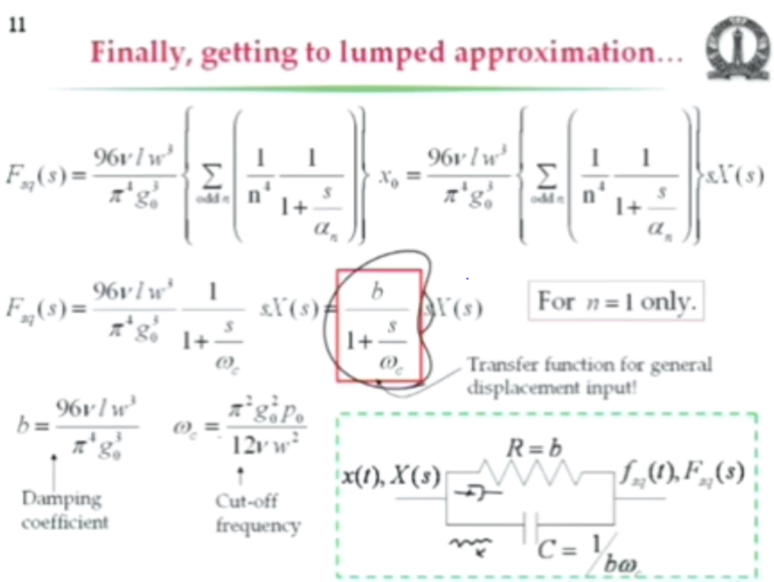

So it comes something like this. Now if you see, n to the force and n is odd that is 1, 3, 9 or 1, 3, 5, 7, 9. If it stopped, then it would be = 1. That is just an approximation to first order. You will get a transfer function that looks like this. Actually we have to include this s also. Which circuit, in terms of resistor and capacitor that we imagine, which these are transfer function and that transfer to be resistor, which is like a damper and capacitor, which is like a spring.

Mechanically this is like a spring and this like a damper. This is the b and this is k that means that this (()) (53:53) effect has 2 things, one is damping, other is the air acts like air spring. So both effects come up and they depend on, depending on the frequency of how fast your plate is moving. If it is moving slowly, you will see more of the air, the viscous effect. If we do it fast, then you will actually see the air spring.

So it is not complete dissipation effect or viscous effect that you feel when you do it very fast, you actually see the air spring so effectively.

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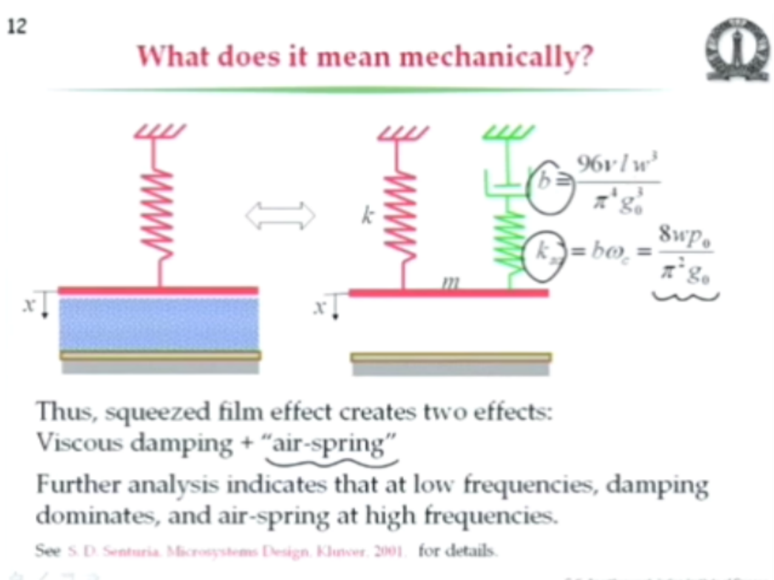

In addition to mechanical spring now, we will have the squeeze film spring and the damper. And this is the effect. So we have to consider the air spring effect also. That is the function of the frequency. And the k is given by this, in terms of p0 and g0. And b is given by these quantities. So this is what we need to do to solve a problem with the lumped model.

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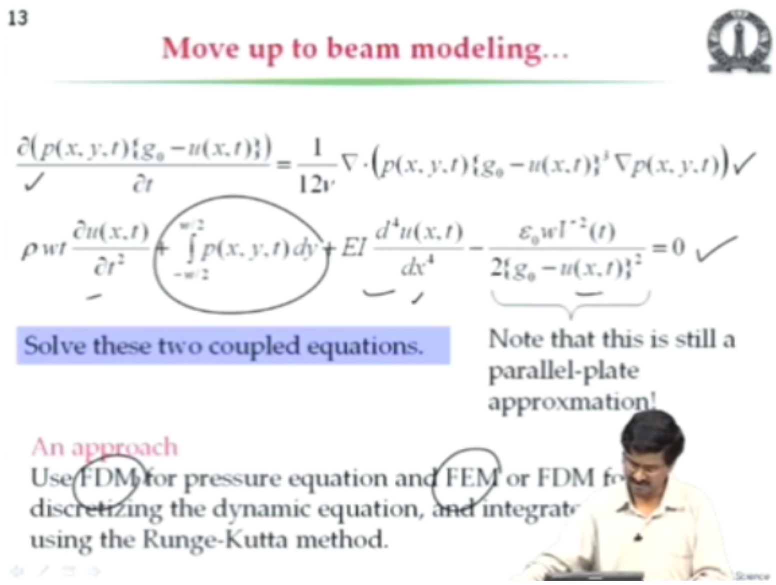

But if you want to do it, you can take this equation, solve this using the isothermal Reynolds equation and our mechanical equation. There is inertia term, there is stiffness term, there is forcing term, then this is the fluid force, the pressure times the area per or per unit length, if you do, this you have to put across the length of the plate and solve this 2 coupled equations. Already we have mechanic electrostatic coupled here.

You need to couple with the fluidic equation also, when you solve it full different, actually for interesting, to make it interesting, final difference method for solving the pressure equation that is that equation, finite element for solving the mechanics equation and electrostatics is just parallel plate approximation right now.

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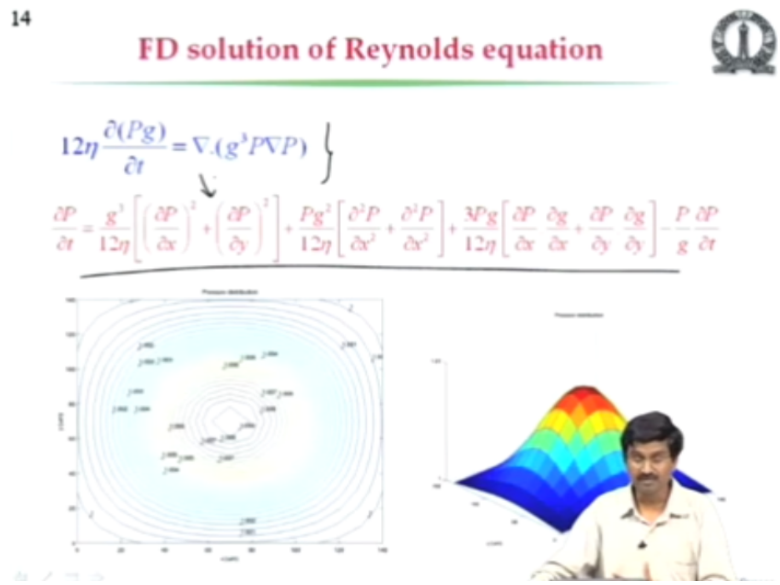

If you are getting a solution, where you can get the pressure field, without linearization. It is a full nonlinear equation can be solved. This is a bit advanced. So it is a long expression from there if you expand, you can solve it numerically, which is what we will do in the things are not amenable for simple electro solutions.

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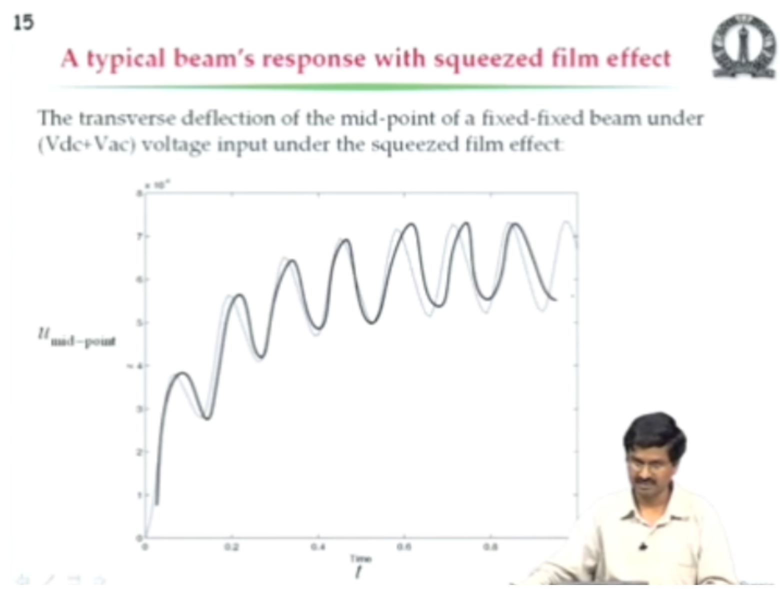

You can get this, the plate, this placement as a function of time, which is quite complicated, with all that it can be done

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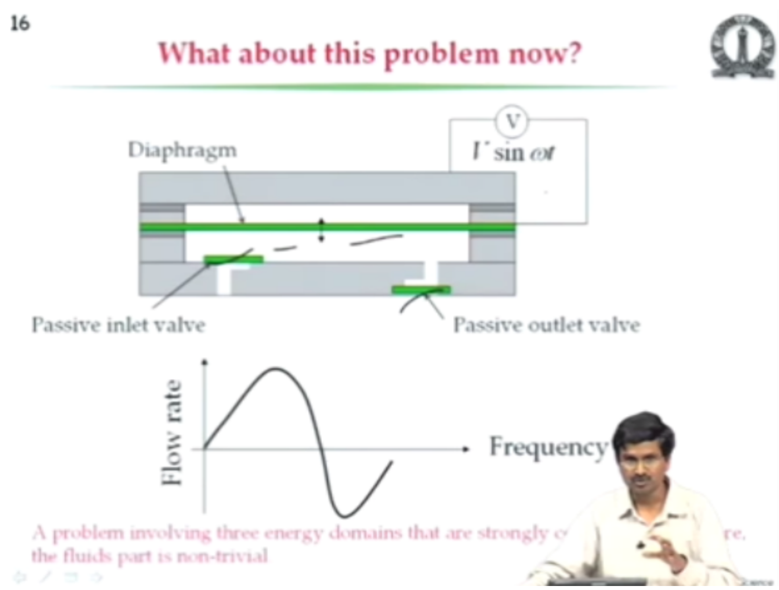

To summarize, let us take a problem such as this, where there is a diaphragm that moves up and down, there are 2 valves that move in and out like this, there is electrostatic force, so there is elastic domain, there is a fluidic domain and electrostatic force, there very strange things can happen and all that can be modeled using basic equation that we have discussed. What we have not done in detail is the fluidic effects.

So you have to read about Navier-Stokes equations and their special conditions has to, when they can be simplified to Quid flow or Poisuelle flow or to squeezed film isothermal Reynolds flow and then use the appropriate simplification to solve the problems or go to the differential equations to solve the complete numerical solution.

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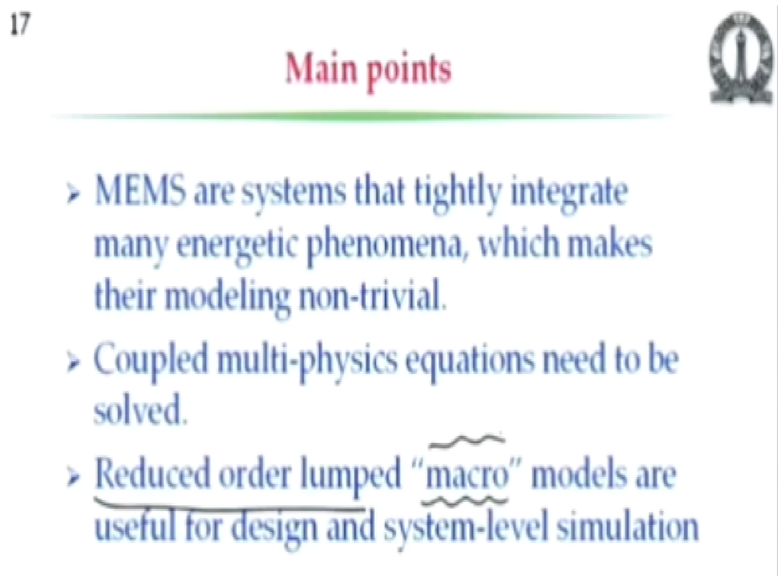

Let us just note down our main points that micro systems involve tightly integrated multiple energetic phenomenon and that leads to differential equations that are coupled to each other. And all of these have to be solved together. But what we can benefit from is reduced order lumped models or what are called macro models, they capture the macro effect of the things and in a simple form that m for inertia, k for stiffness and b for damping.

That also we discussed for couple of cases. In the next lecture, we will talk about electro thermal activators, there again multiple fields coupled to each other. Thank you.