Micro and Smart Systems
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Lecture - 26
Theoretical Basis for the Finite Element Method

So in this lecture we are going to talk about theoretical basis for finite elements. So in the last lecture we introduced finite elements, what is finite elements? And how it is used for microsystems? Today, we are going to establish the basis mathematical basis for finite element analysis.

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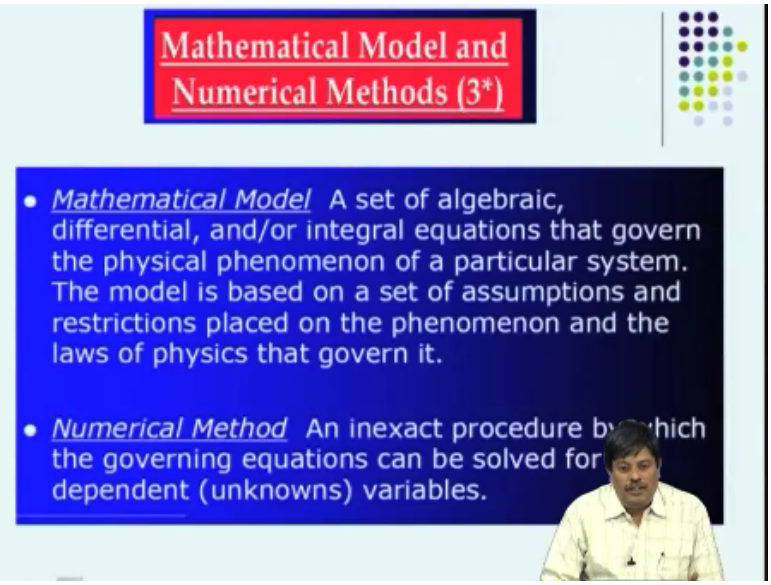

So we will revisit the deformations which we did in the last lecture will first define what is a mathematical model? Mathematical model is a set of algebraic differential or integral equations that govern the physical phenomenon of a particular system The model could be based on set of assumptions as I said it could be based on geometry, kinematics, loads, boundary conditions and also the restriction that is placed on the phenomenon and the laws of physics that govern it.

So in order to solve this mathematical model we need a numerical model we explained this in the last lecture and a numerical model is an inexact procedure by which the governing equation can be solved for dependent variables or unknown so any governing differential equation will have a

dependent variable and an independent radio so what we are solving is for the dependent variable.

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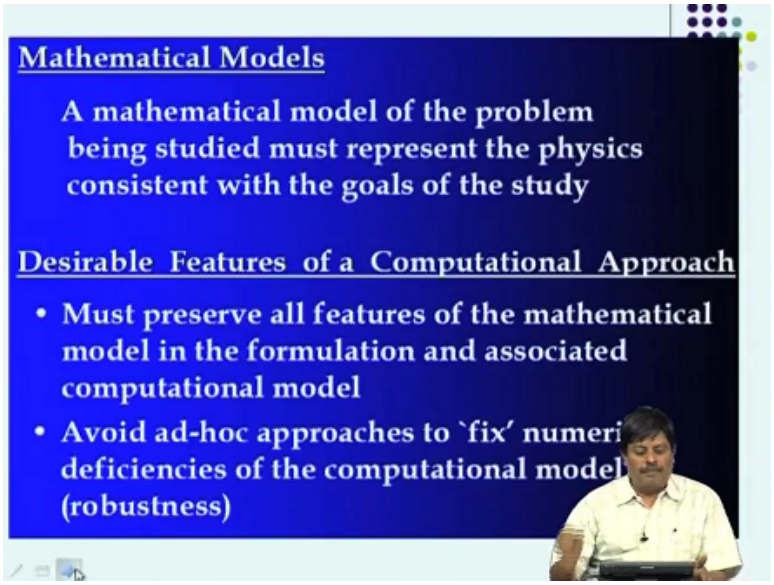

The mathematical model when you come to that is a problem being stated that must represent the physics consistent with the goals of the study if you are talking about vibration problem you have to solve a wavy question if you are talking of electromagnetic problem you have to solve a Maxwell equation and your goal of study in the Maxwell equation is to find the electrical field the magnetic flux the magnetic field in vibration problems.

We are talking about deformation and deformation history so what are the desirable features of a computer Mathematical Model the mathematical model must preserve all features in the formulation associated with the computational model we should avoid any adhoc approaches like the fixing approaches or in other words what we are talking about is to come up with a robust model that will physically represent the physical system.

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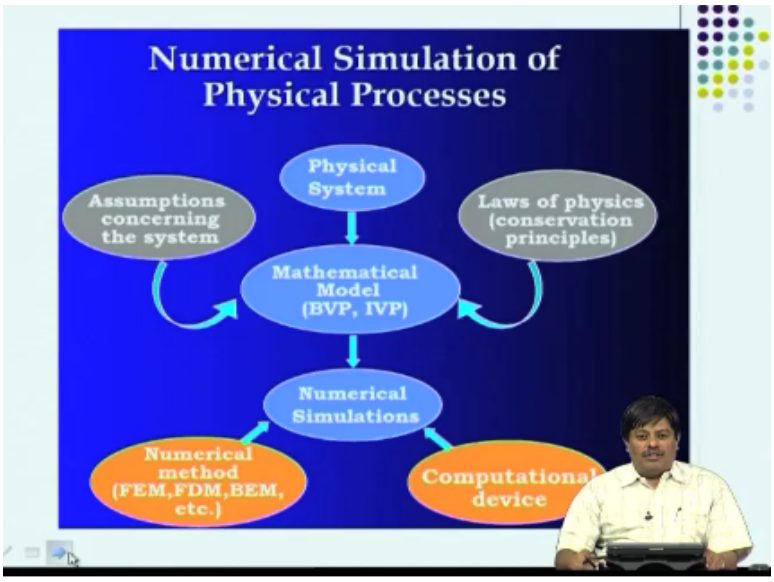

So we will again talk about the physical system so we can construct a mathematical model a mathematical model will be a boundary value problem having a set of governing equations and the boundary conditions and also initial value problem if it is a time dependent problem where you will have initial conditions a time T=0 and these are basically obtained from the laws of physics like the conservation laws energy laws, mass conservation.

And also assumptions concerning the systems so we need a method to solve this mathematical models so we use numerical methods and there are various numerical methods such as finite element that finite difference method, boundary element etc. and these simulations cannot be solved by hand so we need a computational device such as computers to solve this.

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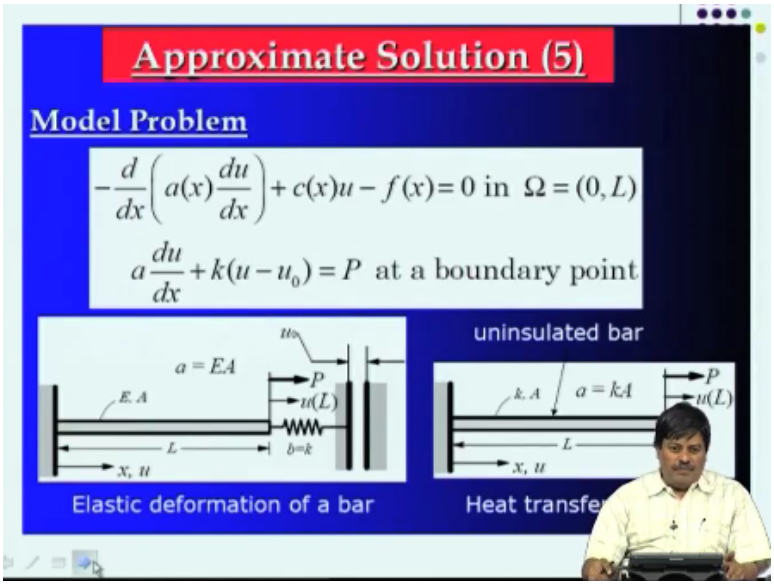

Let us take a second order system The second order system is given by these equations here which is essentially a ordinary differential equation where a(x) or c(x) certain parameter that spatially dependent and this is also subjected to certain boundary conditions which is given by this this equation so what does this equation represent this equation represent many things for example it may or it represents the elastic deformation of a bar that is given here where this P is the force.

K is the spring constant that is here and use the deformation and A is certain material parameter it can be a models of elasticity. The same system can also represent the heat transfer in a bar where P is the heat input or the heat output and in the case of a bar K is constant and K is not zero and A is certain material constants such as the thermal conductivity and the area of cross-section and this is defined over a domain 0 to L.

So that has to be there so if physical system has a governing differential equation which is which is made from the certain basic unknown in the case of a bar it is the deformation in the case of heat transfer it is the temperature and it is subjected to certain boundary conditions as shown here so this is a physical system that we are trying to solve do we have an exact solution probably yes but here what we are trying to see is how we could probably use numerical methods such as finite element method that to solve this problem.

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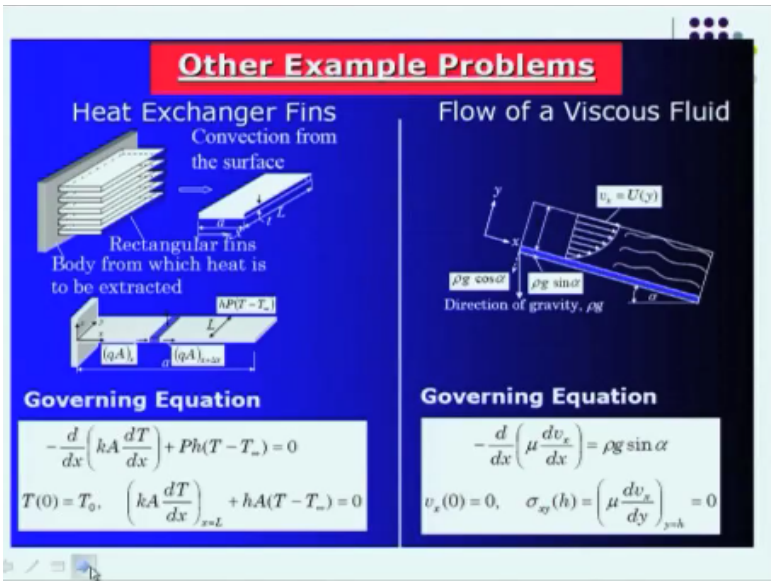

The same equations can also represent many things heat exchangers in a fin where instead of A we have a K times capital A which represents certain parameters of heat exchanger it also represents the flow over a viscous fluid where A is basically viscosity and fluid and the right hand side the forcing function is basically the density and the acceleration due to gravity and both these equations again are set of boundary conditions.

Where in this heat exchange problem the temperature is preserved we have P at certain spatial distance is specified and it is a heat flux is also specified so we see that a governing equation has a domain and over which we need to solve subjected to certain type of conditions called the boundary conditions and all these equations are called the boundary value problems for which we are seeking a solution.

And in this problem T is the dependent variable and X is the independent variable, so what we are trying to solve is the dependent variable is the basic unknown for which we are looking for the solution.

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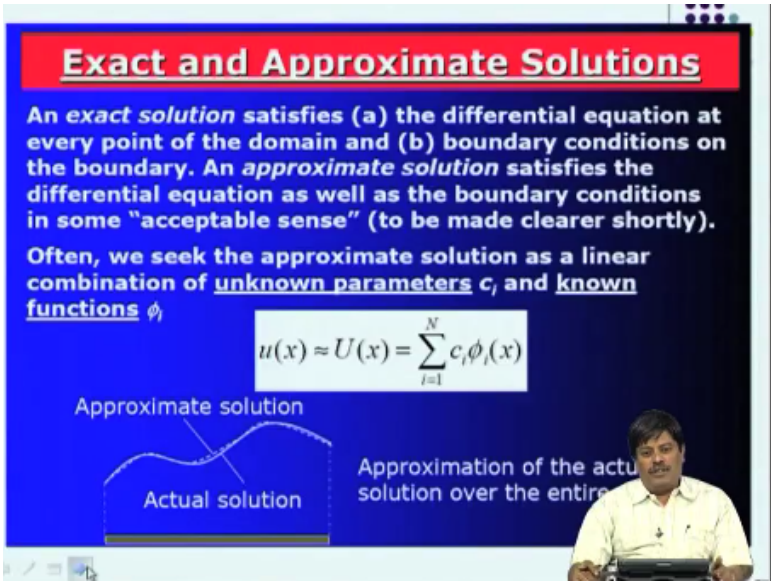

So now let us actually define what is an exact solution and what is an approximate solution an exact solution is the one that satisfies the governing differential equation at every point in the domain the domain 0 to L and also in addition it also satisfies the boundary conditions on the boundary an approximate solution satisfies the differential equation as well as boundary conditions only some acceptable sense and one of the at certain point.

So the goal here what we are doing is not the exact solution we are trying to seek an approximate solution as a linear combination of certain functions coupled with certain unknown co efficient called Ci. So we are looking like a solutions which is given by this which is basically sum of Ci*phi(x) and this is a function we need to choose appropriately such a way that it represents very nearly the exact solution but not the exact it is not the exact solution.

So if you plot if you plot this variation what we are looking at is the difference between an exact solution and an approximate solution which is shown here the dotted line is exactly the approximate solution with does not fall exactly on the exact solution line but it is more or less very near the exact solution.

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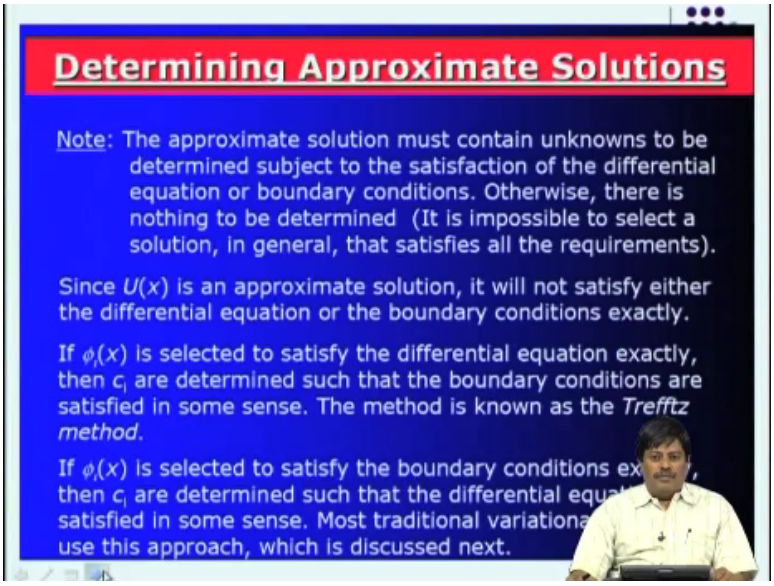

We should note that approximate solution must contain unknowns to be determined subject to satisfaction of the differential equations or the boundary condition of course if we do not have unknowns we have nothing to determine suppose u(x) is an approximate solution to small u(x) that we define for the differential equation it will satisfy it will not satisfy either the differential equation of boundary condition exactly.

And the solution depends upon the function we choose phi(x) if phi(x)is selected such a way that it satisfies the differential equation exactly then such a can do such a solution is called the Trefftz method in the Trefftz method the boundary conditions are not satisfied exactly but only an approximate sense. Let us look at the second possibility suppose we choose phi(x) which is satisfying only the boundary condition exactly.

And that not satisfy differential equation exactly what satisfies only in some approximate sense such as such a solution is also an approximate solution and most of our variations are based on this because it is easier to choose a function that satisfies the boundary conditions rather than the differential equations so most of our methods that I am going to describe in this lecture are based on the premise that we chose the functions Phi(x).

Such a way that it satisfies the boundary conditions exactly and differential and the governing differential equation in some approximate sense.

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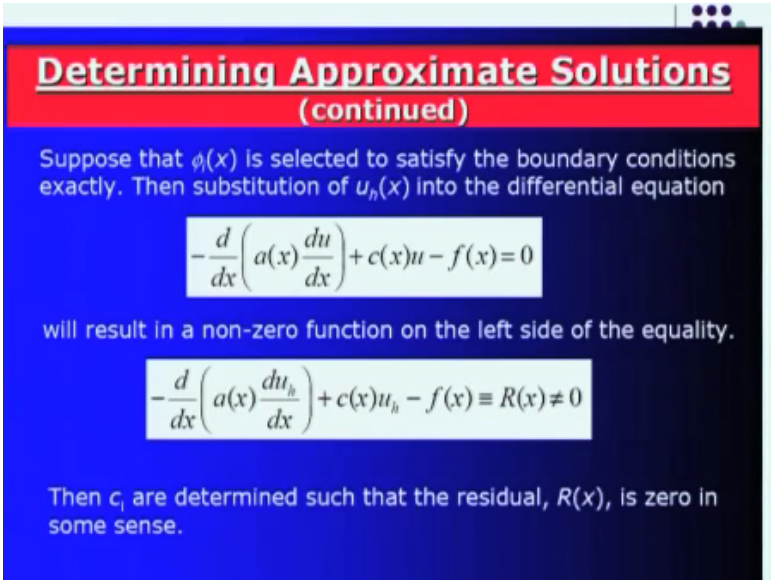

Let us revisit the same equation now suppose we choose some Phi(x) as a summation as a linear combination of this Phi(x) as the solution and we substitute this into this equation obviously this will not be satisfied exactly because we already said that we are choosing phi(x) that satisfies only the boundary conditions exactly but not the governing differential equation So this will result into some residue which we call R(x) because it will not satisfy exactly.

So our objective is to choose the Ci in our summation such a way that there residue of x is minimized or goes to 0 in some sense.

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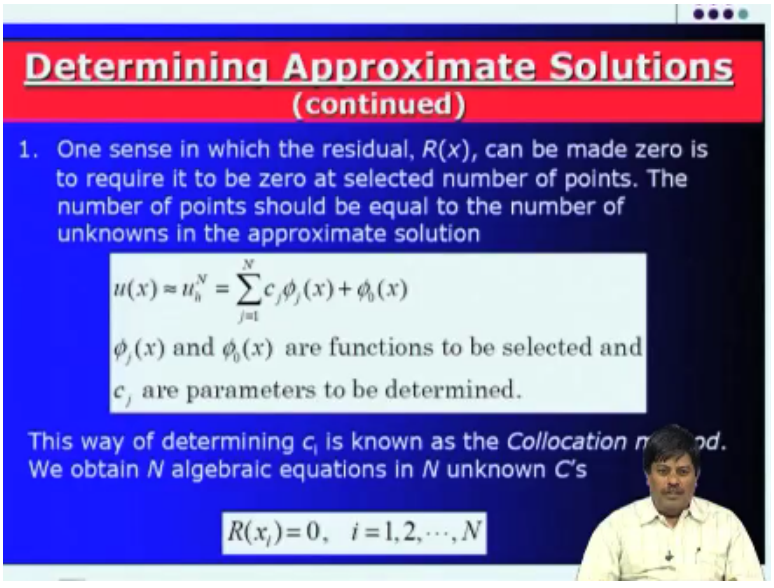

So now let us talk about how to determine the approximate solution in one sense we can take this residue which is a function of all the Ci the unknown coefficients and we can selectively choose that number of points and make it 0. So we will get a set of simultaneous equation in terms of Ci which we can solve and find the solution this way of determining Ci is known as the collocation method.

So this is one of the numerical methods that we can actually do so the number of points we choose to make this residue go to 0 is equal to the number of unknown Ci we have in the solution. So if there are 10 Ci to be determined we need to choose 10 locations where this residue goes to 0 and such a method is called the collocation method.

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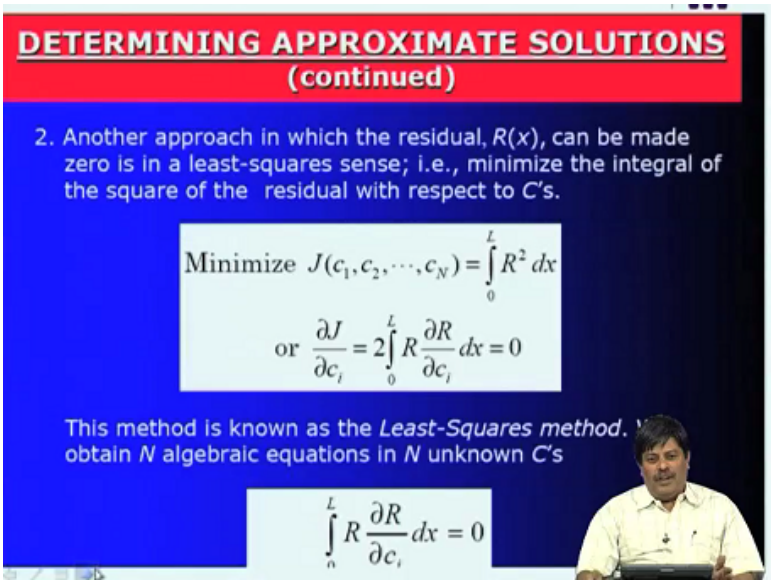

Is there any other method yes there is other methods that are that you can construct one way of doing is make this residual R(x) go to 0 in some least square sense that is we minimize the integral of the square of the residual with respect to the unknown coefficients Ci, So we take this equation we call this J which is nothing but integral of the domain length, the integrant is R square that is square of the residue.

So we minimize this 0 that we take these integral differentiate with the unknown coefficient Ci and we get this so if we make this R*dou R/dou Ci = 0 over the domain we get a set of N

simultaneous equation corresponding to N CIS that we are trying to find out , such a method of determining approximate solution is called the least squares method.

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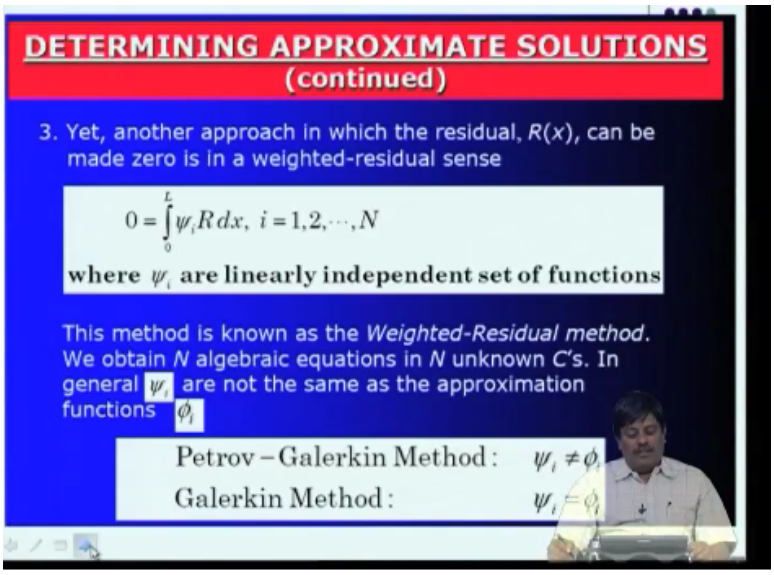

There are many methods that you can find and one of the methods what we talk about is the weighted residual method where we take the residue R weight it with the weighting function psi integrate over the domain okay, there will be N psi corresponding to N unknowns and these psi i essentially the ones which satisfies the governing boundary conditions so this method is called weighted residual method.

And that we obtained N algebraic equation corresponding to N unknown C. So in general the CI are different from phi, phi are basically the functions that are used to approximate the dependent variable that for which the solution we are trying to find out psi are the ones which are the waiting functions which we are trying to use to find out the approximate solution and in general psi and phi are not same If psi is not same such a method is called the Petrov Galerkin method.

If psi are same as phi it is called the Galerkin method or in other words we would see later that Galerkin method is another form of finite Element method that we are going to talk about in this course.

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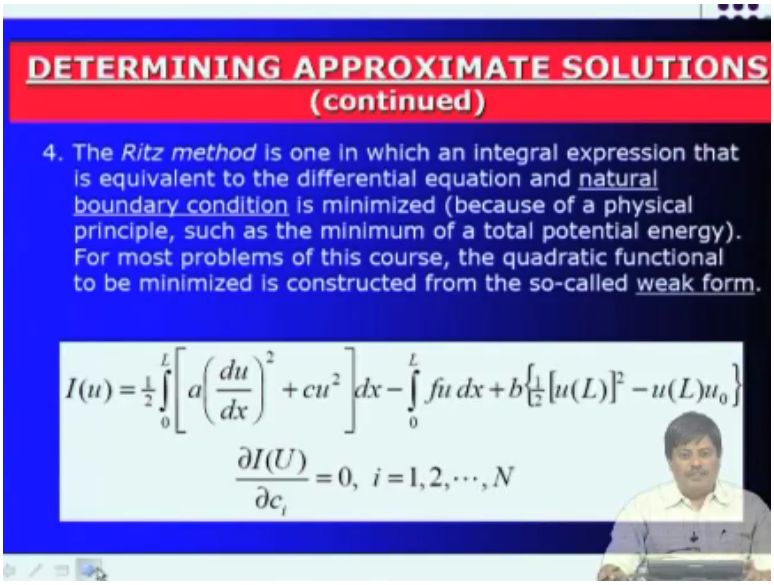

So we come back how we actually approximate the solutions in the finite element, we use what is called the Ritz method. So RITZ method is one in which integral expression that is equivalent to a differential equation and natural boundary condition is minimized because physical principles such as minimum total potential energy is the basis for this minimization and for most problems of this curve.

The quadratic functional to be minimized is constructed from the so called weak form of the solution we will explain this a little later So the quadratic function what we described for the differential equation we just dealt with it is of this form okay. So basically by minimizing this functional=0, we get a set N equations corresponding N unknowns which we solve to get the approximate solution.

So basically what we have done is we have taken the original equation and converted into this form big form of the we will let you know little part that little later in the lecture I will tell you how we can construct a weak form of the solution and this is the method we adopted in FEM rather than taking the weighted route.

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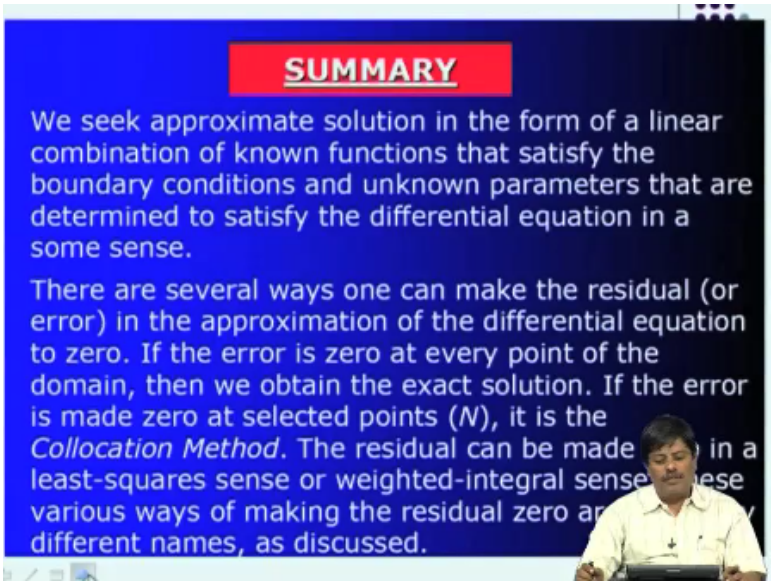

So let us summarize what we have understood now We seek an approximate solution in the form of a linear combination of known function that satisfy only the boundary conditions and unknown pattern meters that are determined to satisfy the governing equation in some approximate sense there are several ways one can make residual or the error due to approximation of the differential equation to 0.

If the error is zero at every point of the domain then we obtained an exact solution however this is not the case if the error is made to go to zero at selected points N it is called t collocation method the residual can also be made to zero in the least square sense using the least square method and these are the various ways of making the residual 0 and by the method by which you use this to make that is it takes a different forms are different names are different numerical methods as we discussed.

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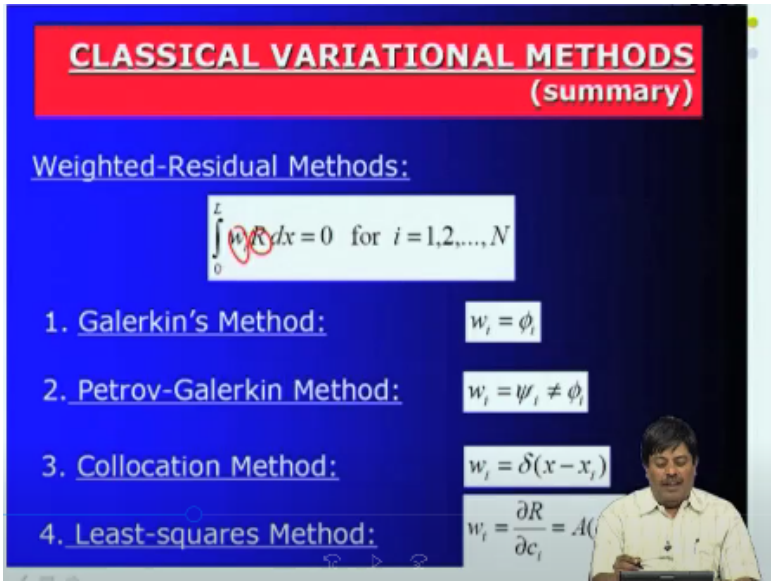

So this is summary so everything can be explained under the weighted residual method so you take the residue R you weighted with the function W integrate over the domain to 0 and there are N number of weights corresponding to N number of unknowns that we are trying to solve if the weighted function is same as the approximation function you use for the dependent variable such a method is called the Galerkin method.

If the weighted material if the weight is totally different from that of the approximation function such a method is called a Petrov Galerkin method if the waited if their weight you used to minimize is equal to a direct delta function which is called the collocation method and if the weight is used as the differential of the residue with respect to the co efficient then such a method is called the least square method.

So each of these are derived from one method which is called the weighted residual method which basically takes some weight function weighted with the residual integrate over the domain so this is a fundamental principle for all the method and finite element is one subset of this which is got from the taking the weak form of the solution.

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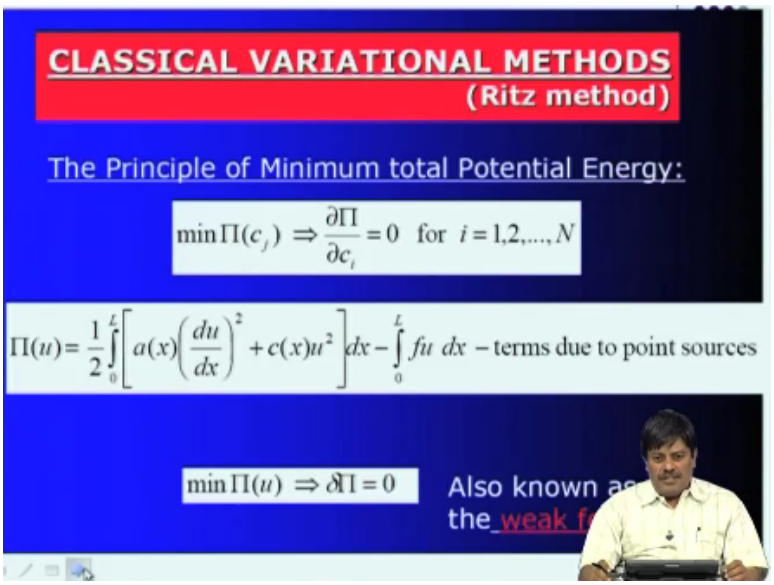

So we will come back to the variational principle One of the fundamental principle that governs the systems is the principle of minimum total potential energy that total energy is the energy due to the external forces acting on the structure and also due to inertia which is called the kinetic energy, if you take the total energy and minimize with respect to unknown coefficient C=0 then we get what is called the form of the solution.

So for example for the problem which we just discussed a second order governing differential equation the weak form of the solution is obtained by this. So when we minimize this =0 such a form is also called the weak form, so how do you construct a weak form we will let you know now.

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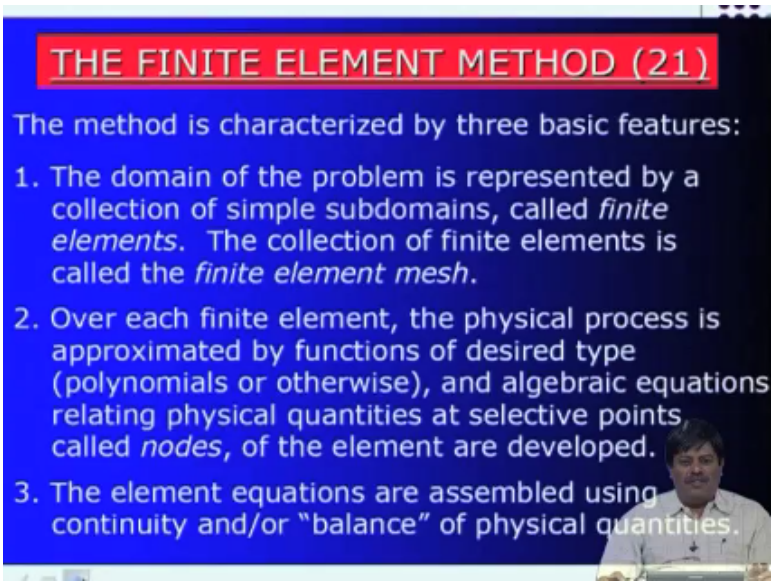

So before we understand what is a very weak form let us talk about what is FEM I discussed this in the last lecture and I am going to revisit here the whole domain is characterized by three basic features So the first feature is the domain of the problem is represented by a collection of simple subdomains called the elements are defined it element the collection of these finite element is called the finite element mesh over each finite element.

The physical process is approximated by functions of desired type the most common type is the polynomials are few other types and the algebraic equations relating the physical quantities at selective points called nodes of the element are developed. The element equations are then assembled together using the principle of continuity of the dependent variable and it gradients or the balance of forces of physical quantities.

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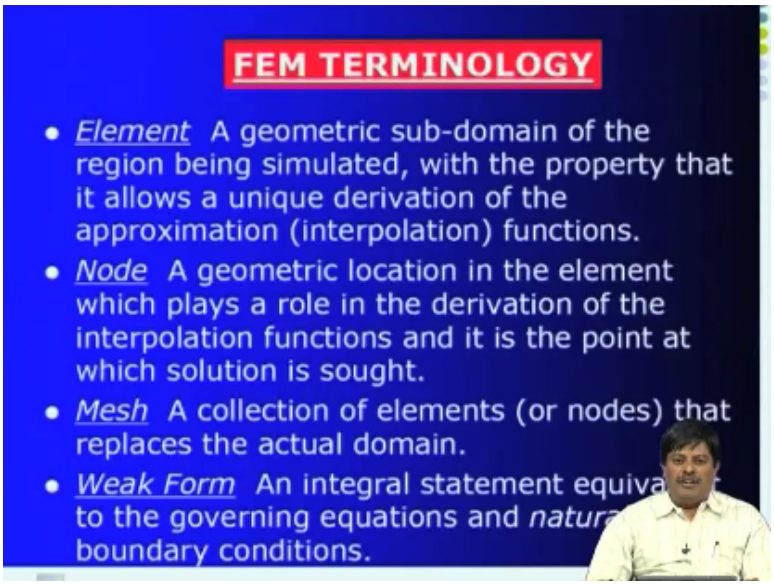

Some of the FEM terminologies what is an element an element is a geometric subdomain of the region being simulated with property that it allows a unique division of the approximation functions. A node is a geometric location of the element which plays a role in derivation of the interpolation function and it is the point at which the solution is sought, we do not because even though what we are talking about is a continuum.

We seek the solution only at nodes mesh is a collection of elements or nodes that replaces the actual domain and the fundamental thing in FEM is writing the weak form of the equation, weak form is an alternate statement of equilibrium what we are looking at a solution exact solution is solving the governing equation the governing equation is not amenable for solution weak form is amenable for numerical solution it is an alternate statement of equilibrium.

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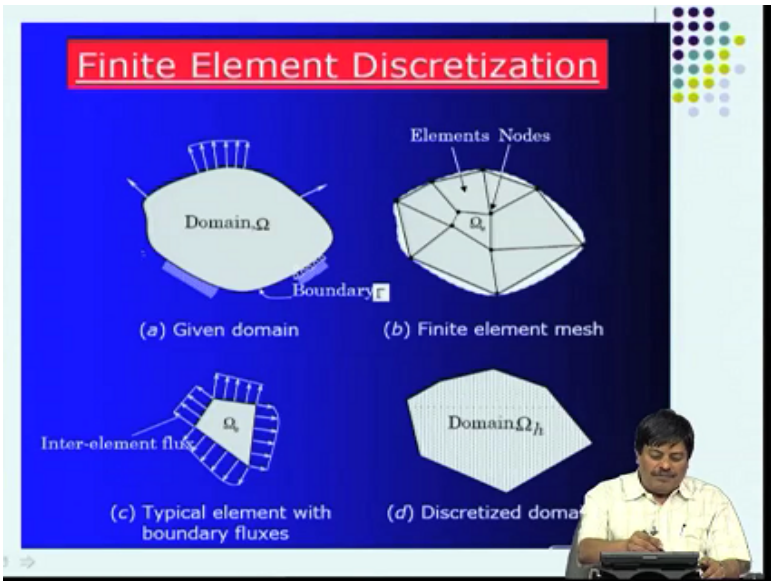

So we take this domain in approximate domain in FEM this is the given domain we mesh it, this is the meshed region into number of elements and if you isolate is that a typical element we have the boundary flues the loads because of the external tractions that are applied on to the boundary that will be part of it and we solve for each of this synthesize the whole thing and get the solution at these nodes.

So these points are called the nodes these points are called nodes and this is a typical element. So we take each one of this fit the approximate function develop all the mathematics concerning this eliminate assemble this whole region and get a solution in terms of nodes that in essence is what we call finite elements and finite elements discretization.

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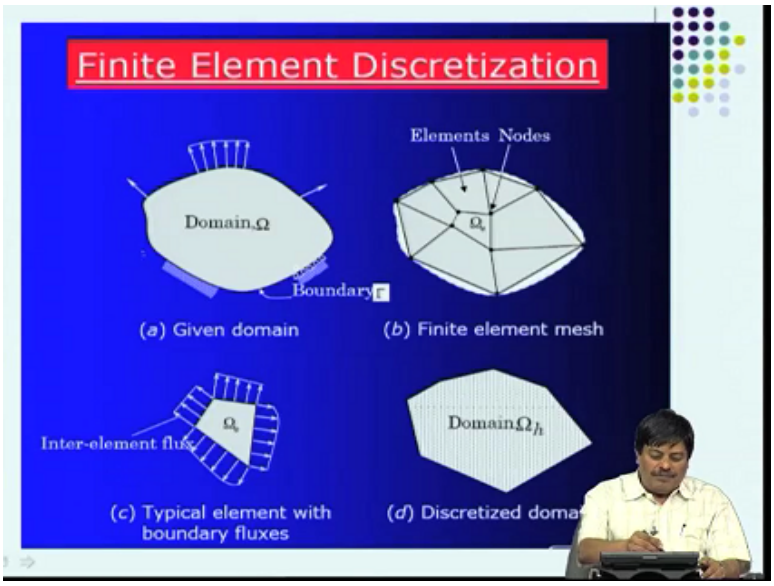

So what is a finite element model, A set of algebraic equation relating to the nodal values of the primary variable here we are said displacement if it is a heat transfer problem it is a temperature it is electromagnetic problem it is either electric field or a magnetic field or both, so it is a set of algebraic equation relating to the nodal values of the primary variables to the nodal values of the secondary variable.

The secondary variable can be forces or the heat flux are the magnetic flux etc. depends upon the problem What is a finite element model finite element read model is not same as finite element method, there is only one finite element method but there can be many finite element model the one model could be based on least square finite elements one model could be based on Galerkin finite element etc.

And the numerical simulation is the evaluation of the mathematical model that is solution of the governing equation using the numerical simulation tools like FEM using a computer.

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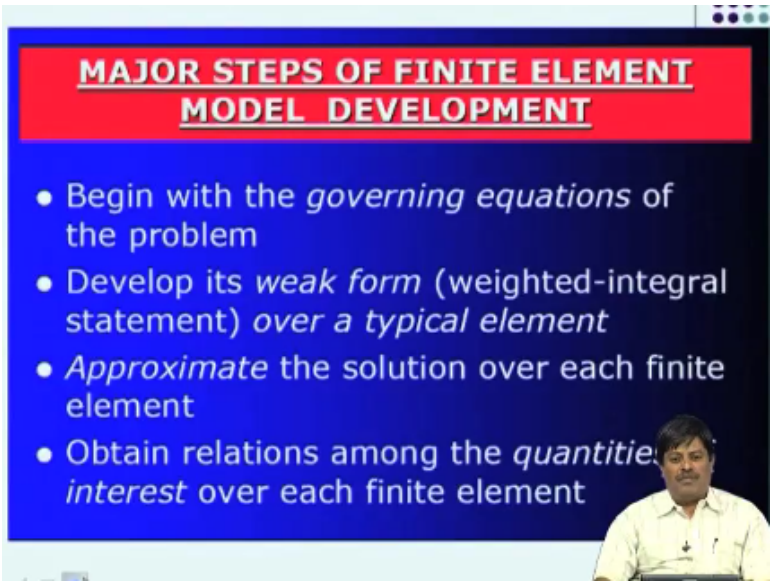

So the major steps in finite element model development we begin with the governing equation of the problem and the problem is converted into a weak form of the governing equation which is an alternate statement of the equilibrium we can use weighted residual statement weighted residual statement or weighted integral statement for developing this we solve the system approximate solution find the approximate solution obtain the relations among the various quantity of interest over each element.

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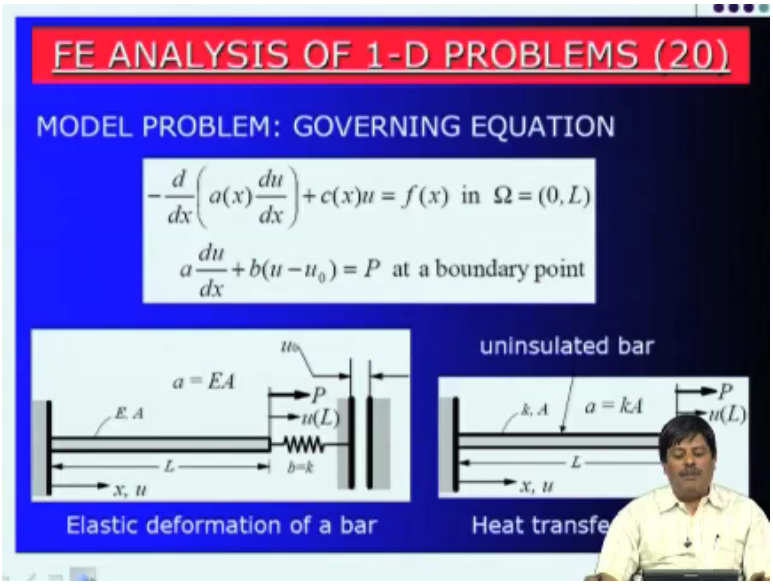

So we will come back here to this problem again and develop the weak form of the solution so this is a problem which we explained before it can represent either the elastic deformation of a bar or a heat transfer in a bar and many more physical phenomena and what you are thinking

about the system is a mathematic is same the physics of the by which these equations was derived is totally different.

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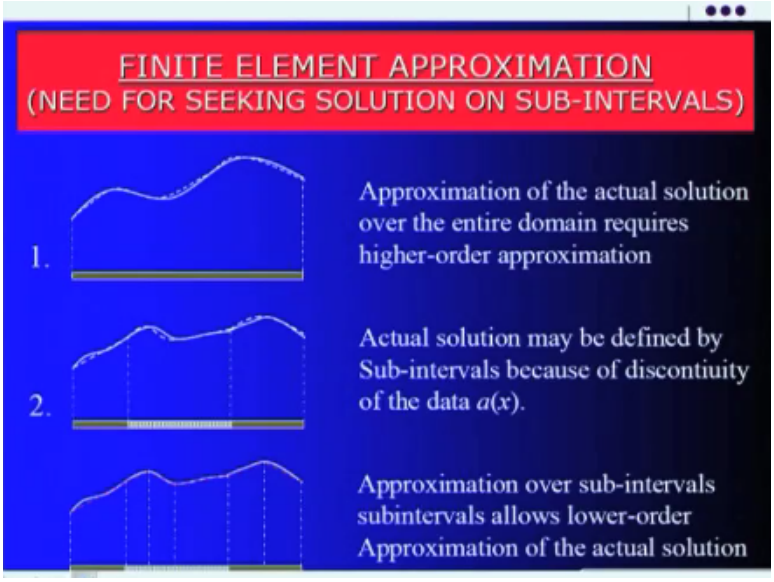

So we had we are looking at the obtaining an approximate solution so if you want to take the whole domain as such which is very long, so the approximation for such solution requires very high order approximation so we need to actually suppose if this varies like a highly like a plying or a knob like a variation we need really a higher order approximation of the variable so maybe we need a fifth order polynomial.

Seventh order polynomial to get the variation over the entire domain on the other hand we split up this into many subdivision as we talked about in our first lecture on FEM. Then is it possible to use lower order approximation over this region so if we use many such elements we can even with lower order approximation we can simulate the actual variation more exactly so that is the philosophy by which we split up the entire domain into many subdomains called the elements.

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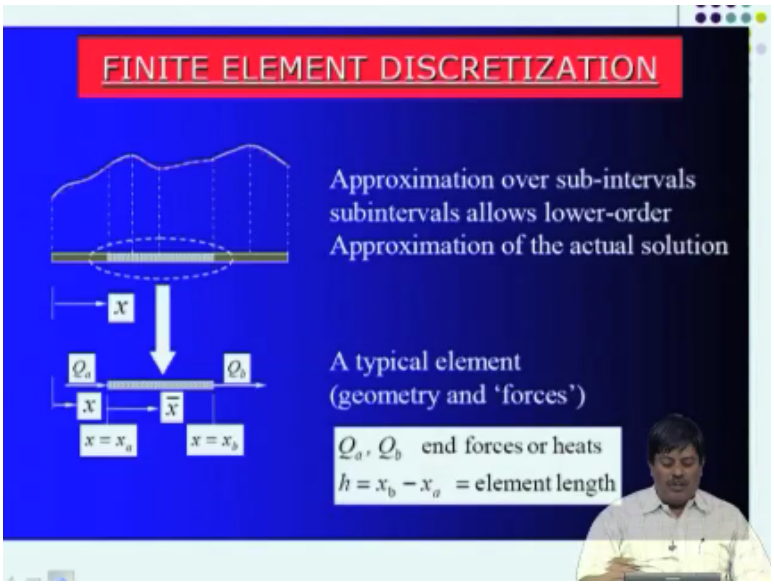

So a typical element is shown here so a typical element will have the dependent variable it can be U, U can be displacement or the wave temperature etc. as the case may be and it was subjected to boundary fluxes which is called Q not, QA and Qb, at the two ends at X= XA and X= XB. So these are called the end forcers or the heat as the case may be if it is a structural mechanics or the heat transfer problem and the domain length H is basically XA- which is the length of the element.

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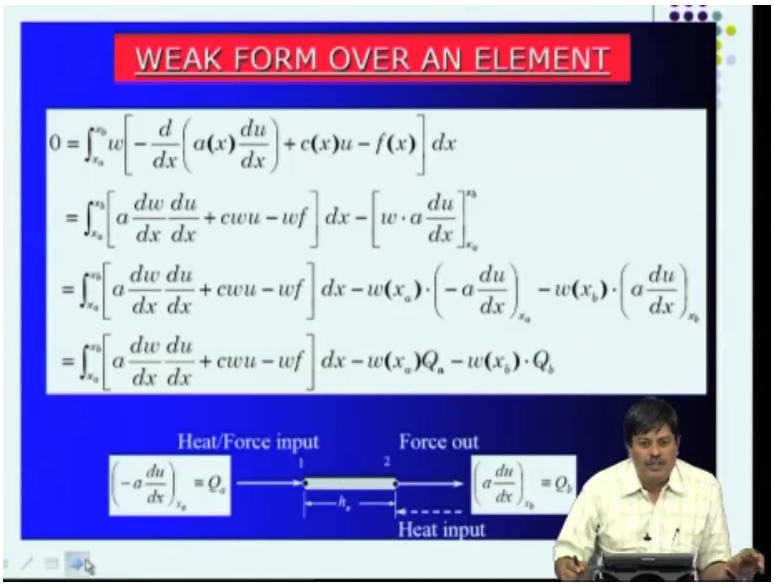

So how do you get the weak form of the solution so the weak form of the solution is basically got by taking the governing equation and weighting it with some function W(X) and integrating over the domain so in doing so we have to in doing so we have to integrate this by parts so while

doing so we can take that while integrating in most of the weighted residual technique or in the variational statement integration by parts is absolute necessity to get more insight into the system.

So we can actually do that by taking the so here W is the first function and this is this quantity is the second function so we can choose this as a first function and this as a second function so when we integrate this this is the first function and this could be the second function so when we integrate this you take adu/dx first function into the integral of the second function which is given here which is evaluated at the domain X here.

And XB into a differential of the first function that is dw/dx and these are grouped together So this statement has a lot of meaning this is the boundary term that is coming here, let us take a brief look at what are the boundary terms, so the boundary terms is W evaluated at XB and into this flux term that is evaluated at X B XA and W evaluated at XB and the flux term evaluated at XP, so basically you see that there is a W term and derivative of U term.

The derivative is basically a flux term which we call it as forces are heat flux as the case may be depending upon the problem you solve and a set of equations in this form, okay So as I said earlier ADU/DX could be thought of a force input at XA and the force output at XB or a heat input at XA or a heat input at XB, so these are the boundary terms and by doing those variational form weighting it with a function we always get the form that this flux input takes in relation to the dependent variable.

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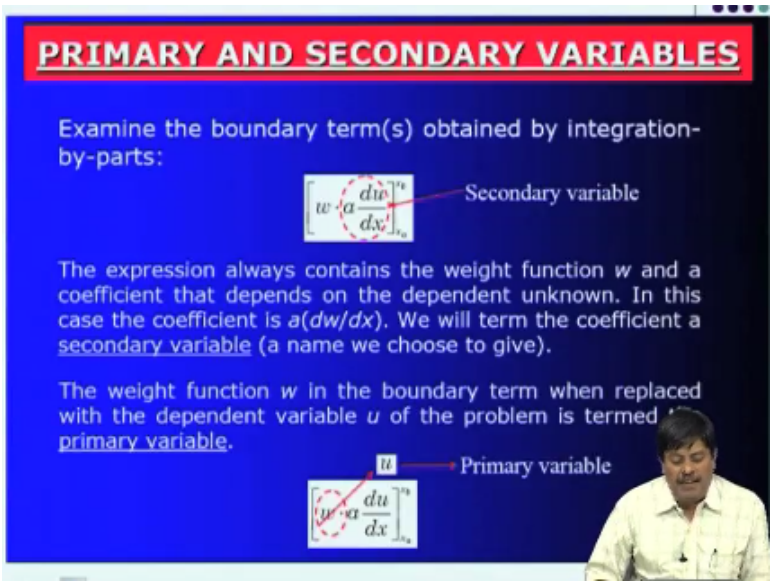

So let us examine this boundary terms, the boundary terms was w*adu/dx, okay so here we considered this adu/dx as a secondary variable and this is obtained by integrating the actual governing equation by parts the expression always contains the weight function W and a co efficient that depends unknown, in this case the dependent co efficient is adu/dx this is adu/dx we will term the co efficient as a secondary variable.

The weight function W in the boundary term when replaced with the dependent variable U becomes a primitive variable so you see that the boundary term contains both the primary variable and the secondary variable which is a directive were the primary variable and it is very necessary to understand this primary and dependent and the secondary variable because these form the boundary value problem for which we need to solve.

We need to satisfy our assumed solution exactly, so because we said that the fundamental principle on which FEM is based is how we satisfy this equation exactly the boundary condition exactly.

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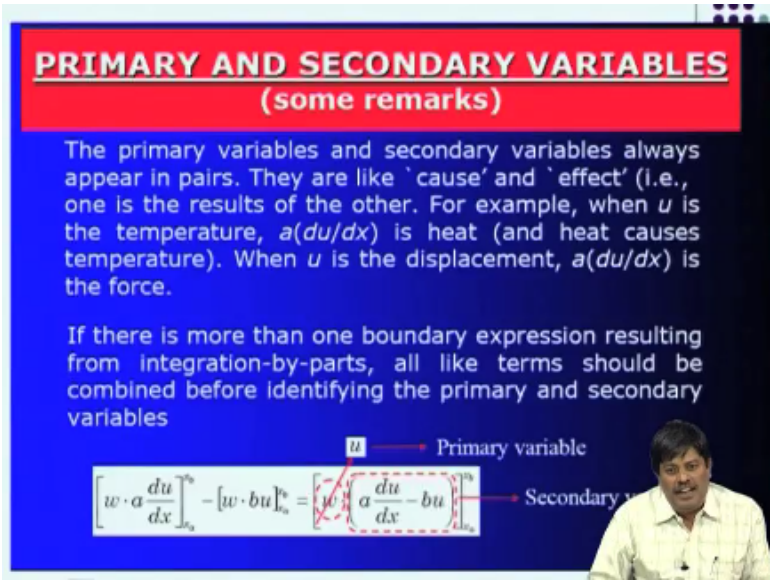

So the primary variables and secondly variables always appear in phase they are like cause and the effect that this one is a result of other for example U is the result is that there is a deformation caused by adu/dx which can be heat or a force. So they always occur in pairs and if there are more than one boundary expression resulting from the integration by parts all like term should be combined before identifying the primary.

And secondary variable as we have done here So that we get the form of the secondary variable which is very crucial for us in our finite element developing of the finite element approximation to the problem.

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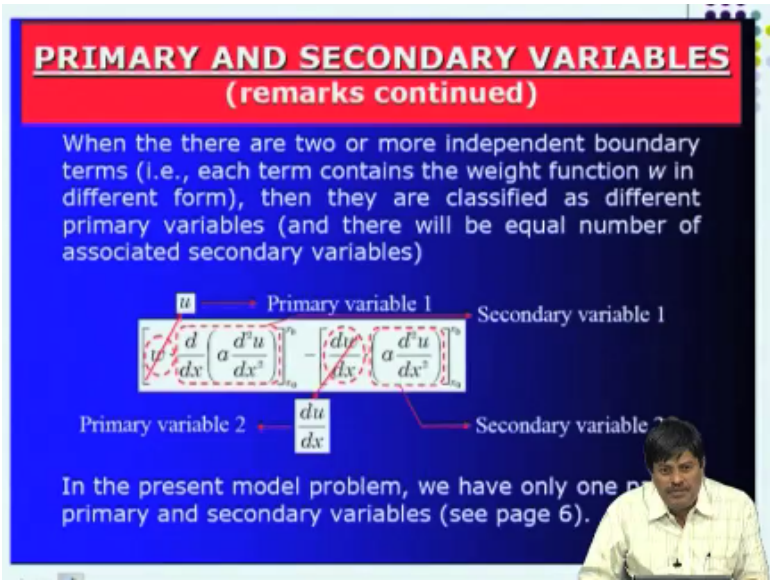

When there are two or more independent boundary terms that is when each term contains a weight function W In different form for example if it is if you are talking about the fourth order system such as beam you would get the boundary terms of this form where you have a primary variable in W And dw/dx and a secondary variable which is second order and third order as shown here then there will be two primary variables and two secondary variables for this problem.

However, in the present model we have only one pair of primary and secondary variable that is U and adu/dx.

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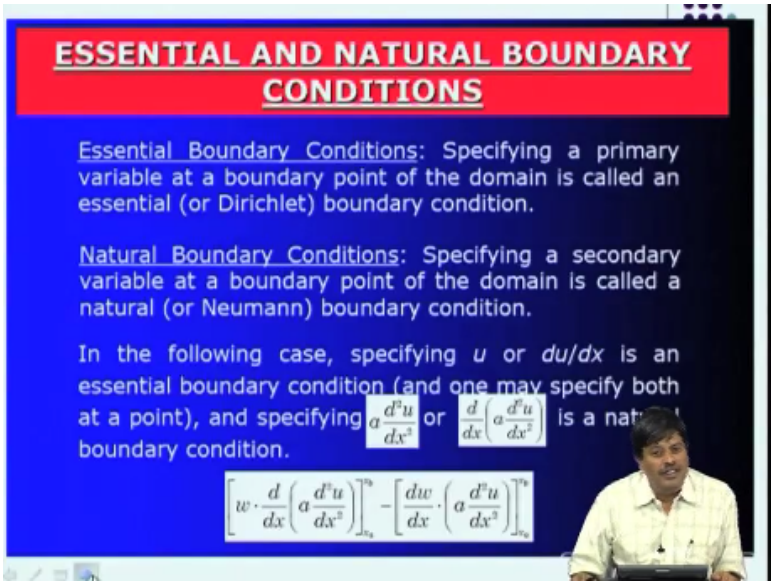

So what are these primary and secondary variable do they have any meaning physical meaning the answer is yes The primary variable essentially prescribes the what is called an Essential boundary conditions in mathematical terms it is called the Dirichlet boundary condition The secondary variable which is like a cause and effect as I said it is caused because of the primary variable which is basically a derivative of the primary variable is called a natural boundary conditions.

Or in mathematical terms it is called the Neumann boundary conditions So in the following specifying U or DU/ DX is an essential boundary condition and specifying D square u / DX square D cube u/ DX cube is the secondary variable, so in the case of beam each has a physical

meaning Ad square U/ DX square says I and D cube U/DX cube is a bending moment and A D Cube U/ DX cube specifies it is a shear force.

So by taking the governing differential equation and weighting with a function and integrating by parts we get a set of boundary terms and these boundary terms has physical meaning and there are essentially the essential boundary conditions or the natural boundary conditions.

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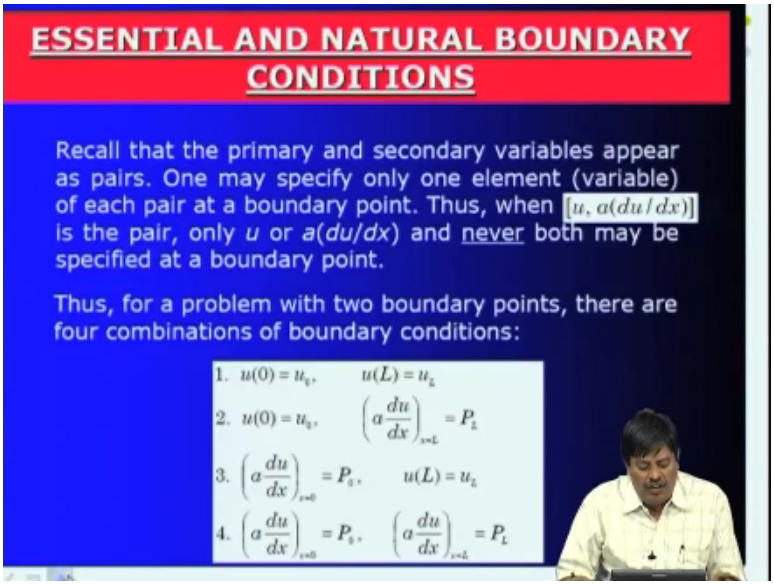

So let us recall that the primary and secondary appear in pairs one may specify only one element variable of each pair at a border point thus we have U and ADU/DX is a pair. So either only U or ADU/ D or never both may be specified that is when U is specified at a particular boundary ADU/ DX cannot be specified or if ADU/DX is specified you cannot be both cannot be specified simultaneously.

So thus far a problem what we have just considered a second order problem with two boundary points there can be four combination of boundary points that is when U is specified at X=0 or u is specified at X=L or U is specified at X=0 and the flux is specified at X=L or the flux is specified at X=0 or U is specified at TL or only flux is specified at both X=0 XM, there are four possible boundary conditions that you can see.

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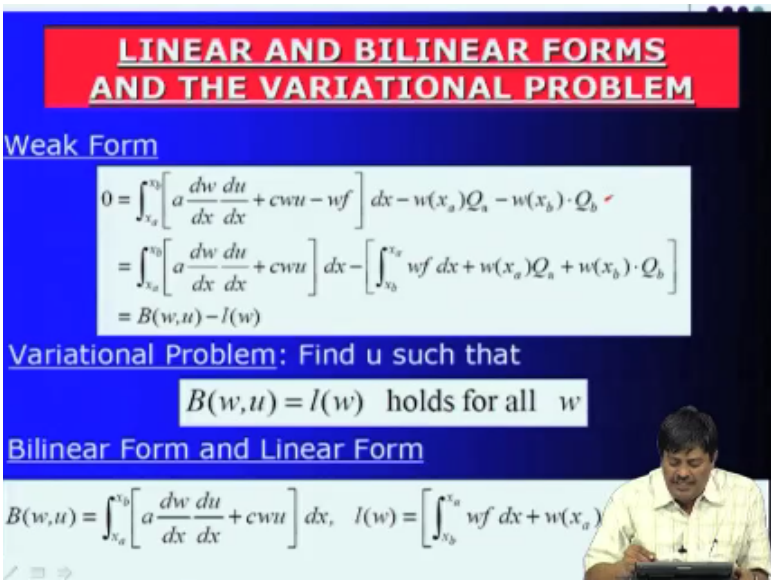

So now this equation is the basically the weak form of the equation and in the calculus of various terms this is basic daily basically called the inner product which is represented by B of W and you it is inner product between W and U and these are the flux terms are the force terms So basically the variational problem is you take the inner product or the wave form of the equation=L of W and that is the variational statement of the problem that we are trying to solve.

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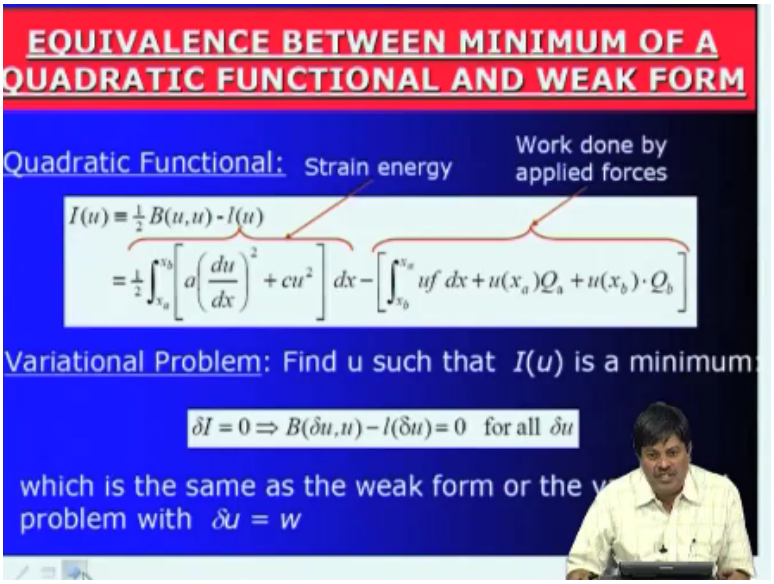

Now we revisit our theorem of minimum potential energy where we wrote a functional which is called the energy functional which is nothing but the inner product of U with U that is when W Is replaced by U we get this is the inner product so the when we minimize this inner product we get

the wave form of the solution So essentially you can convert so we can hear we can see we adopt this for FEM and how this is related to the weighted residual technique.

So in summary we could say that we take a quadratic functional which is in the weighted residual and replace the weight function with the dependent variable we get the quadratic function minimization of the quadratic functional is essentially the problem we are looking for and that is an approach we take it for finite elements.

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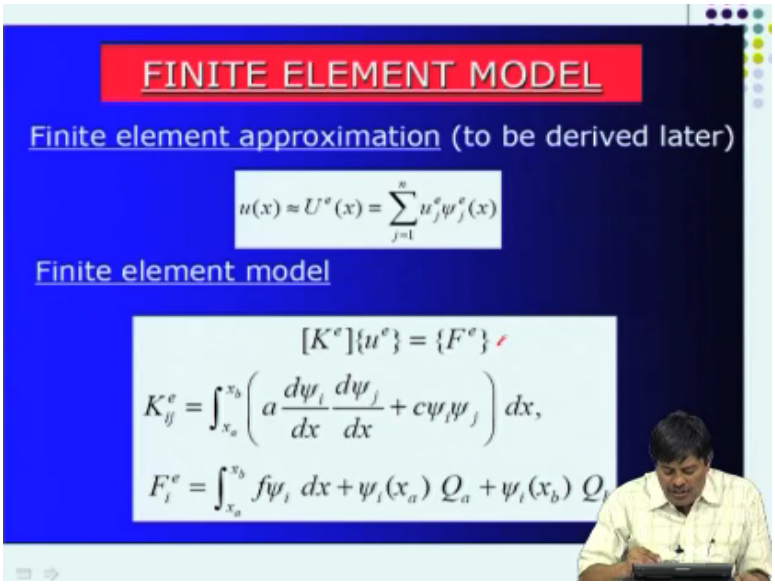

So the finite element model essentially uses this converts this governing differential equation in the algebraic equation we will go through this motion a little later how we can construct these elements like stiffness matrix and the force vector from the governing differential equation as we go along here we are trying to establish only the theoretical basis of it.

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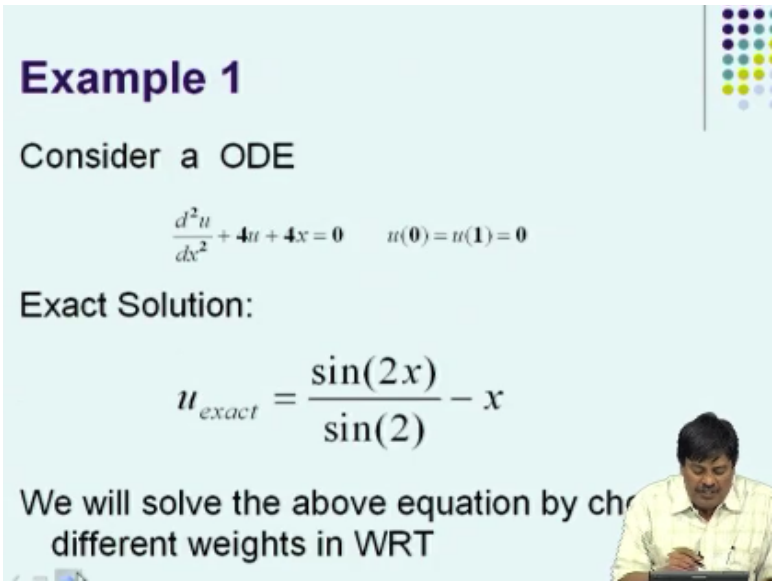

Let us take some examples here So consider a differential equation this is a differential equations with constant co efficient with U as a dependent variable and X as the independent variable it is subjected to two boundary conditions that is U u at X=1 is0 There is no need for an approximate solution here in per say mainly because this exact solution exists and the exact solution is given by this term.

Now what we will do here is we will pursue the approximate solution for this method using the weighted residual technique by choosing different weight and see what kind of different numerical methods we can construct from this weighted residual technique.

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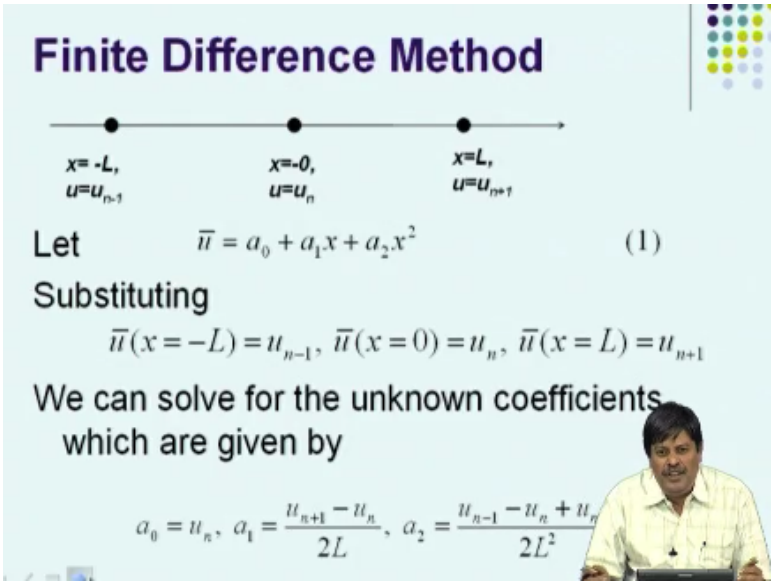

So let us see how we can construct finite difference method which is a very common method used for solving the differential equation So we now choose the first thing that we need to choose is an approximation to the solution so we choose the approximation to the solution as given here the question is why we have you choosing quadratic because there are three point at X=0, X=N/N-1n and N+1 which is located at 0-L and +L.

So the first thing is we need to relate this approximate solution to the nodal displacement are the nodal quantities UN -1 and UN+1 which are located at –L0 and L, so we substitute at X=-L the quantity is un-1 at X=0 it is UN at X=-L it is UN, so when we plug this into this equation 1 we get a relationship between the un coefficient and the unknown co efficient A0 A1 A2 with UN-1 UN and UN+1. So that is your A0 will be UN A1 will be given by this and A2 will be this so we plug it back here.

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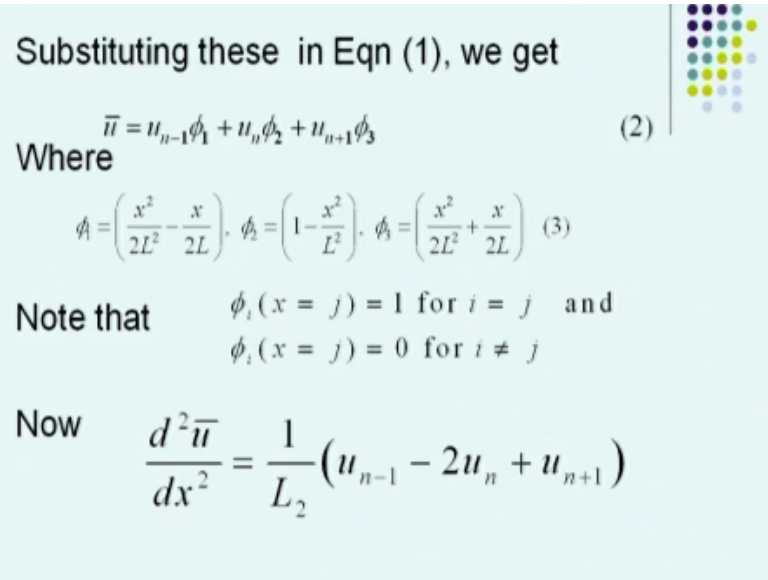

And we can write the U component has some functions of phi *UI which is UN-1*phi1 UN*phi2 and UN+3phi3 where phi1, phi2, phi3 are given by these expressions now let us look at the property of this when we substitute X=-L we see that phi2 and phi3 goes to 0 but phi1 exists which is equal to 1 when X=0 phi and phi3 are 0, phi2 =1and when x=L, phi1 and phi2 are 0 and phi3 =1 so it shows some property later we will see in FEM term these are called shape functions.

So now We take the derivatives, the second derivative and the first derivative, the second derivative is given by this which is nothing but the finite difference approximation using central difference so of this is called the central difference the central difference to the problem.

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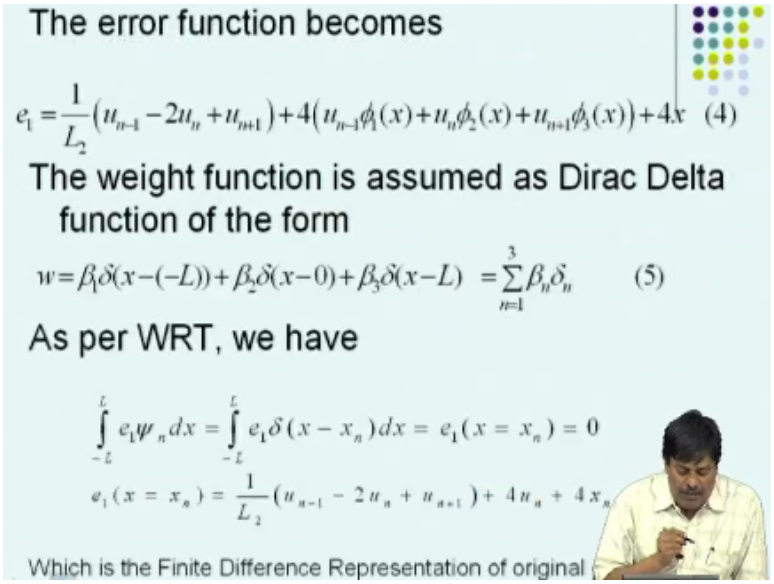

So when we plug this back here we take the error in the function the error in the function is given by this., when we plug this approximate solution to the governing differential equation now we choose direct delta function as the weight function when we choose the direct delta function as the weight function that is we assume the weight functions of this form and each one of this function there are three unknowns are there, UN-1 UN+1.

So there are three weight functions the function should contain three different functions that is three direct delta functions are considered here we weight each one of this with the error function as shown here and we get a set of equations so when we do that the governing differential equations can be reduced to this form this is basically the finite difference approximation for the given differential equation. So we can derive the finite difference method from the weighted residual technique.

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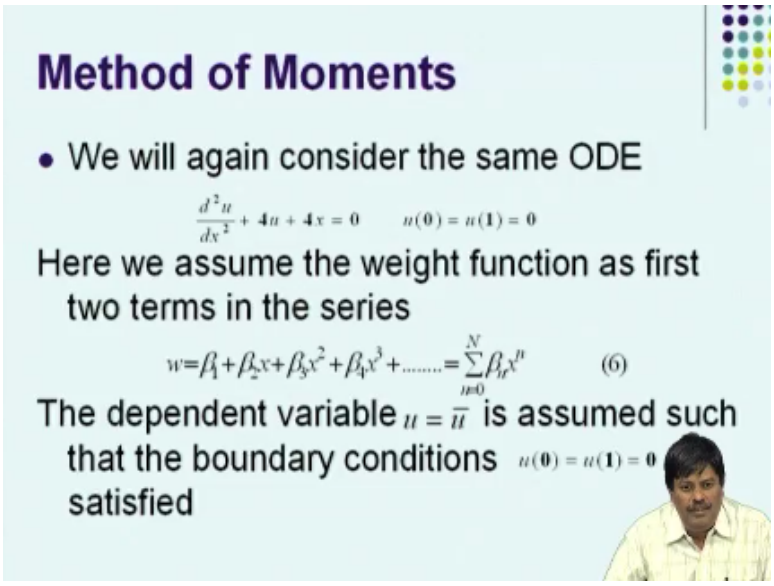

Let us know consider the method of moments again we will take the same equation U for which we have shown that there exist exact solution the same domain U(0) and U(1) varies the domain length is one and the boundaries the dependent variable goes to 0 and now for this we need to choose a weight function and the weight function is taken as the increasing order of polynomial That is beta n*X to the power of n.

And the number n we choose depends upon how many unknowns we choose for the dependent variable U So we assume dependent variable U bar such a way that the boundary conditions are satisfied exactly.

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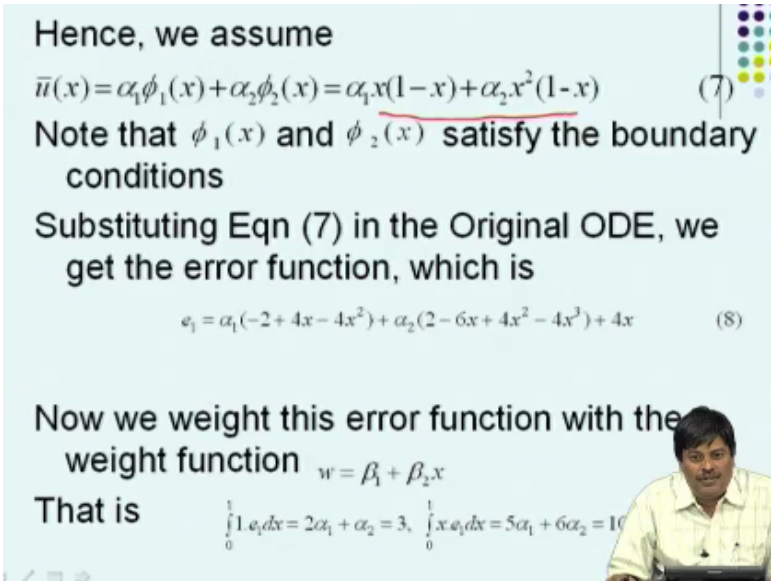

So we use some functions of this form where alpha 1 and alpha 2 are unknowns that needs to be determined from the weighted residual technique and we can see that X*1-X is the first function which goes to 0 both at X=0 and X=1 X square*1-X is again a second function which again goes to 0, at X=O and X=1 now here substitute this equation into this equation and we get a residue is given by this it is a function of alpha, alpha 1, alpha 2 which we need to determine.

So we take this residue and weighted with the first function our first weight function which is 1 we have used this as the weight function So we get this equation when we weighted with the second function X into this error function we get the second function so we have 2 unknowns and 2 equations which can solve alpha1 and alpha 2.

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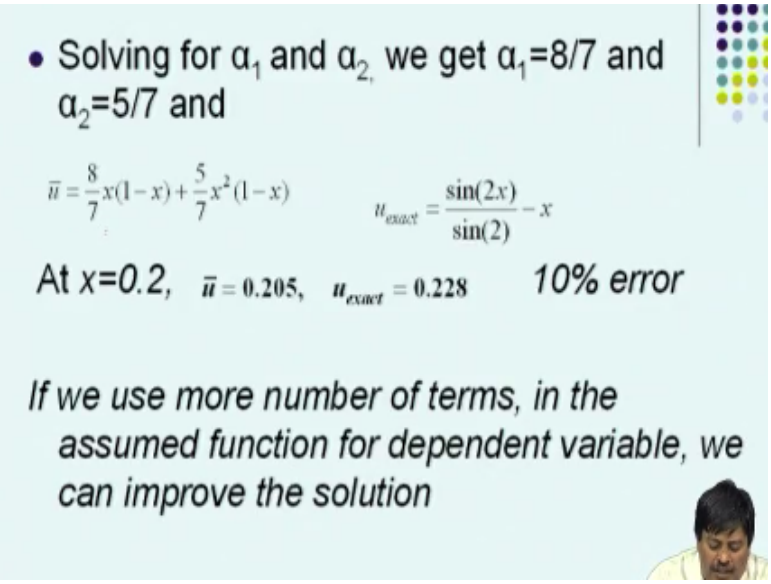

If we do that we get my U bar will be given by this solution the U exact is given by this let us compare how do we compare we have to substitute the value of X at some point in both these solutions and find out what is the value of U, you get at X=0.2 u bar is given by this value 0.205 and u exact is given by 0.228 we find that 10 percent error not a bad thing considering we just consider only two terms.

So what happens if you use more terms suppose we use our functions as U bar=alpha 1 X81-x+alpha 2 X square *1-X+alpha3X cube*1-X obviously this error will decrease further. So if we

use more number of turns in the assumed function for the dependent variable we can improve the solutions considerably.

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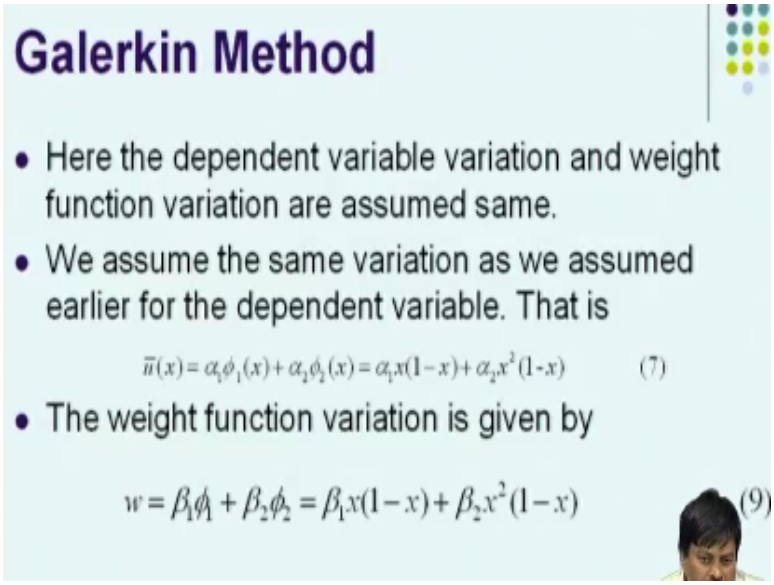

Let us go to the next method that is the Galerkin method, what is the difference as we said earlier that in the Galerkin method the dependent variable and their weight functions are assumed as same So here we assume the same variation as we assume earlier for the dependent variable that is we assume dependent variable of this form and we make sure that this form satisfies the governing equations.

I am sorry satisfies the boundary conditions exactly that is a fundamental requirement it has to satisfy the boundary condition exactly So instead of using a polynomial form of weight function we used the same function with B Instead of alpha we attach beta1 and beta 2 attached to it. So Galerkin method both the weight function and as well as the dependent variation variable variation satisfy the boundary conditions this is not true in other methods.

In other methods, there is no constraint that is placed on the weight function.

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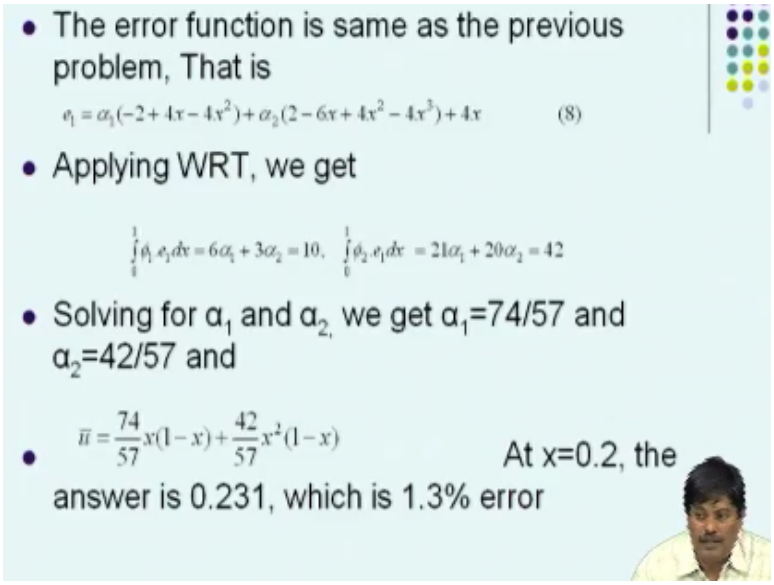

So if we substitute this function we get the error function which is of this form into the governing differential equation now we take the two functions we weight it weight this function with the error function, weight function is exactly the same here as we said the first function is this one so we take X*1-X is my first function phi1 we weight it with this and we get an integrated we get an integrated we get an equation in terms alpha 1 and alpha 2.

We take this second function here and substitute it here phi2 and integrate it we get the second function so when we solve these two functions there are two simultaneous equations we get alpha 1 and alpha 2 are given by this and this solution is given by a so this solution is vastly different from the what we have obtained for the method a moments vastly different So we tried to substitute at the same point X=0.2 and find out what is the answer for U bar.

And we get 0.231 as opposed to 0.228 which is the exact solution so we have the error is completely considerably minimized what is the reason for getting smaller error here as compared to the method of moment the major difference in the method of moments we used a lower order approximation for weights here we use higher order so what we have used essentially is a cubic approximation whereas we just use a quadratic. So it is drastically reduced so that is expected basically.

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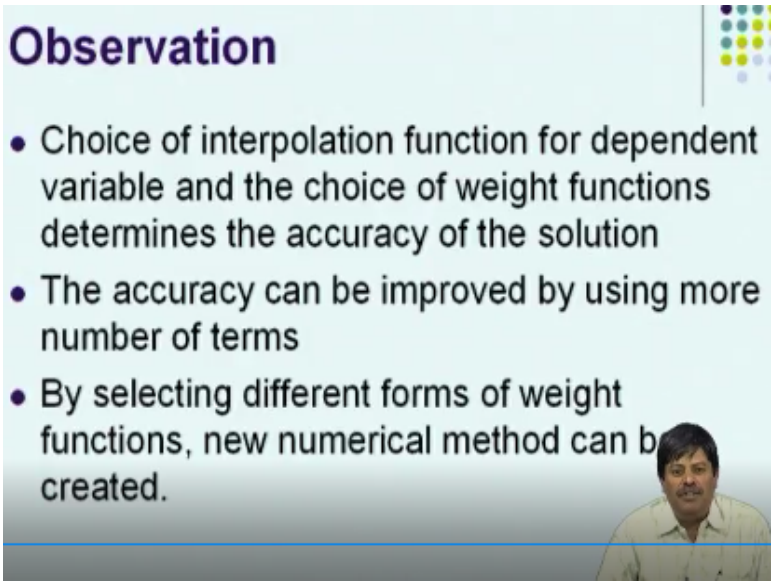

So what are the observations that we see the choice of interpolation function for the dependent variable and the choice of weight functions determines the accuracy of the solution that we saw in the three examples we saw and we also see that the accuracy can be you improve either by using more number of terms or by using higher order approximation as the case may be and the higher approximation depends upon the what is there on the right hand side.

The forcing term is it going to be linear force or a quadratic force or a cubic force etc. and by selecting different forms of weight functions we can construct new numerical method so that is what we have seen here in this lecture.

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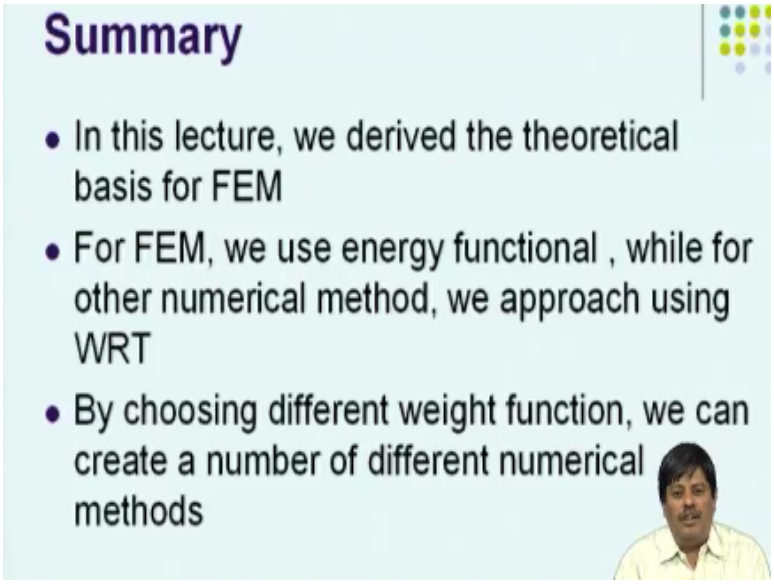

So let us summarize what we understood in this lecture we in the last lecture we said what is FEM without giving it the mathematics and science behind the FEM in this lecture we established the theoretical basis of FEM and the theoretical basis of FEM is essentially a weighted residual technique and we also establish a direct relationship between the weighted residual technique and the FEM, in weighted residual technique, we weight the residue.

We force it to 0 in some form either using collocation method or by using least square method or by using Petrov Galerkin method in the FEM we write the weak form of equation, so the weak form of equation can be constructed by taking the governing equation and weighting it with sum assumed weighting it with the assumed solution the governing equation then integrating over the domain.

And while doing that when we say that the dependent variable is equal to the weight function we get the energy functional or thee weak form of the equation so the minimization of the energy functional is same as turning the weak form of the governing equation through weighted residual method as I said for FEM use energy functional for other numerical method we approach using weighted residual technique.

And by using different weight functions we can create a number of different numerical methods we have seen how we can kind of create the finite difference technique the method a moments technique the Galerkin technique the least square FEM technique etc. and also the collocation technique.

So we have seen the whole range of different methods and many more methods can be created by using different weights. So there is a direct relationship between the weighted residual technique and the finite element technique well the finite element technique uses the minimum potential energy and the weak form from the weighted residual technique goes by rating the way by taking the governing equation.

And weighting t with the sum weighted functions. So here we have established a theoretical basis in the subsequent classes we will talk about how we can create a different elements by taking the physics of the system Thank you.