Micro and Smart Systems
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Lecture - 30
Finite Element for Structures with Piezoelectric Material

This is the lecture number 30 of the Smart and Microsystems course, where we would be studying about the formulation of Finite Element for the smart material structures, and in this case we are considering piezoelectric materials structures. So in the previous lectures we basically started studying about the finite element formulation in general, taking mechanical system into the basic aspects of finite elements for mechanical structures.

And in this lecture we were basically be extending this to smart materials structures and typically although there are many materials as we discussed in our earlier lecture, piezoelectric material structures is considered for demonstrating the finite element method for such structures.

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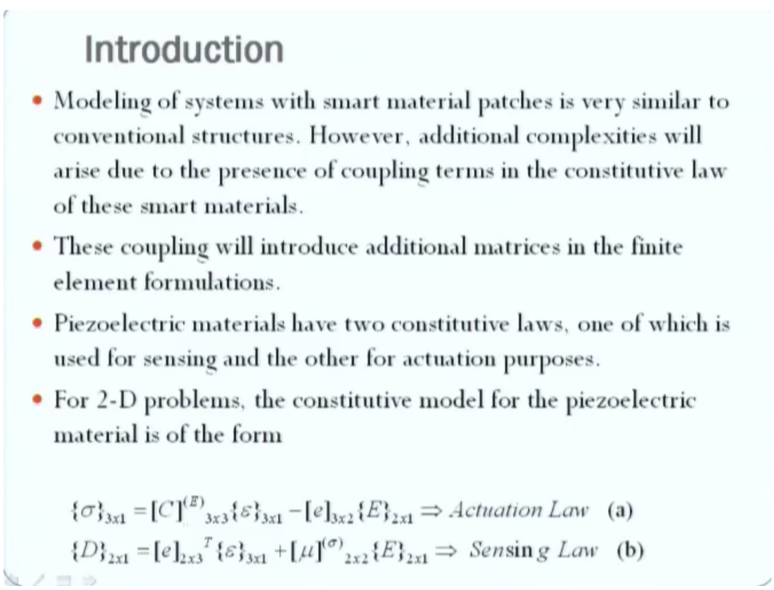

So just to introduce we studied that that in lecture number initial lectures that smart materials typically like the Magnetostrictive material like Tufnell or Piezoelectric material like PZT has 2 constitutive law, one is the sensing law and the actuation law. So it is this 2 laws give rise to in addition to the mechanical component there is also an electrical component the additional complexity arises in the finite element formulation.

So the additional complexity is due to the coupling that couples the mechanical energy to the electrical energy, so what does this mean in the finite element terms, it introduces additional matrices. So here we have given 2 law where the sigma is the stress is related to the mechanical part C is the stiffness which is basically material properties, and epsilon which is the strain.

And the electromechanical coupling coefficient e that is what we call the piezoelectric coefficient which is coupled to the electrical field. And here we have said that the mechanical stress is a 3x1 vector for a 2-D problem that a sigma x, sigma y and tau xy and the 3x3 is the material property matrix and epsilon is 3 strains corresponding to the stress that is epsilon x epsilon y and gamma xy.

And the 3 electric field and the 2 electric field in the 2 coordinate direction in Ex and Ey similarly, the electrical displacement is related to the strain through electromechanical coupling coefficient piezoelectric coefficient and the permittivity of the medium and the electric field. So this is the first is an actuation law, second is the sensing law.

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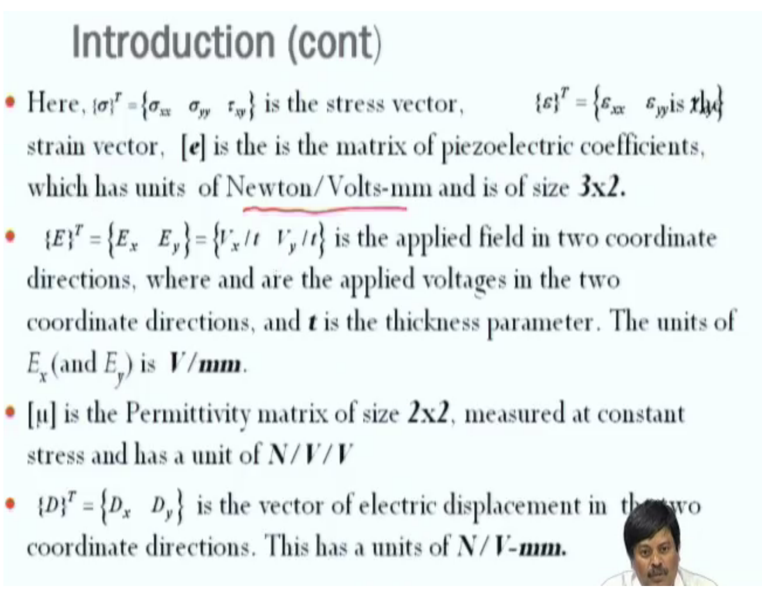

So in the above expression e is essentially the matrix of piezoelectric coefficient which has a unit of Newton per volt millimeter and it is of the size 3x2, and the electric field is basically in the 2 direction Ex and Ey which is related to the voltage, and the thickness of the structure t is the thickness of the structure, Vx and Vy are the voltage in the 2 directions the co-ordinate directions applied voltage.

And the electric field has unit of volt per millimeter because it is V/t t is the thickness in the millimeter, mu is the permittivity matrix of the size 2x2 and measured at constant stress and has a unit of Newton per volt per volt, and these the vector of electrical displacement which has a unit of Newton per volt millimeter.

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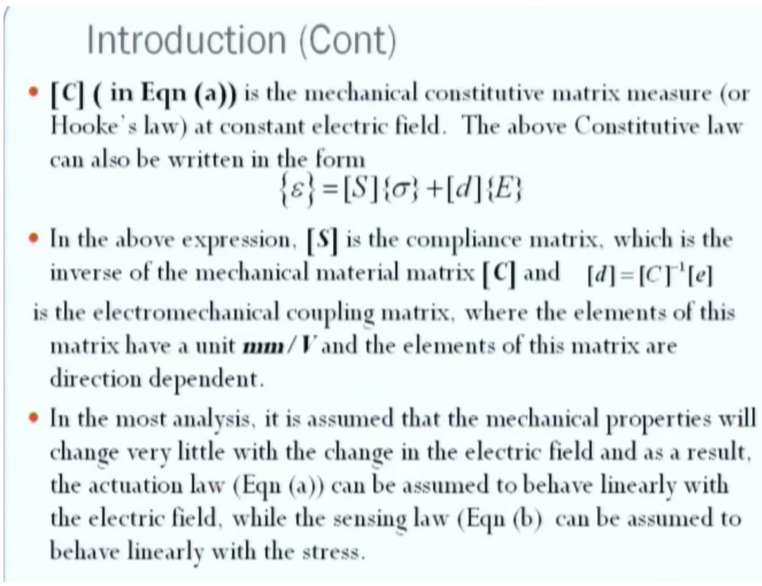

So that C basically is the mechanical constitutive law which we already said what it is in our previous lectures in finite element. And the above constitutive law can be rearranged by taking the epsilon on the left hand side, and S as the compliance matrix which is nothing but C inverse*sigma and d matrix is electromechanical coupling coefficient d, where d=C inverse*the piezoelectric coefficient matrix.

So where the coupling matrix that is d has a unit of millimeter per volt because it is taken an inverse here, so in most of our analysis it is assumed that the mechanical properties will change very little with the electrical field, and as a result we can uncouple both the sensing law and actuation law which is not true in the case of magnetosrictive material it is highly coupled.

But in the case of piezoelectric material it makes our life little simple mainly because this laws can be uncoupled because of the behaviour that the mechanical properties change very little with the change in the electric field.

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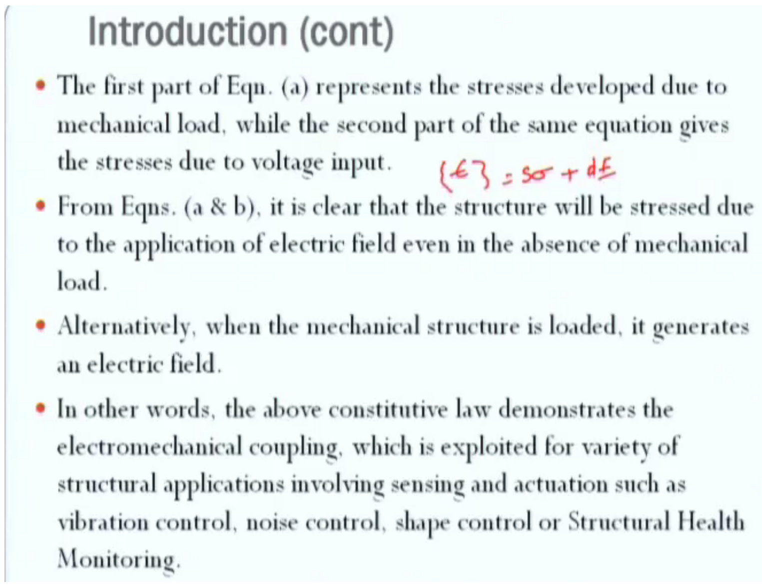

So the first part of the equation that is the actuation equation represents the stress developed due to the mechanical load, while the second part is the same gives stresses due to the voltage input. So we have that epsilon =S sigma+d*E, so the first part is because of the mechanical and the second part is because of the electrical field, so it is clear that the structure will be stressed due to the application.

Suppose we do not have a mechanical loading and if we pass an electric field we see that it is going to be stress or strain and the vice versa, suppose we apply an electric field it will be stressed so it will also elongate. That is what alternatively when the mechanical structure is loaded it generates an electric field, so this is what we call the smart concepts.

So we can actually use these aspects of the additional terms that are coming to our actuator application or sensing application. So I have said that in other words the above constitutive law demonstrates the electromechanical coupling which is exploited for a variety of structural applications involving vibration control noise control and health monitoring.

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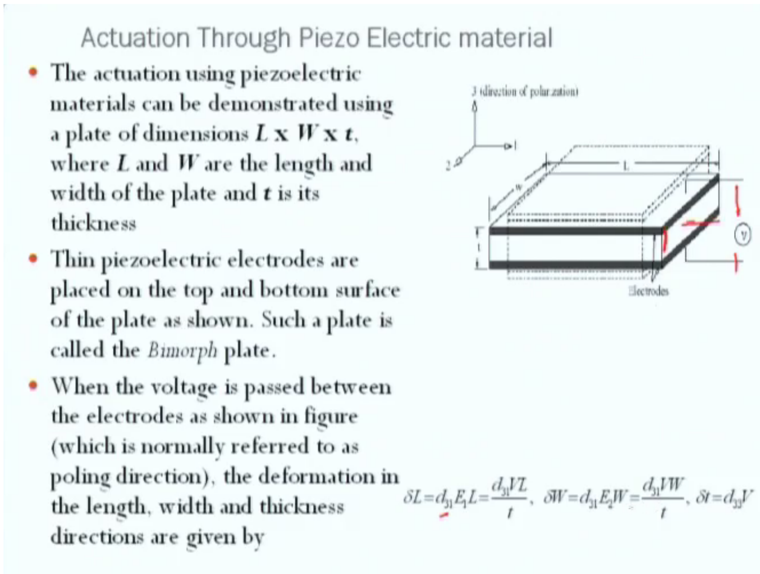

So now let us talk about actuation, so as I said when we pass on the electric field the mechanical structure elongates causing the strain, so in this crystal aspect let us consider the structure shown here which has a length width and the thickness in this direction, and let us consider 2 thin piezoelectric electrodes which are bonded to the top and bottom surface and such a structure is called bimorph structure.

And here taken a bimorph plate of thickness t this is the thickness t having length L, with W. And suppose we pass on an electric field through a system here where we pole it this way, when we pole it basically we will have the strain developed in the direction perpendicular to the poling direction, so this is the direction of poling, so in the current position it is shown we will have a stress in this x direction and also in this direction which are the perpendicular direction to the poling field.

So when we do that so we can actually relate the change in length that is strain to the d31 this is the electro because the 31 3 says that it is in the direction 3 with the applied voltage in the 1 direction, so we have actually poling it in the 3 direction and we are getting the stress in the 1 direction, so that is why we have d31, and E1 is electric field and L is the length.

So this is basically d31 V L/t is basically the strain in the length direction and d31 V W/t in the width direction and in the direction of poling it is d33*V. So we know that if we can actually pole in the appropriate direction and we can actually pole in the direction if you know in what direction we want the stress or strain, we can actually pole the piezoelectric plate in the appropriate direction.

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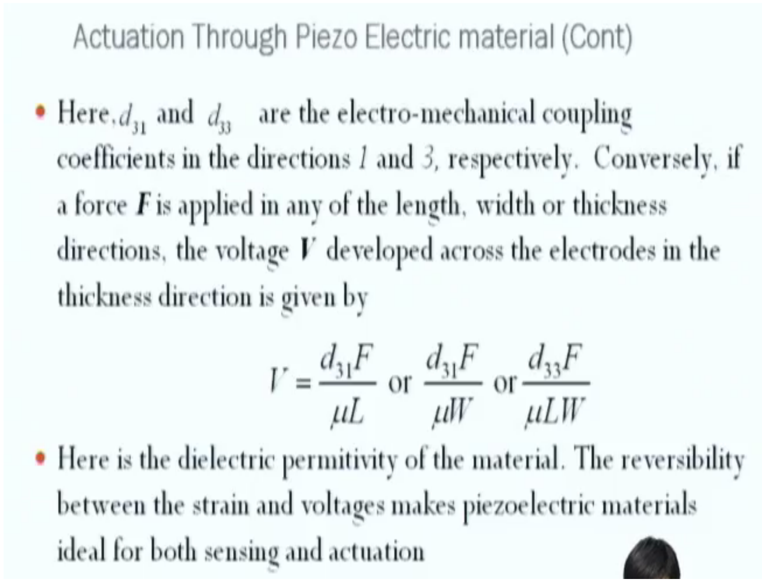

So as I said here d31 and d33 are the electromechanical coupling coefficient in direction 1 and 3 respectively. Conversely, if the force F is applied in any of the length along any of the length width or thickness direction, the voltage V that is applied will be d31 F/mu L that is coming from the second constitutive law where mu is the permittivity of the medium.

So it can be in the x direction it is d31 F/mu L, d31 F/mu W in the W direction and d31 F/mu L W is in the thickness direction. The reversibility between the strain and voltage makes the piezoelectric material an ideal for both sensing and actuation, so we will actually see how we can do as we go along in this lecture.

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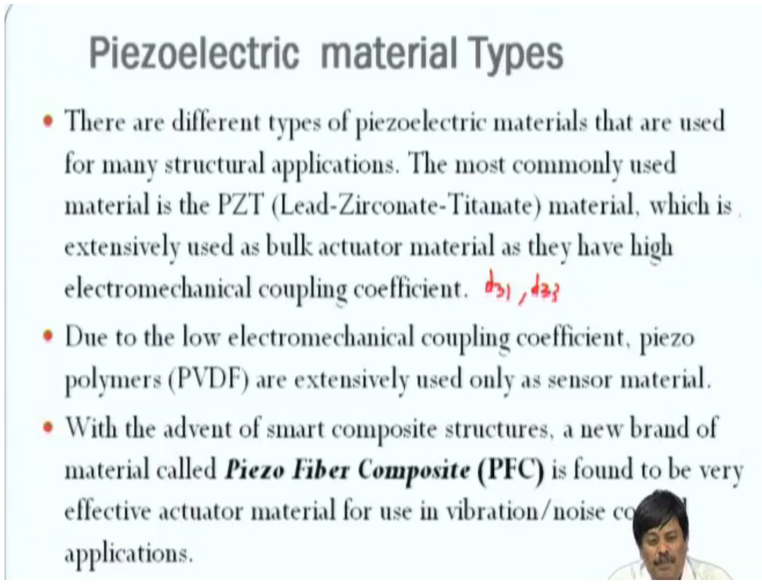

So what are the piezoelectric types we did discuss this in our earlier lecture, there is our lecture number 2 where we said there are different kinds of piezoelectric material and it is available in different forms either in the ceramic form or in the crystal form or in the polymer form. The most commonly used is the ceramic form that is PZT.

Where PZT stands for Lead Zirconate and Titanate material, which is extensively used as a bulk actuator material as they have high electromechanical coupling coefficients, here we are talking about d31 d33, so the material which has very high d31 and d33 we call it as a very good actuator material.

However, the polymer form of the piezoelectric material which is called the PVDF that PVDF is polyvinyl difluoride which has a very low d31 and hence extensively used as a sensor material it cannot be used as the actuator material. And we also talked about the other forms of the piezo material it is in the form of fiber composite, the Piezo Fiber Composites basically it is used extensively as an actuator material.

Because it can be embedded into the composite structure and it is found that it is very useful to do vibration and noise control. Here we would not be talking about PFC here but we will see how the same constitutive law can be used for PVDF and PZT material and how it can be used for sensing and actuation application.

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Finite Element Modeling

- Finite element modeling of the mechanical part is very similar to what was discussed in the previously except that the coupling terms introduce additional energy terms in the variational statements, which results in additional coupling matrices in the FE formulation.
- Introduction of Piezoelectric material, introduces an additional degree of freedom in the FE formulation. This additional DOF can be the electrical potential (Normally referred to as Φ, which is related to the Electrical field vector $\vec{E} = E_x i + E_y j + E_z k = \nabla\Phi$) or electrical field itself.
- Alternatively, the analysis can be performed by using conventional beam, or plane stress element derived earlier and the effects of coupling terms can be translated as equivalent concentrated nodal loads. Such an approach is possible when we assume that the sensing and actuation law are not coupled, which is mostly true in piezoelectric material. We will demonstrate both approaches in this lecture

Now coming back to the finite element modeling, see the finite element modeling of the mechanical part which is epsilon =S times sigma is very similar to what we discussed in the previous lecture, except that the coupling terms introduce additional energy terms in the weak form of the governing equation which would result in the coupling matrices in the FE formulation, so what does it mean in the finite element modeling.

So the introduction of piezoelectric material introduces an additional degree of freedom which is in the form of the electric field E, this additional degree of freedom can be basically can be an electric potential phi which is related to electric field E by taking the gradient of the potential phi or we can take the electric field itself either way it is fine.

Alternatively, since the sensing and actuation law are not coupled analysis can be performed by using conventional beam type elements or plane stress type elements or even plate element derived earlier and the effects of coupling terms are translated into equivalent concentrated loads on to the beam structure and the conventional beam element which we actually use can be a beam or plate element or plane stress elements can be used to actually solve both sensing and actuation problem.

We will deal with these 2 and see how each of them can be used in this lecture.

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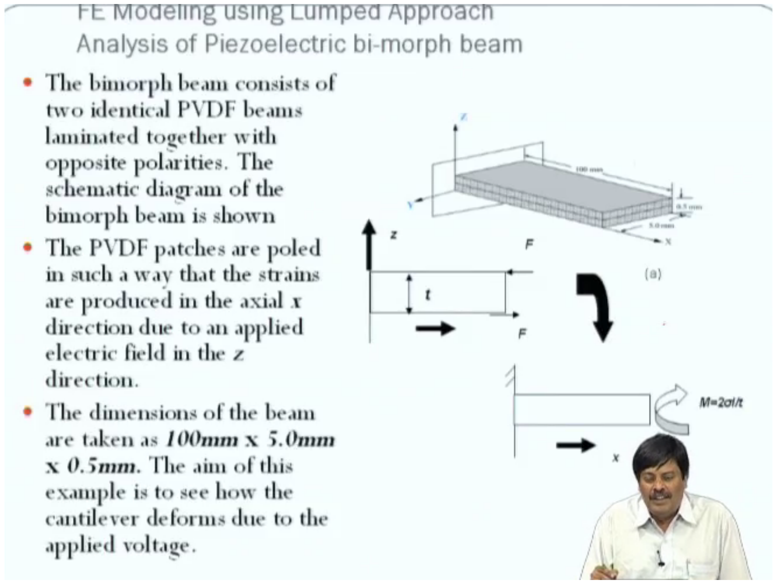

So the first approach we call it as the lumped approach, where we have doing an FE analysis of a piezoelectric bimorph beam, so here there is bimorph beam where there is 2 layers one is the top layer and the bottom layer are stitched together, and we have taken 100mm x 5 mm x 0.5 mm the width is 5 mm.

So here the 2 bimorph beams are stitched together and we consider this bimorph material is made up of polymer type of piezo material that is a PVDF beams with opposite polarities, so the PVDF patches poled in such a way that the strains are produced in the axial direction that is we poled in the wiser plane that is in this direction, so that we get the stress in x direction due to the applied field in the z direction.

So we get the field in the z direction, so we get the poling in z direction will introduce a strain in the x direction, so these are the dimensions, so what does it mean? If I use a lumped approach so this is going to cause a force F at this end and a force F in this end, which is basically equal to a couple, so we can actually calculate this force from the piezoelectric constitutive law and apply a couple to the cantilever beam, and then study the response using our beam element, let us see how we can do in the both approaches.

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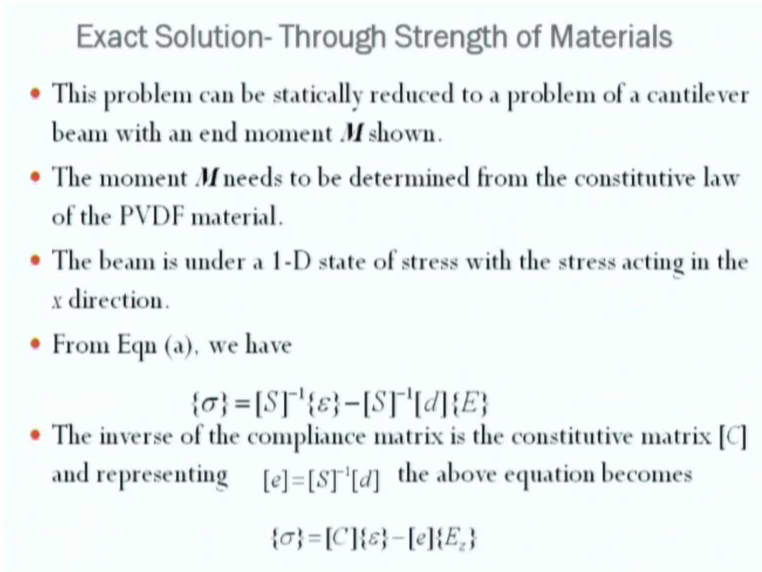

So before doing that we can actually derive the exact solution to this using the strength of materials approach, so as I said that the problem can be statically reduced to a problem of cantilever beam with an end moment M as shown in the figure. The moment M needs to be determined from the constitutive law of the PVDF material. So the beam is under the 1-D state of stress with the stress acting in the x direction.

So we have simplified the problem considerably, so that we can actually go ahead and do a simple exact solution and we will actually check this exact solution with our beam element formulation, so from the constitutive that is the actuator law we have sigma =S times sigma-S inverse d*E, so we can write the constitutive law into this form where E is the piezoelectric coefficient which we have derived it earlier.

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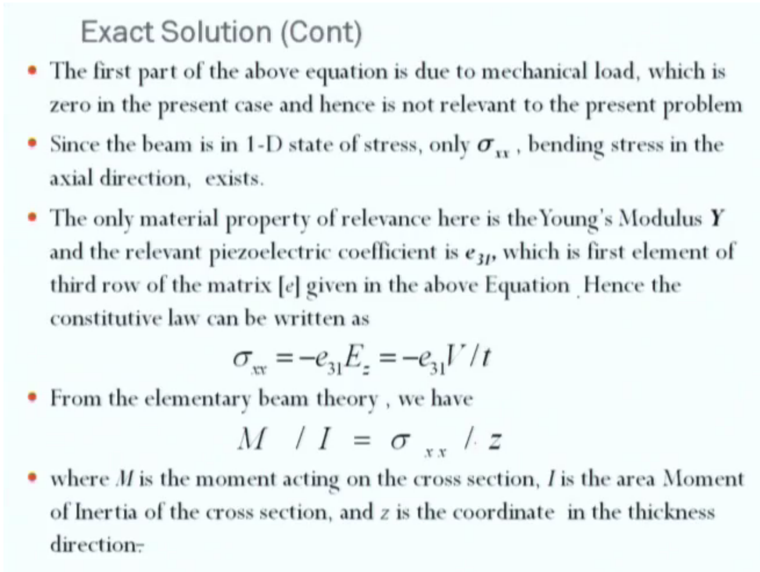

So the first part of the above is due to the mechanical part, which is 0 in the present case because there is no mechanical loading we have applied, we have applied only purely the electrical loading which is converted into equivalent that is equivalent moment. And hence the first part is not relevant to the present problem, since the beam is in 1-D state of stress only sigma xx has and the bending stress in the axial direction exists.

The only material properties here of relevant to the Young's modulus y, and the relevant piezoelectric coefficient is e31 that is the piezoelectric coefficient because it is in the direction 1 due to the field in the direction 3 that is the z direction, which is the first element of the third row of the matrix e. Hence the constitutive law here becomes sigma xx =e31*Ez or e31*V/t V is the applied field in the z direction.

From the elementary beam theory we said that M/I =sigma xx/z, z is the thickness co-ordinate, so where M is the moment being acting on the cross section and we have found the moment, we need to find this moment to be lumped on to the to be use in the equation, and I is area moment of inertia of the cross section and as I said z is the thickness coordinate.

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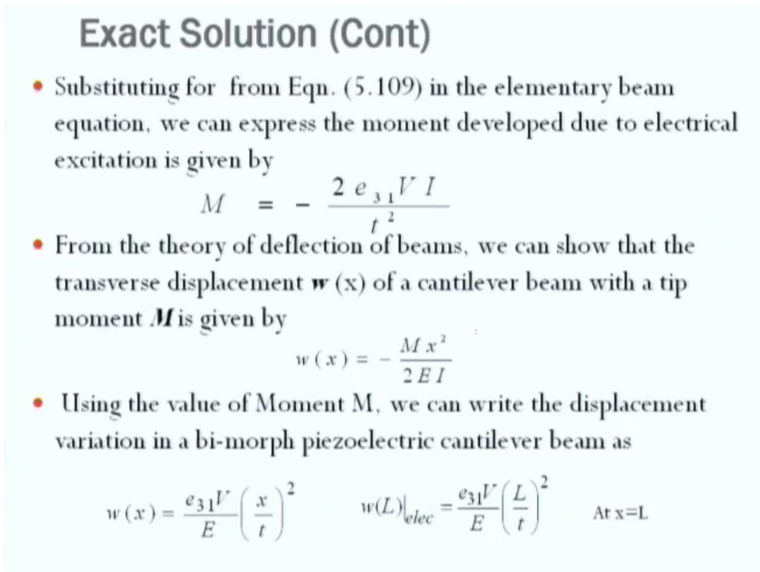

So substituting this into the equation we get basically we take this the moment we substitute sigma x here so the M/I =V*e31/ t z and we get and at the midpoint of the thickness at the 2 top ends z ranges from –t/2 to +t/2, when we substitute this the moment becomes 2*e31*V I/t square.

So from the theory of deflection of beams we can show that the transverse displacement w of a cantilever beam is given by w M due to a cantilever is M x square/2 E I with M substituted from e31 coming from the piezoelectric constitutive law, so when we actually plug this in equation we get that the equation for a bimorph piezoelectric PVDF beam is given by this equation. So we can substitute x at any point and we can get the deflection due to this effect of piezoelectricity.

So supposed we take that at x=L that is at the tip e31 V/E*L/t whole square, we get that so we can basically see how this whole thing when we increase the voltage for a given material there is the deflection increases or when we decrease the thickness or increase the length the deflection increases.

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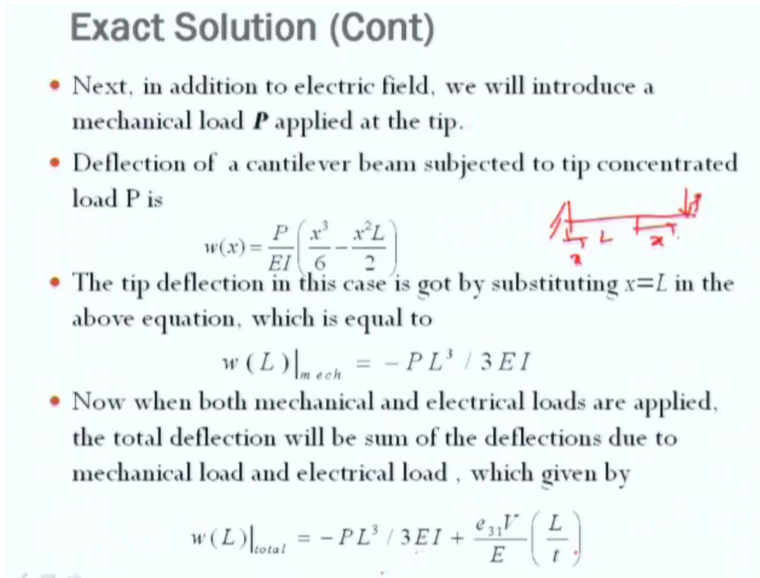

Next, what we do is we will add in addition to electric field, we will also introduce a mechanical load P, so when we do that now we have 2 problems, one is due to mechanical load, one is due to electrical field, we have already found out what is the deflection due to electric field. Now we will take the mechanical load and we will take we will solve this problem of cantilever beam with a mechanical load L.

And we know that this can be easily solved by any of the deflection methods are used in strength of materials and that exact deflection at any point x from here can be got by x cubed/6*-x squared L/2 multiplied by P/E I, and when we substitute at x=L, so we have taken x= here, so then x=L we get the deflection is P L cube/3 E I. Now the total deflection at the tip is due to mechanical load which is P L cube/3 E I and due to the electrical.

So we see that basically the electrical deflection opposes the mechanical deflection, so when we increase the voltage we it will be increasing it will decrease the total deflection to such a level that we can actually choose a voltage which will make the total deflection 0, and that is precisely the actuation that we are talking about.

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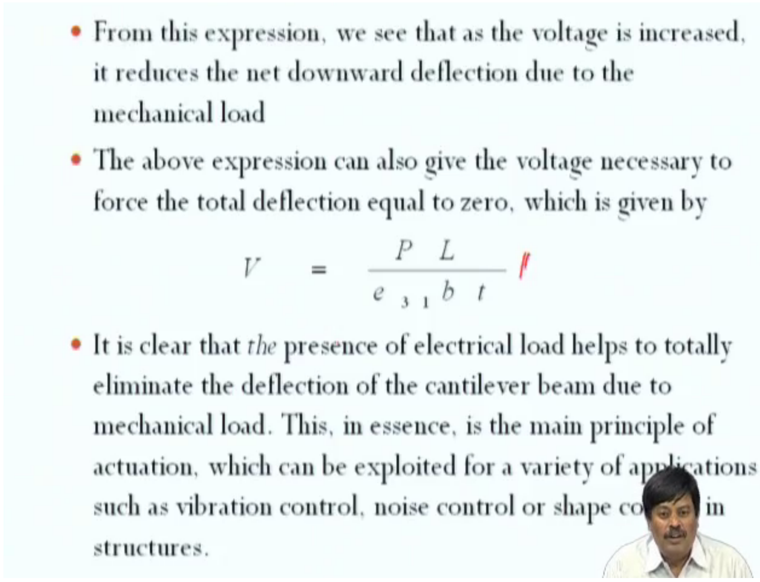

So from this expression we see that the voltage is increased, it reduces the net deflection due to mechanical load, so we can actually get an expression that what is the voltage that is necessary to make the deflection 0, so basically we can equate this term equal to 0 and we can get what is V that makes the total deflection equal to 0 and that is given by here.

So we can actually make this equal to 0 by using this value of voltage. So it is very clear that the presence of electrical load helps totally eliminate the deflection in a cantilever beam due to the mechanical load, this is essence is the main principle of actuation which can be exploited to a variety of application as I talked before.

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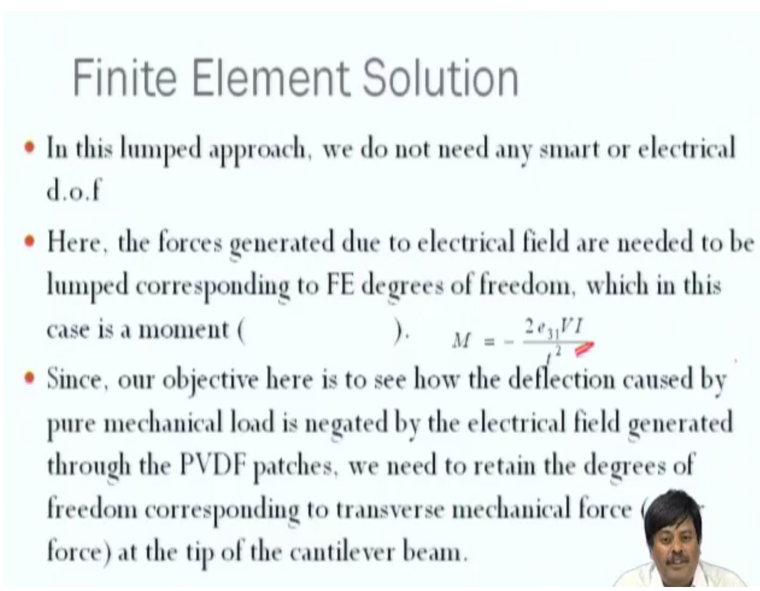

Now let us do the finite element solution, so as I said here in this lumped approach we need not use any smart or electrical degree of freedom. Here, the forces are generated due to electric field are needed to be lumped onto the corresponding finite element degrees of freedom, which in this case is the moment M which is given by this expression.

Since, our objective here is to see how the deflection is caused by a pure mechanical load is negated by the electric field generated through PVDF patches, we need to retain the degree of freedom corresponding to the transverse mechanical force at the tip of the cantilever beam.

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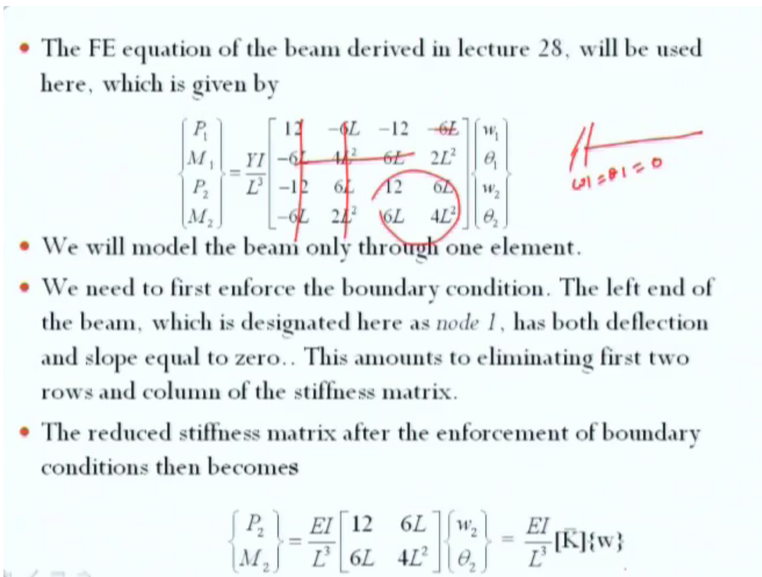

So we go back to the stiffness matrix which we derived in the lecture 28 which will be use here, which is basically Y I/L cube Y is the basically the Young's modulus given by this matrix related to the w1 is the transverse degree of freedom of theta 1 is the slope at the node 1, w2 is a transverse at node 2, and theta 2 is the rotation at node 2. So we will model this beam only through one element here, and we need first to enforce the boundary conditions.

Since, it is a cantilever beam we are talking about, so you have w1=theta 1=0 here, so we apply this and eliminate these 2 rows and columns, and we will have only these the reduced stiffness matrix will contain only these terms here, so which is written here P2 and M2 is related to w2 and theta 2 by here, and we know M2 which is got by our reducing the electrical load into equivalent moment which can be applied here.

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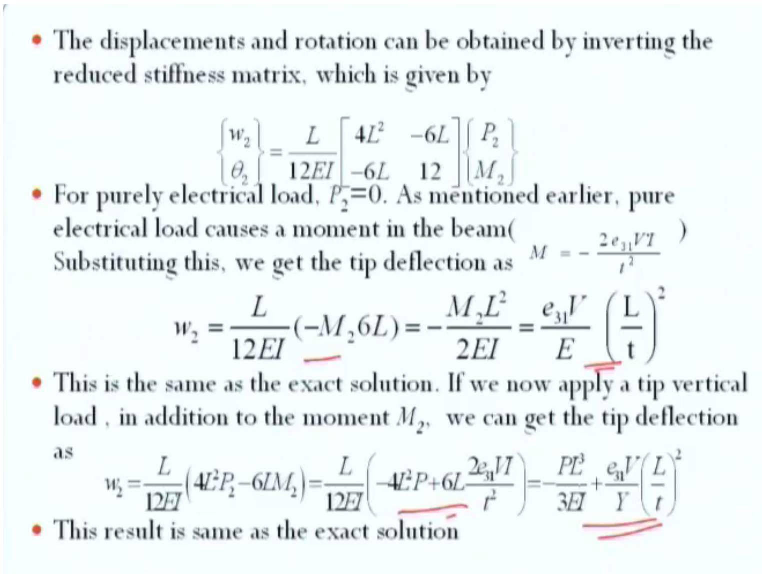

So we take the inverse of it w2 theta 2 is the inverse of this matrix which is given here, and we substitute M2=2 e31 V I/t square and from which we can compute the transverse displacement which will be equal to this expression, and when we substitute for a moment we get this, which is same as what we got from the exact solution. So because the stiffness matrix for beam is exact we are able to get the exactly reproduced the sensor material solution here.

And now if you apply a vertical load P here at the transverse degrees of freedom, so we also get that the total solutions can be so here this will not be 0 in the first case it is 0, but in this case we have a P here, and when we get the total solution is given by this when we put this equation we get this, and which is same as what we got with sensor material solution.

And again we can equate this to find out what is the total voltage required to make the total reflection 0. So what we have seen here is what we derived from the exact solution we are able to reproduce exactly with the finite element solution.

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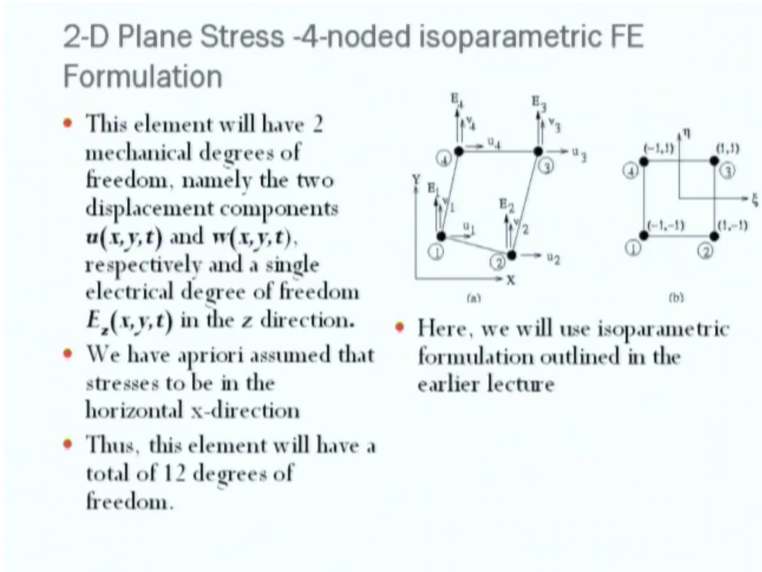

Next, let us see whether we can actually do a much sophisticated element formulation that is in terms of the 2 dimensional 4 noded isoparametric finite element formulation. We discussed isoparametric finite element formulation in lecture number 29, where we outlined the procedures that are required that is to recap, we require both the variation of the dependent variable that is deformation u and v and also the variation of the coordinates.

In addition now because we have a smart degrees of freedom, we are not assuming anything with regard to the uncoupling of the sensing and actuation law, we take it as it is we introduce also a smart degree of freedom in to the formulation, and this smart degree of freedom will be only in the that direction so we use this.

So the general degree of freedom may shown here, so you have at each node it can take 3 degrees of freedom that is the axial deformation u, the transverse deformation v and the electrical displacement E, and this is mapped onto the rectangular isoparametric co-ordinate system psi and eta, and using a coordinate transformation, so the psi and eta which is mapped onto a square of unit 2, so totally this element will have 12 degrees of freedom.

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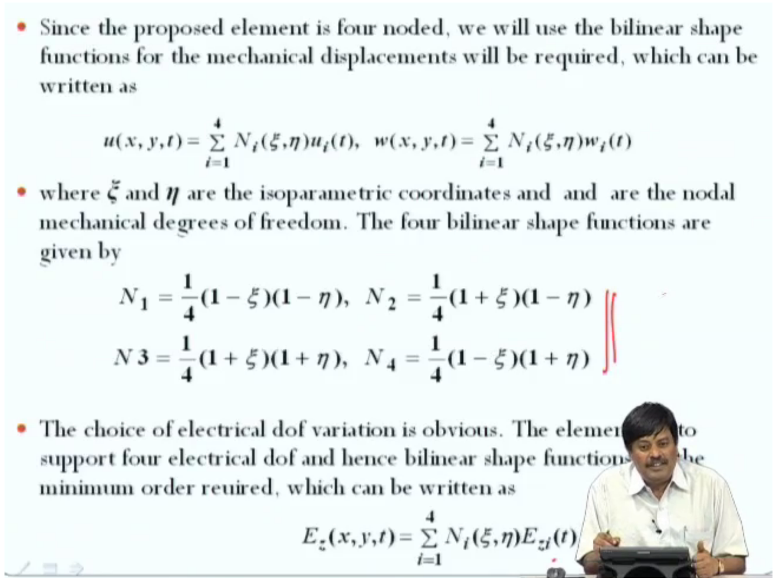

So now we need to assume the variation for the 2 mechanical degrees of freedom that is u and v and u and w here and the electrical degrees of freedom, the mechanical degrees of freedom is assume exactly the way we did for the conventional mechanical structure, so we take it in the form of Ni ui, w=Ni wi where ui and wi are the nodal coordinates of the given element, and psi and eta are the isoparametric co-ordinate system as I said.

And we write the shape functions in the form of in the isoparametric co-ordinate system and these functions which are given here are derived in the lecture number 29. So now the question is how do we actually interpolate electrical field, which we have also introduced as additional degrees of freedom.

The electrical degrees of freedom is exactly assumed as the variation is assumed exactly same as that of the mechanical degrees of freedom, that is we take that Ez or the E as a function of psi and eta Ni*Ei where Ni is again given by these 4 degrees of freedom, we have 4 E's at the 4 nodes, so that means it will have 4 shape functions.

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- Here, the same shape functions used for mechanical dof is also used here, and E_{zi} are the nodal electrical degrees of freedom at the four nodes.
- In isoparametric formulation, we map the actual geometry of the element to a square of size 2 defined in the generalized coordinate system through a Jacobian transformation
- This requires the variation of the coordinate system in the generalized coordinates in terms of the nodal coordinates of the actual element geometry. Hence, one can use the same displacement shape functions to describe this variation and can be written as

$$x(x,y)=\sum_{i=1}^{4} N_i(\xi,\eta)x_i, \quad z(x,y)=\sum_{i=1}^{4} N_i(\xi,\eta)z_i$$

Here, the same shape functions is used for mechanical degree of freedom as well as the electrical degree of freedom. The isoparametric formulation as I said we map the actual geometry to a square of size 2 using the generalized co-ordinate system through a Jacobian transformation. So for which we need to assume the variation of the coordinates and the coordinates are assumed exactly as that of the mechanical degrees of freedom and the electrical degrees of freedom.

Because it is an isoparametric formulation where the same number of nodes that participates in the deformation will also participate in the co-ordinate transformation.

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- Jacobian is computed using the chain rule . This was explained in lecture 29.
- The strains are evaluated by using strain-displacement relationship. That is,

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{xx} \\ 2\varepsilon_{xz} \\ E_z \end{Bmatrix} = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial z & 0 \\ \partial/\partial z & \partial/\partial x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u \\ w \\ D_z \end{Bmatrix}$$

- Using the assumed variation of field variables in the above equation enables expressing the strains in terms of nodal displacement vector with $\{u\}_e=\{u_1 \quad w_1 \quad u_2 \quad w_2 \quad u_3 \quad w_3 \quad u_4 \quad w_4\}^T$

and electric field vector $\{E_z\}_e=\{E_{z1} \quad E_{z2} \quad E_{z3} \quad E_{z3}\}^T$

So the Jacobian is computed using a chain rule, I am not going into this, this was explained in lecture number 29 because the Jacobian relate makes the coordinates the derivative with respect to x is related to derivatives with respect to psi and eta. So once we have established this relation then we are in a position to evaluate the strain displacement matrix.

And the strain displacement matrix in addition to the mechanical degrees of freedom we also introduced the electrical degrees of freedom which has no strain except that it is directly related to the electrical displacement, so this aspect becomes the strain displacement matrix, so now using the assumed variation of field variables in the field variables here are the displacement u w and the electrical field E.

We write the element vector displacement vector as u1 w1 u2 w2 u3 w3 u4 w4 that is the mechanical vector and the electrical vector is E1 E2 E3 E4 are 4 nodes.

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- That is strain can be written as

$$\{\varepsilon\} = [B]\{u\} = \begin{bmatrix} [B]_{u(3X8)} & 0 \\ 0 & [B]_{E(1X4)} \end{bmatrix}$$

where $[\boldsymbol{B}]$ matrix, is given by

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} & 0 & \frac{\partial N_4}{\partial z} & 0 & 0 & 0 & 0 \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial z} & \frac{\partial N_4}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix}$$

So now the strain displacement matrix after it is computed will be of this form B*u and it has 2 explicit components one coming from the mechanical degrees of freedom that is u and w, and one coming from the electrical degrees of freedom E, and the mechanical degree of freedom size is 3x8 corresponding to 8 degrees of freedom and the 3 constitutive, and the electrical degrees of freedom has only one constitutive law and it has 4 degrees of freedom.

And the expanded form of this B matrix is given by this equation here, where the B matrix will contain only the shape function matrix corresponding to the electrical degrees of freedom, and this part is very similar to that what we derived in the earlier lectures. So we just directly put it here and these are filled with 0's, and only this part corresponding to the electrical degrees of freedom gets added to the strain displacement matrix.

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- The weak form of the governing equation for this problem is given by taking the variation of the total energy, which is given by

$$\delta\left(\begin{array}{l}\frac{1}{2}\int_{t_1}^{t2}\int_V \{\dot{u}\}^T \rho\{\dot{u}\}dV dt + \frac{1}{2}\int_{t_1}^{t2}\int_V \{\sigma\}^T\{\varepsilon\}dV dt + \frac{1}{2}\int_{t_1}^{t2}\int_V E_z D_z dV dt + \\ \int_{t_1}^{t2}\{u\}^T\{F_c\}dt + \int_{t_1}^{t2}\int_{S_1}\{u\}^T\{F_s\}dS_1 dt + \int_{t_1}^{t2}\int_{S_2} E_z D_s dS_2 dt\end{array}\right)$$

where S_1 and S_2 are the surfaces in the structure where the surface forces, and residual displacements act. Using the constitutive model, we can re-write the weak form of the governing equation as

$$\int_{t_1}^{t_2}\int_V \{\delta u\}^T \rho\{\dot{u}\}dV dt + \int_{t_1}^{t_2}\int_V \{\delta\varepsilon\}^T[\bar{\bar{C}}]\{\varepsilon\}dV dt - \int_{t_1}^{t_2}\int_V \delta E[\bar{\bar{e}}]^T\{\varepsilon\}dV dt$$
$$+\int_{t_1}^{t_2}\int_V \delta E_z[\bar{\bar{e}}]^T\{\varepsilon\}dV dt + \int_{t_1}^{t_2}\int_V \delta E_z \mu E_z dV dt + \int_{t_1}^{t2}\{\delta u\}^T\{F_c\}dt$$
$$+\int_{t_1}^{t2}\int_{S_1}\{\delta u\}^T\{F_s\}dS_1 dt + \int_{t_1}^{t2}\int_{S_2}\delta E_z D_s dS_2 dt = 0$$

The next part is because there are additional complexities that has arisen due to the coupling, we need to rewrite the weak form of the governing equation and the weak form of the governing equation can be got from the Hamilton's principle, which we actually derived in the earlier lectures. So here we take the weak form and minimize it to get the finite element matrix, just similar to our principle of minimum potential energy.

So the first part is the inertial part due to mechanical degrees of freedom that is coming from the kinetic energy of the system, this is the strain energy of the system, this is the energy due to, this is additional energy that comes into picture because of the piezoelectric material, that is coming because of the electrical displacement, and this is the force vector due to concentrated load, this is due to the surface traction and this is the additional force that is coming here.

So here S1 and S2 are the 2 surfaces where the surface forces and the residual displacement act. Now using the constitutive model we can rewrite the weak form of the equations in this form by taking the minimum variation, and here this is coming because of the constitutive model, we are not deriving it get here mainly because it is too complex, but assume that this can be done by taking care of the embedded system as we do it for laminated composite.

So these are the total matrices that are going to come into play because of the constitutive matrix the weak form of the governing equation.

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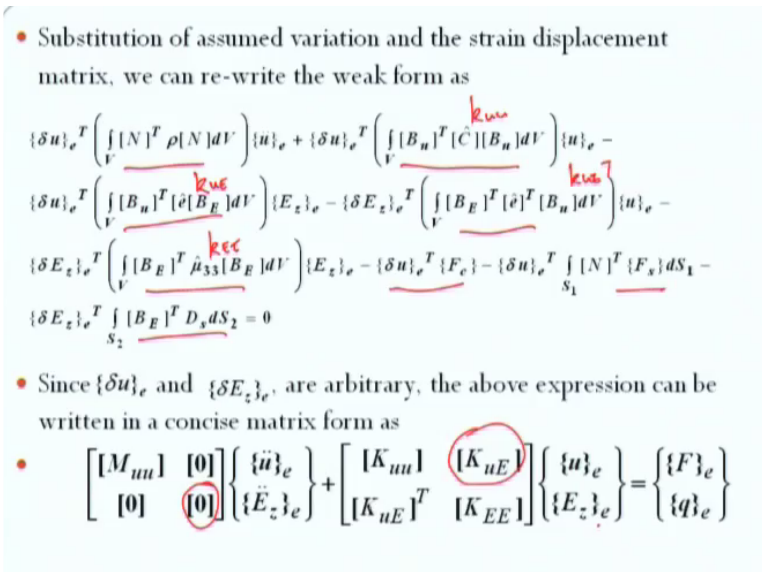

Once we group these things together we can write we can now in this situation we can substitute the variation u=Ni ui, then we have w=Ni wi and E=Ni*Ei, when we substitute this into this equation and write it in matrix form, this is the mass matrix, this is the matrix corresponding to mechanical degrees of freedom, which we call it as Kuu, this is due to coupling between mechanical and electrical which you call it as KuE, this is due to again kuE transpose

And this is due to purely electrical field which we call it as KEE. And these are the force vector due to concentrated load, surface load and also the body force due to electrical field. So we put all these things into matrix form, we get here in this form, we know that there is no entry due to the inertial component coming from electric field because we still take that it is not an electrodynamics problem where there is a coupling between the dynamics, the inertial force also getting coupled here.

Here, we say that the mass is only totally due to the mechanical degrees of freedom, so there is no component of mass that will come into the due to electrical degrees of freedom, and we see that the mechanical degrees of freedom or electrical degrees of freedom are coupled through this matrices, this is the coupling that gives us the actuation or sensing element that we want for various applications.

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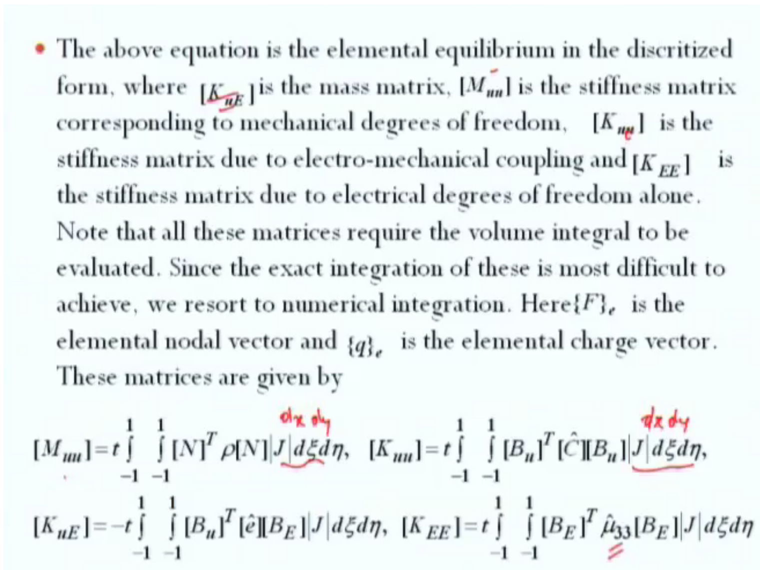

- The above equation is the elemental equilibrium in the discritized form, where $[K_{uE}]$ is the mass matrix, $[M_{uu}]$ is the stiffness matrix corresponding to mechanical degrees of freedom, $[K_{uu}]$ is the stiffness matrix due to electro-mechanical coupling and $[K_{EE}]$ is the stiffness matrix due to electrical degrees of freedom alone. Note that all these matrices require the volume integral to be evaluated. Since the exact integration of these is most difficult to achieve, we resort to numerical integration. Here $\{F\}_e$ is the elemental nodal vector and $\{q\}_e$ is the elemental charge vector. These matrices are given by

$$[M_{uu}]=t\int_{-1}^{1}\int_{-1}^{1}[N]^T\rho[N]|J|d\xi d\eta,\quad [K_{uu}]=t\int_{-1}^{1}\int_{-1}^{1}[B_u]^T[\hat{C}][B_u]|J|d\xi d\eta,$$

$$[K_{uE}]=-t\int_{-1}^{1}\int_{-1}^{1}[B_u]^T[\hat{e}][B_E]|J|d\xi d\eta,\quad [K_{EE}]=t\int_{-1}^{1}\int_{-1}^{1}[B_E]^T\hat{\mu}_{33}[B_E]|J|d\xi d\eta$$

So the above equation is the elemental equilibrium equation as I said where the KuE where the Muu I am sorry this is the Muu is the mass matrix and the Kuu is the corresponding to mechanical degrees of freedom, Kuu is the stiffness matrix stiffness matrix, KuE is the stiffness matrix stiffness matrix due to coupling. I have explained all these things previously, and each one of them can be explicitly written now in the isoparametric form.

These are all in terms of over the volume which is dv, dv we can be written as T*da, da is d psi*d eta multiplied by Jacobian this is basically =dx*dy. So similarly, we have for each one of these which is equal to dx*dy multiplied by T becomes the volume. So once we get this we know the shape functions, we know the density, we know the matrices here, by using all these things and mu 33 is the permittivity.

Now we know the complete system of matrices and we are in a position to solve the same problem using the 2 dimensional approach not the lumped approach which we talked a little bit earlier.

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- The elemental load and charge vectors are given by

$$\{F\}_e = \{F\}_c + \int_{S_1} [N]^T \{F\}_s dS_1, \quad \{q\}_e = -\int_{S_2} [N]^T D_s dS_2$$

- The matrices in FE Equations are assembled to obtain their global counterparts and solved for obtaining solutions for displacements and electric field.
- Note that it has a zero diagonal block in the mass matrix, which requires special solution schemes
- The method of solution for sensing and actuation problem is quite different.
- For sensing problem, for a given mechanical loading, we need to determine the voltage developed across the smart patch
- This is done by first obtaining the mechanical displacement due to the given mechanical load, which is then used to obtain the electric field and hence the voltage developed in the sensor patch

So here the elemental followed is due to the mechanical load, concentrated load and the surface load, and qe is the charge vector due to electrical displacement. The matrices in the finite element equations are assembled as in the case of the mechanical structure we did some examples in the last lecture, and we find out the global stiffness matrix, then we enforce the boundary conditions, and we solved for the displacement and the electric field.

Note that it has 0 diagonal in the block matrix which requires special solutions schemes, so we cannot directly solve because it becomes a positive semidefinite system, so we need special solution schemes. The method of solution sensing and actuation problem are quite different, now we will see where how we can do a sensing problem or actuation problem.

For a sensing problem for a given mechanical loading we need to determine what is the voltage and this voltage will exhibit certain features which we can extract to do the sensing using a smart patch. So this is basically done by obtaining the mechanical displacement due to the given mechanical load which is then used to obtain the electrical field and hence the voltage developed across the sensor patch, as I said this will introduce certain features and these futures can be extracted to actually do a sensing problem.

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- In order to solve this, the global matrix equation can be expanded and written as

$$[M_{uu}]\{\ddot{u}\}+[K_{uu}]\{u\}+[K_{uE}]\{\dot{E}_z\}=\{F\}$$
$$[K_{uE}]^T\{u\}+[K_{EE}]\{E_z\}=\{q\} \qquad \textbf{(A)}$$

- We can write the second of the above equation as

$$\{E_z\}=[K_{EE}]^{-1}\{q\}-[K_{EE}]^{-1}[K_{uE}]^T\{u\} \qquad \textbf{(a)}$$

- Using the above equation in the first equation and simplifying, we get

$$M_{uu}]\{\ddot{u}\}+[\bar{K}_{uu}]=\{\bar{F}\} \qquad \textbf{(b)}$$
$$[\bar{K}_{uu}]=[K_{uu}]-[K_{uE}][K_{EE}]^{-1}[K_{uE}]^T, \quad \{\bar{F}\}=\{F\}-[K_{EE}]^{-1}\{q\}$$

- Note that Equation (b) is only in terms of mechanical displacements, which can be solved using conventional solution techniques. Using this solution, electrical fields are obtained using Equation (a), from which voltages can be obtained.

So in order to solve this the global matrix equation which is given in the previous case here, this equation we can actually write it as 2 different equation, we can eliminate this portion for getting the voltage, for example this can be rewritten in this form, the first equation can be expanded as given here, and the second equation can be expanded like here. Now from the second equation because this is a square matrix we can take an inverse.

So we can take KEE inverse into q-KEE inverse into KuE transpose into u, so we substitute back here and we totally eliminate the electrical degrees of freedom and entirely we leave it as the mechanical degrees of freedom. So in the above equation is the only in terms of mechanical displacement which can be solved using the conventional solution techniques, so using this solution we can obtain the electrical field, once we know the mechanical through this equation.

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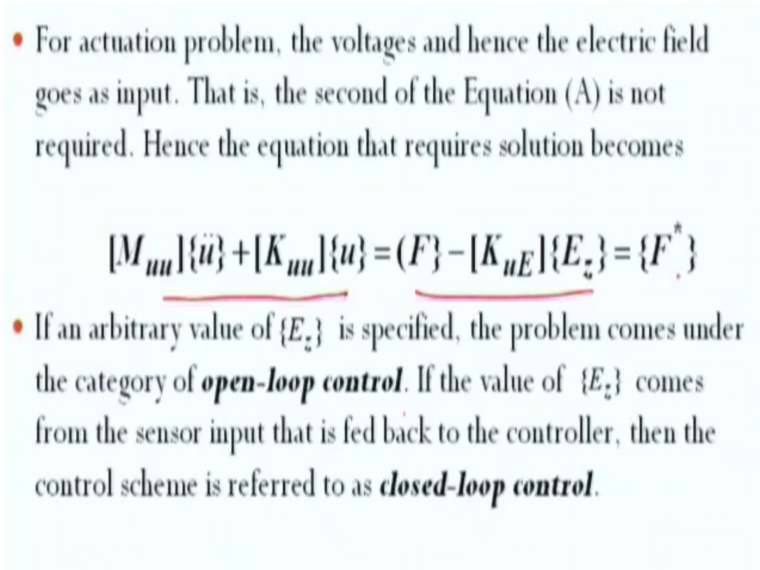

So for actuation problem the voltages and hence the electric field goes as input for the actuator, because we have to feed the electric field in order to actuate, so that is the second equation in A, that is this equation is not required for the actuation problem. Hence, the equation becomes this equation for actuation problem we have this one and this goes as an input here which is equivalent load, and now we can actually perform the actuation.

If any arbitrary value of E is specified the problems comes under the category of open-loop control, if the value of E comes from the sensor input and which is fed back into the controller, then the control scheme is referred to as closed-loop, we can do both open-loop and closed-loop from the finite element analysis, and this is how we do it for using the 2-dimensional problem.

So as we see here that this formulation are using the isoparametric is a level higher and more sophisticated than the lumped approach we did. The lumped approach only gives as an approximate value how this the piezoelectric material operates, because we make this assumption that the sensor equation sensor law and the actuation constitutive law are uncoupled.

But in actual case what we have done in the second approach we do not use any such assumptions, we take it as it is then we simplify the equation and then we can construct the plant matrix or the output matrix which are used for control for especially for closed-loop control from the finite element formulation.

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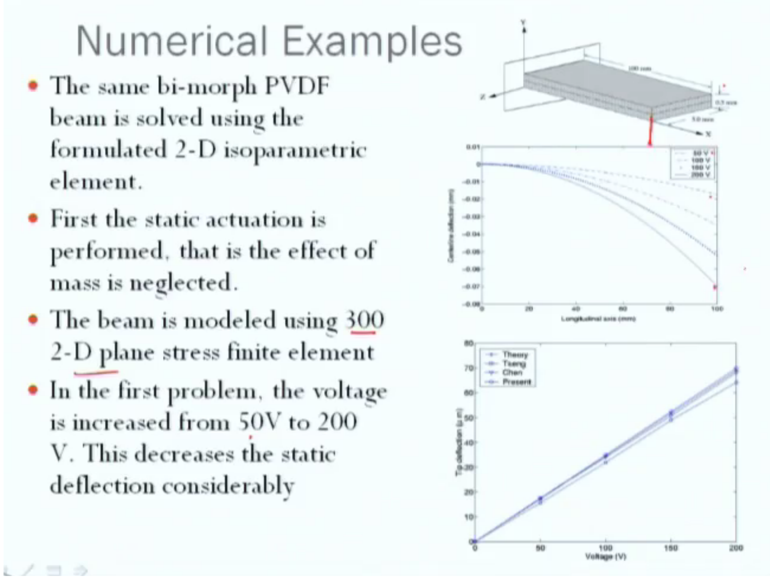

Let us now do the numerical example, we will consider the same numerical example which we solve using the exact equation and also using the lumped approach, and here we will see how more information can be got especially for the actuation problem. So this is static problem where the effect of mass is neglected, so we have to show this example we have model this structure only along this plane exact plane using 300 2-D plane stress finite element, with a thickness t here given here.

In the first problem, the voltage is increased from 50 volts to 200 volts, so when we actually do that from 50 volts to 200 volts we that the deflection, we know that when we apply this load the tip is going to reflect maximum in this direction, so there is no mechanical load here, so as we increase the load we see that the deflection increases as we increase the voltage here. The other one which we have said is for a given voltage how does this whole the theoretical tip deflection vary.

So we have given in the form of the deflection shape of a beam and also the graph relating to the voltage versus the tip deflection, so as the voltage increases we see that the tip deflection increases.

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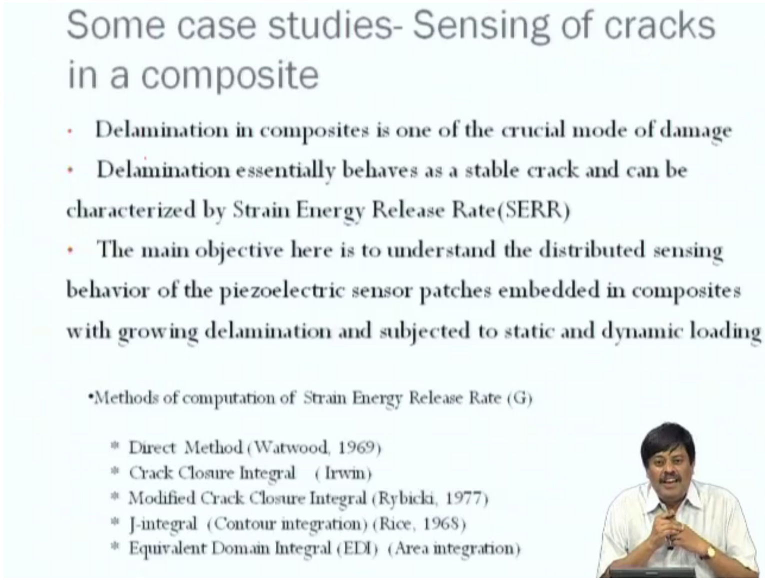

So now let us actually do some case studies using the formulated element, so here what we have done here is to show how the piezoelectric material can be used as a sensor now, what we did is the actuation problem we actually used to actually see how we can make the deflection 0, but here how we can actually sense the presence of the crack, what happens when the piezoelectric material is put on a cracked beam.

So here we have taken a composite beam where the crack is in the form of delamination, composite is constructed as a laminate construction and due to excessive loading the laminate can peel of causing delamination, so is it possible to actually determine the predict this delamination well in advance that is the goal and for which can we used to this piezoelectric material.

So here that delamination essentially behaves as a stable crack and can be characterized what is known as in fracture mechanics called the strain energy release rate. The main objective here is to understand the distributed sensing behaviour of the piezoelectric sensor patches embedded in composites with growing delamination.

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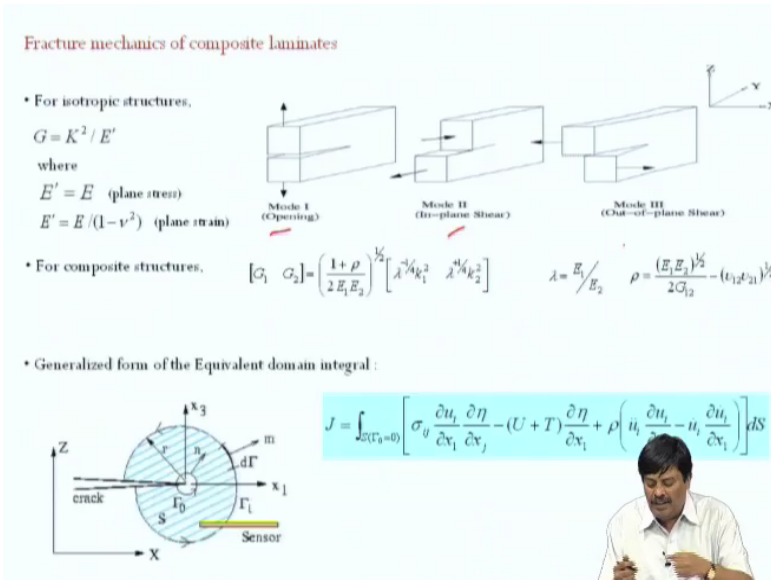

So in the composites there are 3 modes of failures, whether it is in composite or metal, one is the opening mode where the crack opens this is called the mode one crack. When the other one is when mode shears due to shear loading which is put in other terms it shears, so that is called the mode 2. One is the outer plane loading that is the tearing out one against the other which is called the mode 3. And the crack is characterized by the basically the stress intensity factor.

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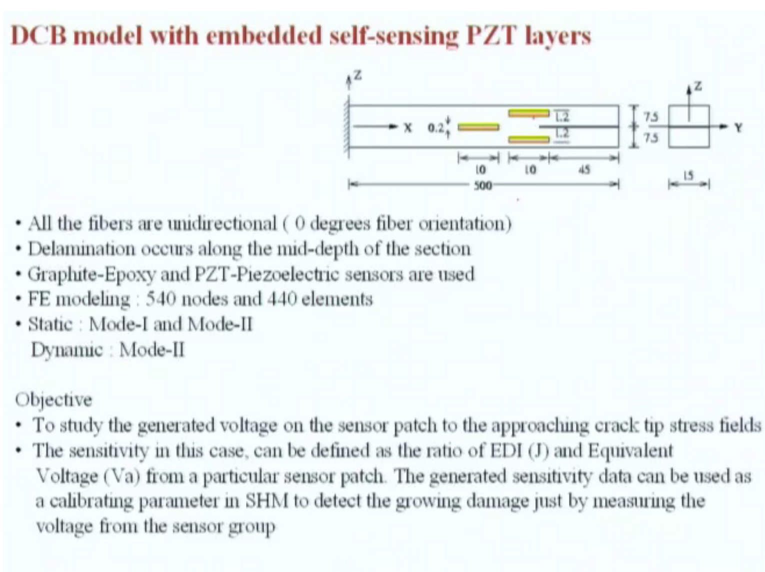

So here what we have done is we have taken a cantilever beam with the central delamination through with and we want to see what is the piezoelectric sensor response, that is what is the voltage that is developed as the crack starts propagating and goes inwards. What would be the sensor voltage that will be predicted in these 3 cracks, which is very useful to actually see whether there is something going on?

So hence the object is used to study the generated voltage in the sensors to the approaching cracked tip stress fields. And to study the sensitivity of this defined by the J integral, which is a measure of the crack sensitivity or the stress intensity factor with the voltage generated by the sensor patch. So this is basically used in structural health monitoring study, so here we have not defined what it is, it is just a case study to make us understand how this can be used in the sensing application.

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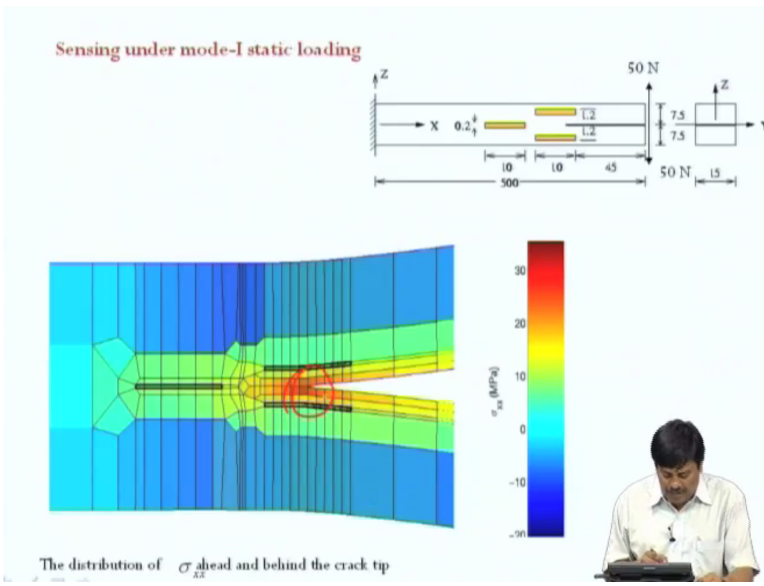

So if you before doing it if we model this with our formulated element and predict this stresses, this is the basically the axial stress and we see that there is a very high stress is found in this region which is basically a singular region where the stresses will be very high because the crack is emanating from here.

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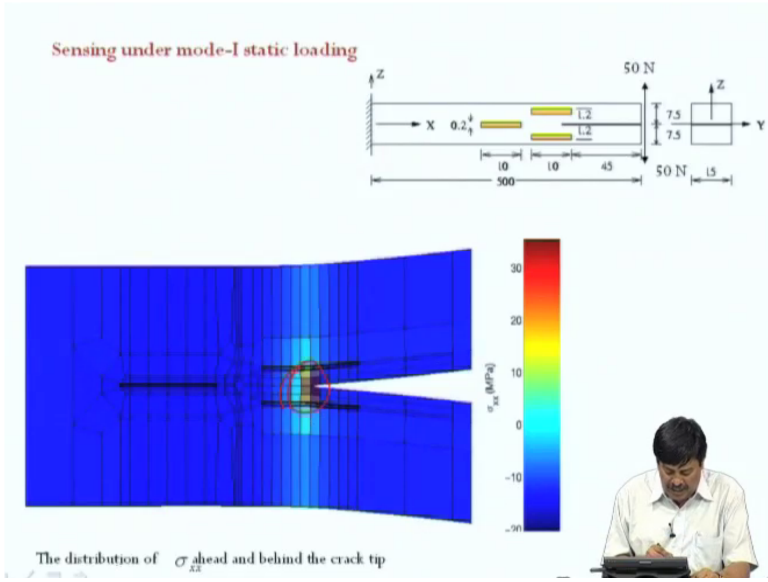

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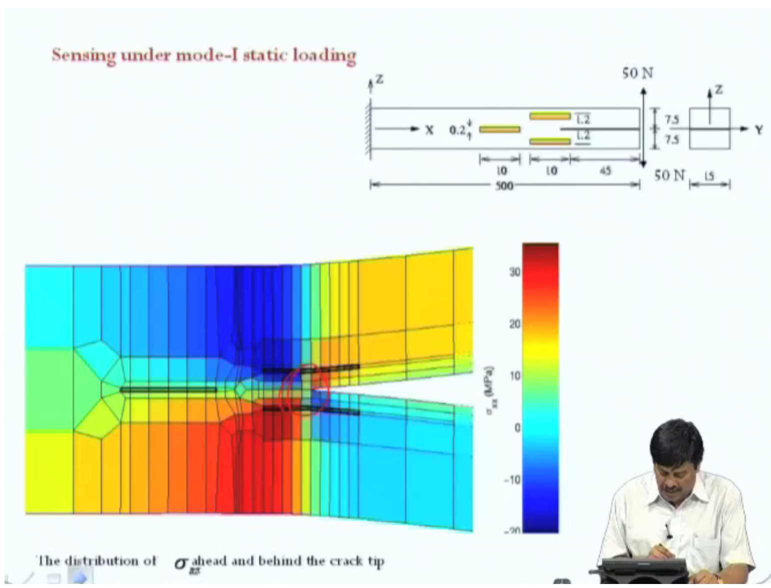

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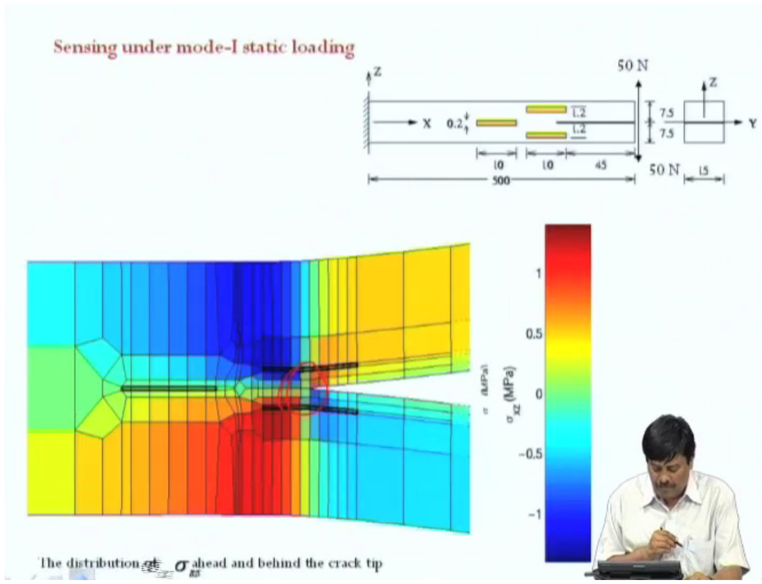

So we can actually predict the sigma x, sigma z which is the other stress that is there which is also pretty high here, and the tau xy basically.

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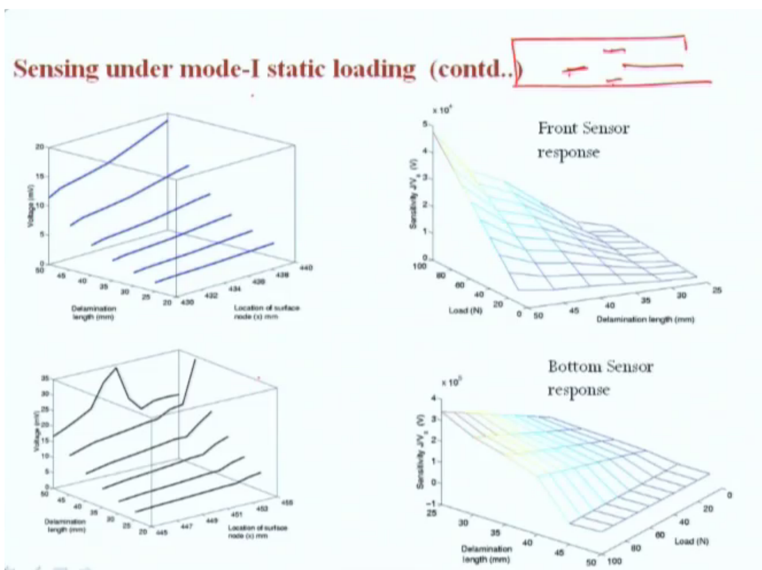

So when we actually study the voltage here under the static load, so this is the length of the delamination which is increased from 20 to 50, the 20 is very small is approaches 50 which is very near to the top and the bottom sensor and also the sensor right across the crack tip. So we see that the voltage increases as the delamination increases as the location from the surface node okay.

And this is basically the voltages that is found and we will see that this is on the first sensor this is the tip sensor, so we have 3 sensors here. So basically this is the crack here, the sensor here, sensor here and sensor here. So this is on the top node there is a front sensor response, this is the bottom sensor response and this is the response at the approach, you see there is a sudden peak of the sensors.

So based on the increase in voltage the state of stress changing and based on this concert we can basically say that the crack is there, which it is approaching fast, and that is possible that was made possible only because we had the piezoelectric patch here. So this is very tremendous for us for in terms of using this for the sensing application, this is one of the sensing application there are many applications that we can actually construct using the piezoelectric material.

So here we clearly see there is an increased peak in the response in the sensor 3 and based on that we can basically say that there is something happening, even here in the front sensor we see that as the load is increased as the delamination is nears 50, there is a huge increasing the load, so this is the sensitivity of the J integral.

So basically what we are saying is if you just measure the voltage and if we have this curve, we can say how severe the crack is and what are is there any necessary repairs that we need to do to avoid catastrophic failure, which is very very important from the structural health monitoring. So this was done by using a load, which was in this direction.

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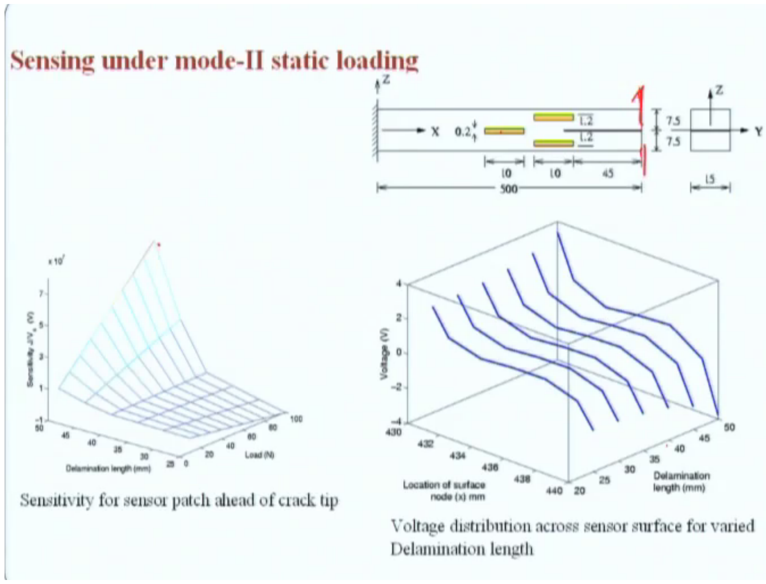

So now we do the same thing when the load is in this direction where we apply here, so this is going to cause a shearing, so the shearing is basically the mode 2 load, so in the shearing load here is the sensitivity of a response you see there is a tremendous increase in the voltage when this is approaching very near the crack tip, so this is on this sensor here. So basically by measuring this we will know that whether the approaching is it going to be catastrophic or not.

So we can actually generate a lot of this curve and correlate the fracture parameters with the sensor voltage, this is basically the voltage distribution across the sensor surface for a vary delamination length. So we see that it increases tremendously as the length of the delamination is increased, so become basically we have shown that we can basically use this concert of sensing using piezoelectric for structural health monitoring applications. We can again do this for variety of other applications.

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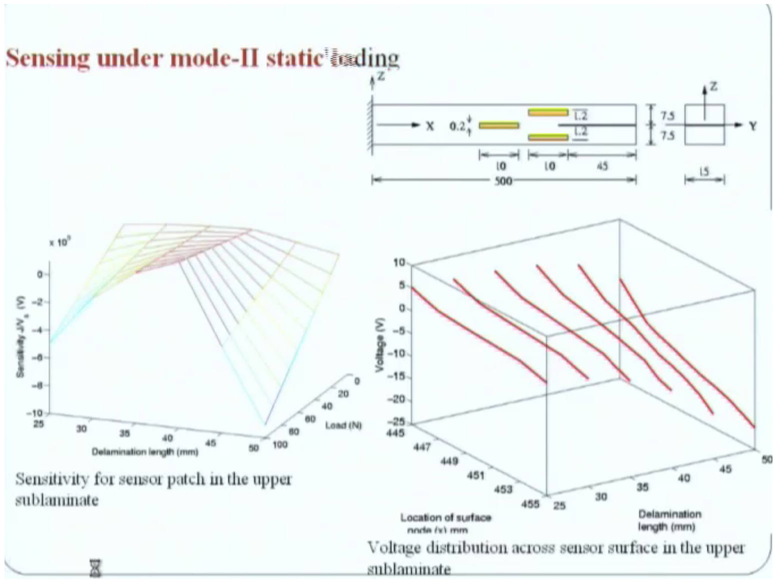

So this is another way where the sensitivity is done for the patch when the upper portion of this sensor here, and same thing is the voltage distribution this is the sensitivity, so we see how the voltage, in most cases there is a linear variation in some cases where the level of stresses are very high especially in this sensor we see a significant nonlinear variation. So basically what we are trying to see here is how we can use piezoelectric material for the sensing application.

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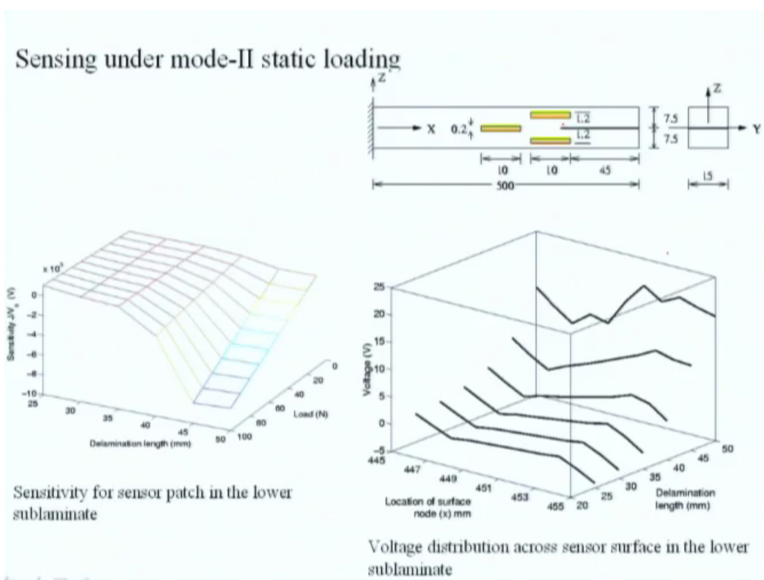

This is another aspect of in the lower sensor, so we have done for this sensor, this sensor, this sensor. You see there is a complete non-linearity especially for the mode 2 loading, how can the voltage varies, so basically this change in voltage variation is a measure of the state of stress in the happening in the laminate which can basically be used to actually make an opinion about how severe the crack is, and that is basically a sensing problem.

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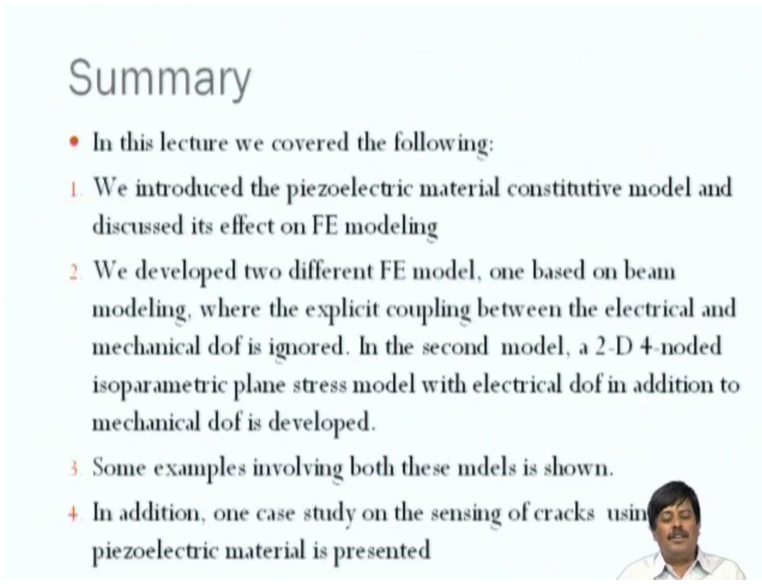

So let us summarize what we studied here, so in this lecture we covered the following, we introduced the piezoelectric material constitutive model and discussed its effect on the FE modeling. So basically the effect is coming in terms of additional coupling matrices because additional energy that goes into the weak form of equation due to piezoelectric material coming from the electrical field.

We developed 2 different finite element model. One based on beam modeling where the explicit coupling between electrical and mechanical degree of freedom was ignored, so we basically reduce the problem into a statically equivalent load lumped onto the beam model, and we studied that as the actuation problem.

So in the second model we developed at 2-D 4 noded isoparametric plane stress model with additional degree of freedom in the form of electrical field, and we also again studied the same actuation problem that is giving a voltage and reducing the total deflection in the beam structure. Then we introduced another case study of how this can be used as a sensing.

Where we actually related showed how this can be embedded into a composite and measure the voltage because of the ensuing mechanical load and tell about the state of the structure especially with regard to the presence of crack. So basically we can use the piezoelectric material as the crack sensor for sensing application.

So essentially what we have covered in the entire finite element method is we basically said we started basically on how the finite element modeling can be helped in the microsystem design, then we covered the theoretical basics for finite elements and we basically said how a second order system can be characterized, how many numerical methods can be characterized using weighted residual method.

And how we can derive finite element method from the weighted residual method. Then we actually established the theoretical basis of the entire finite element modeling for say using the mechanical system as the example, this method is not related to mechanical alone it depends upon the governing differential equation and the physics based on the governing equation and the degree of freedom is physically based on what the physics modeling.

Suppose you are modelling a Maxwell's equation instead of the displacement you would have electric field and the magnetic field as basic unknowns, otherwise if you have the corresponding forces would be the electrical displacement and the magnetic flux if you are modeling a thermal problem then the temperature and the temperature flux will be the dependent and the first variable.

So like that any physics of the problem we can actually use it, if we are solving a fluid problem we need to idealize the whole Navier-Stokes equation in to finite element modelling, where instead of displacement we will have velocities as the basic degree of freedom. So it depends upon the physics what you are modeling.

So in microsystem we need the modelling of electromagnetics, electrical field, electrostatics where Maxwell's equation need to be modelled, wave equation for mechanical system, Navier-Stokes equation for the basically the fluid system and the Fourier law for the thermal systems.

So it does not matter what the governing equation, the approach what we are derived is exactly same only the dependent variable meaning is changing, so in short the finite element modeling what we have developed can be exploited for all other domains which are prevalent in the smart and microsystems, thank you.