Robotics: Basics and Selected Advanced Concepts
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Lecture – 04
Elimination Theory & Solution of Non-linear Equations

Welcome to this NPTEL lectures on Robotics: Basics and Advanced Concepts. In the last 3 lectures, we had looked at the kinematics of serial robots. We had looked at the direct kinematics problem and the inverse kinematics problem. In the inverse kinematics problem, we showed that you need to derive a single equation in one of the joint variables, which we can then solve to obtain the joint variable given the position and orientation of the last link or the end effector.

So, in this lecture, we will look at this concept of Elimination Theory and Solution of Non-linear Equations. So, this will help us understand how we can derive in general a single equation in one of the joint variables. So, quick review of inverse kinematics of serial robots.

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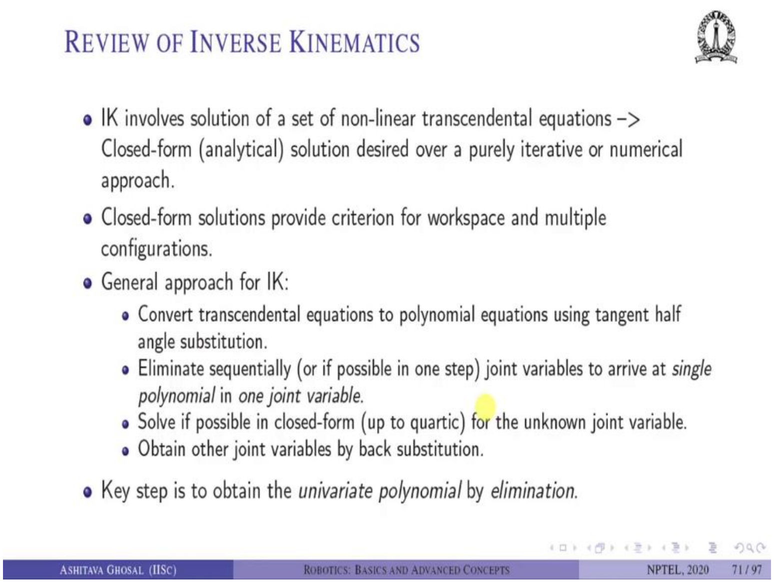

The inverse kinematic involves solution of a set of non-linear transcendental equation. We desire a closed-form or analytical solution over a purely iterative or numerical approach. And the reason is this closed-form solutions provide criteria for the workspace of the robot and how many configurations are possible to reach a given position and orientation, all the multiple configurations, which are possible solutions of the inverse kinematics problem.

So, the general approach for inverse kinematics of serial robots is first, convert the transcendental equations to polynomial equations using tangent half angle substitution ok. So, the kinematics involves sine and cosine θ's and we can convert that into $\tan(\theta/2)$ which is denoted by x and then we get the equation in terms of polynomial in x.

Then, we need to eliminate sequentially or if possible, in one step, joint variables to arrive at a single polynomial in one joint variable. And then, we solve if possible, in closed-form for example, up to quartic equations, we can solve in closed-form for the unknown joint variable and then, obtain all other joint variables by back substitution ok. So, the key step is to obtain the univariate polynomial by elimination, and this is what we will see in this lecture.

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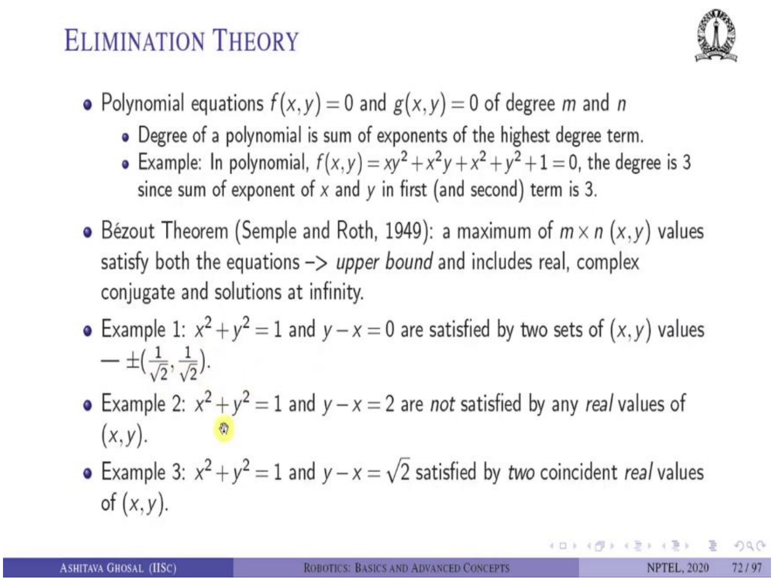

So, let us start what is elimination theory? So, the first important concept is if you are given two polynomial equations, let us say $f(x,y)=0$, and $g(x,y)=0$ of degree m and n ok. So, the important concept is what is the degree of a polynomial?

So, the degree of a polynomial is the sum of the exponents of the highest degree term. So, for example, if $f(x,y)=xy^2+x^2y+x^2+y^2+1=0$, the degree of this polynomial is 3 because the sum of the exponents of x and y in the first and second term is 3 ok.

What do we know? If you are trying to eliminate an equation between two such polynomials, there is a very well-known theorem called Bezout's theorem ok. So, if I have two polynomials, $f(x,y) = 0$, and $g(x,y) = 0$ of degree m and n, there is a maximum of $m \times n$ (x,y) values which satisfy both these equations ok. Important word is maximum of $m \times n$.

This is an upper bound. It includes all the real possible values of (x,y), complex conjugate values of (x,y) and also solutions at infinity. So, you know there is this whole idea that there are some solutions which are at infinity.

So, for example, example 1: if I have an equation which is $(x^2 + y^2 = 1)$ so, this is nothing but a equation of a circle origin (0, 0) and radius 1 and we have another equation which is $(y - x = 0)$. So, this is a line passing through the origin at an angle 45 degrees. So, we can see that these two equations, $(x^2 + y^2 = 1)$ and $(y - x = 0)$ are satisfied by two sets of values of x and y. So, these are nothing but $\pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Example 2: if you have the same circle, but the line is now $(y - x = 2)$. So, now, you can see that these two equations do not have any real (x,y) which satisfies these two equations ok, there are no real values of x and y possible. Finally, again we look at this circle and the line is now $(y - x = \sqrt{2})$. So, then, you can draw and check that it is satisfied by two coincident real values of (x,y) basically, this line is tangent to this circle and both the solutions are at one place.

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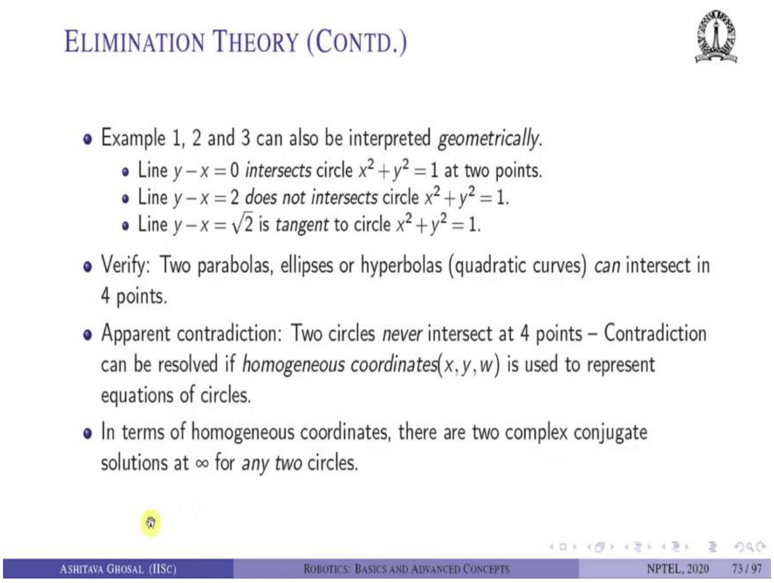

Example 1, 2 and 3 can also be interpreted geometrically. So, the line $(y - x = 0)$ intersects the circle at two points quite obvious, line $(y - x = 2)$ does not intersect the circle at real points. And $(y - x = \sqrt{2})$ is tangent to the circle $(x^2 + y^2 = 1)$ ok.

You can also verify; that if you take two parabolas, or two ellipses, or two hyperbolas ok, these are called quadratic curves ok. So, we have power of x and y as two. So, they can intersect at 4 points. One apparent contradiction is if you have two circles so, we can sketch two circles on a sheet of paper and then, you can see that it will never intersect at 4 points, the circles are quadratic, but they never intersect at 4 points.

So, this contradiction can be resolved if homogeneous coordinates x, y, w is used to represent the equation of the circle. Remember, we had this homogeneous transformation matrix where we added a $(1 = 1)$ in terms of a generalization of that, we could have added an equation $(w = w)$ so, that is what makes x, y which are normal coordinates and if you add another coordinate w, so that is what is called homogeneous coordinates ok.

So, in terms of homogeneous coordinates, you can show that there are two complex conjugate solutions for two circles at infinity for any two circles ok. So, whatever is the radius of the circle, if you draw two circles, they intersect at two points, you can show and two other points are at infinity.

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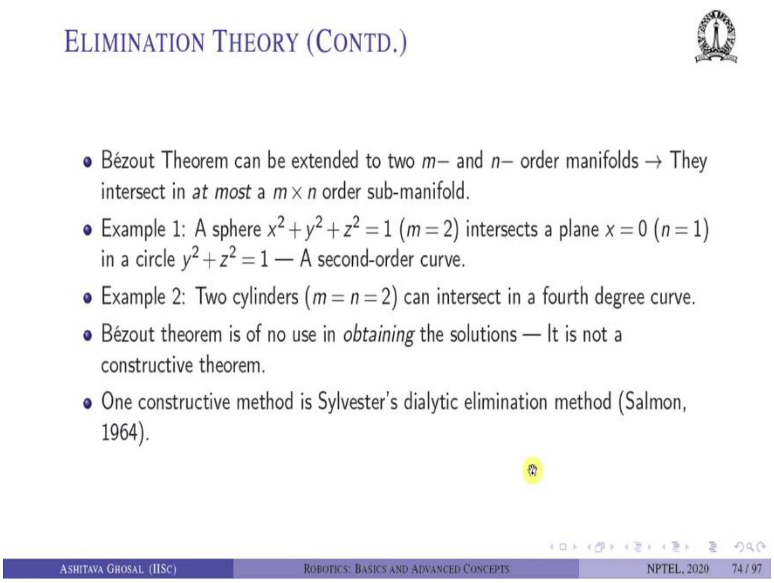

So, Bezout's theorem can be extended to two m- and n- order manifolds. It can be shown that, or it has been shown that they intersect in at most a $m \times n$ order sub-manifold ok.

So, example 1: A sphere which is $(x^2 + y^2 + z^2 = 1)$, here $(m = 2)$ and intersects a plane $(x = 0)$, which is $(n = 1)$ in a circle this is obvious. If you take a sphere and you cut it by a plane through its origin, you will get a circle which is $(y^2 + z^2 = 1)$ and this is a second-order curve ok. So, the sub-manifold is 1×2 which is second-order.

You can also have two cylinders which is $m = n = 2$ and we know from CAD or we know from our basic geometry that it intersects in a fourth-degree curve ok, two cylinders intersect at a fourth-degree curve. Bezout's theorem is of no use in obtaining the solution, it is not a constructive theorem. All it is telling you that there are $m \times n$ possible (x, y) or $m \times n$ sub-manifolds ok, but it does not tell you what are these (x, y) points ok.

So, one constructive method is given by Sylvester's ok, it is available in this book by Salmon in 1964, but this was developed in the 1800's. It is called the Sylvester's dialytic elimination method and I want to show this in a little detail.

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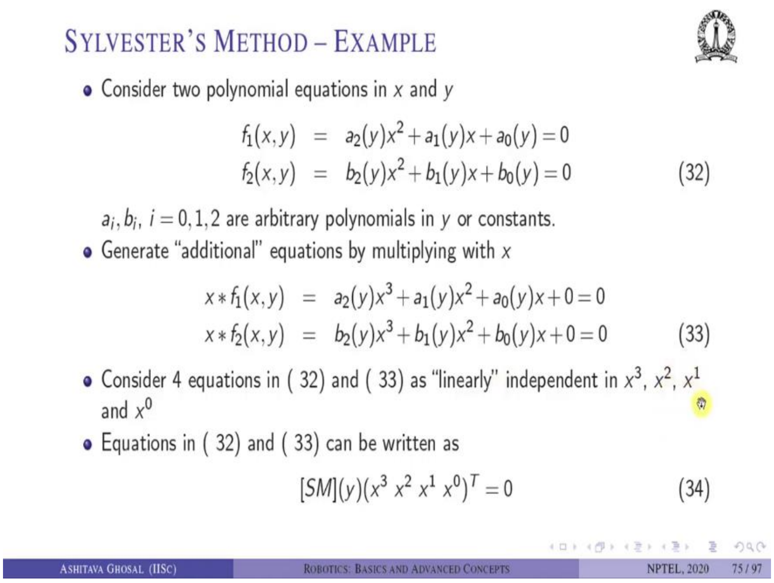

Through one example first. So, consider two polynomials, in x and y so, $f_1(x,y) = a_2(y)x^2 + a_1(y)x + a_0(y) = 0$. So, it is quadratic in x^2, but the coefficients could be any functions of y ok, arbitrary functions of y. So, we have these two polynomials and a_i, b_i are arbitrary polynomials in y or constant ok.

So, what does the Sylvester's method tell you? It tells you that we generate additional equations by multiplying both these two equations by x ok. So, if I multiply $x \times f_1$, we will get $a_2(y)x^3 + a_1(y)x^2 + a_0(y)x + 0 = 0$. Similarly, if I multiply the second equation by x, I will get $b_2(y)x^3 + b_1(y)x^2 + b_0(y)x + 0 = 0$ ok. So, I have generated 4 equations ok.

The second step in Sylvester's method is very important. It tells you that we consider x^3, x^2, x^1 and the constant which is x^0 as "linearly" independent variables ok. So, the word linearly is important, they are clearly not independent because so, second x^2 is clearly square of x^1.

So, once you consider these as linearly independent variables, these four equations can be written as a matrix equation which is $[SM](y)(x^3, x^2, x, x^0)^T = 0$ and I will show you what is $[SM]$ next ok. So, it is basically what you will see is that we will have $a_2, a_1, a_0, 0$ in the first row, $0, a_2, a_1, a_0$ in the second row and so on ok.

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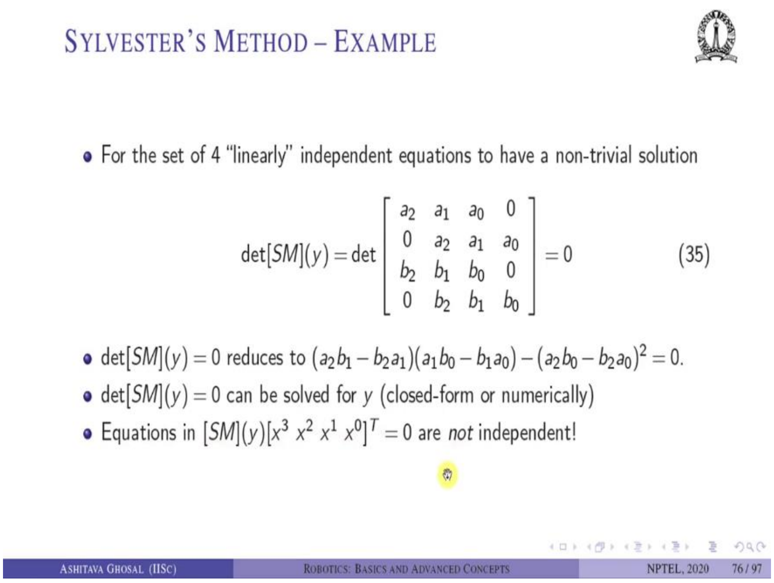

So, the set of 4 linearly independent equations are $(a_2, a_1, a_0, 0) \times$ some column vector, $(0, a_2, a_1, a_0) \times$ some column vector and so on ok, $(b_2, b_1, b_0, 0) \times$ same column vector and $(0, b_2, b_1, b_0) \times$ in the same column vector. So, this equation which is of the form $AX = 0$, it is a set of linear equations of $AX = 0$ ok.

So, this set of equations have a non-trivial solution when $\det[SM] = 0$, this is from linear algebra. So, basically, we have determinant of that matrix equal to 0 ok. So, how did I get this matrix ok? There is a trick. What you can see is that the second equation is nothing but the original f_1 equation. So, $a_2x^2 + a_1x + a_0x^0 = 0$.

When you multiply this equation by x, you basically shift this row to the left. So, similarly, the first equation was $f_2(x, y) = b_2x^2 + b_1x + b_0x^0 = 0$ and we multiplied by x and we shifted to the left. So, what we have by multiplication is four equations in four unknowns so, we have basically a square system of equations and hence from linear algebra, we say that the determinant of that matrix must be equal to 0.

In this case, we can find the determinant, it is quite simple, we just use the rules of determinant, it reduces to this expression which is $((a_2b_1 - b_2a_1)(a_1b_0 - b_1a_0) - (a_2b_0 - b_2a_0)^2 = 0)$ ok. So, remember, all the a's are constant or functions of y ok.

So, hence $\det([SM]) = 0$ is only a function of y. So, what have we done? We have managed to get rid of x ok. We have obtained a single equation only in y ok. It is important to note that this equations $([SM](x^3, x^2, x, x^0)^T = 0)$ they are not strictly independent ok, they are linearly independent because we are considering these x^3, x^2, x^1 and x^0 as linearly independent variables, but they are actually non-linearly dependent.

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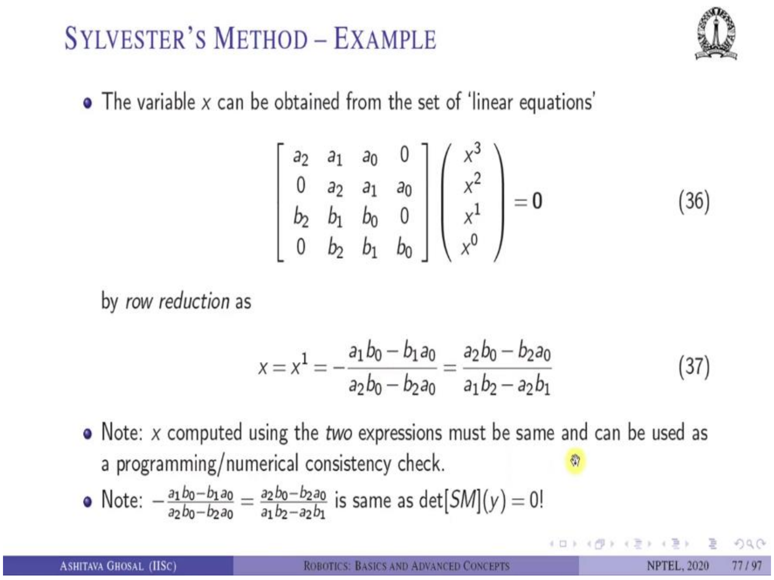

The variable x can be obtained from the set of linear equations ok. So, we have $[SM](x^3, x^2, x^1, x^0)^T = 0$. We can do row reduction ok. So, basically, multiply the first equation by something, subtract from the second and so on ok, standard approaches in linear algebra to solve a set of equations.

And what we will get is that the third row will become $x^1 \times$ something will be equal to 0 and we can simplify that and see that

$$x^1 = -\frac{a_1 b_0 - b_1 a_0}{a_2 b_0 - b_2 a_0} = \frac{a_2 b_0 - b_2 a_0}{a_1 b_2 - a_2 b_1}$$

So, the x computed using the two expressions here, right-hand side must be same and can be used as a programming, numerical consistency check. So, one thing what you can see is this $(a_1 b_0 - b_1 a_0)$, this left term here should be equal to this, which is nothing but if you expand it out, you will get $\det[SM](\mathrm{y}) = 0$.

So, we are not getting any new information or inconsistent information. So, we get an x in two ways, both must be same, and we also show that these two terms must be related by $\det[SM](\mathrm{y}) = 0$ ok. So, what have we done? We have got a single equation in y and we also have solved for x.

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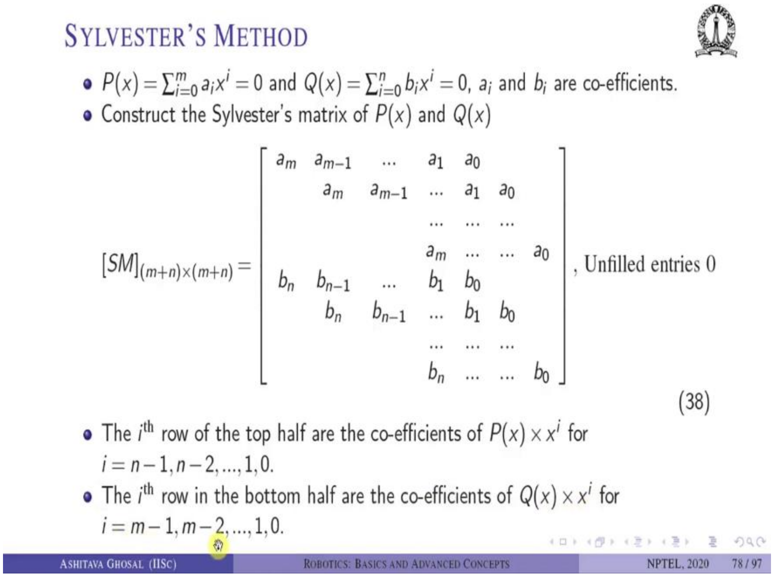

So, in general, if I am given an m^{th} degree polynomial $P(x) = \sum_{i=0}^{m} a_i x^i = 0$ and an n^{th} degree polynomial $Q(x) = \sum_{i=0}^{n} b_i x^i = 0$ where, a and b are co-efficients could be functions of y or constants.

We construct the Sylvester's matrix of $P(x)$ and $Q(x)$ by basically starting with this original equation, multiplied by x first, then multiply by x^2 and so on till you get some terms top half like this. And similarly, you take the equation $Q(x) = \sum_{i=0}^{n} b_i x^i = 0$, then you again multiply by x as many times as possible to eventually get a square matrix ok.

So, we have to multiply by correct number of times, the first equation $P(x)$, such that we get a Sylvester's matrix which is $(m+n) \times (m+n)$, it is a square matrix. So, all the unfilled elements in this matrices are 0, ok.

So, as I have said, we are multiplying by x so, the i$^{\text{th}}$ row of the top half are the co-efficients of $P(x) \times x^i = 0,\ \ i = n-1, n-2, \ldots, 1, 0$ and the i$^{\text{th}}$ row of the bottom half are the co-efficients of $Q(x) \times x^i = 0,\ \ i = m-1, m-2, \ldots, 1,$. So, this matrix is of dimension $(m+n) \times (m+n)$.

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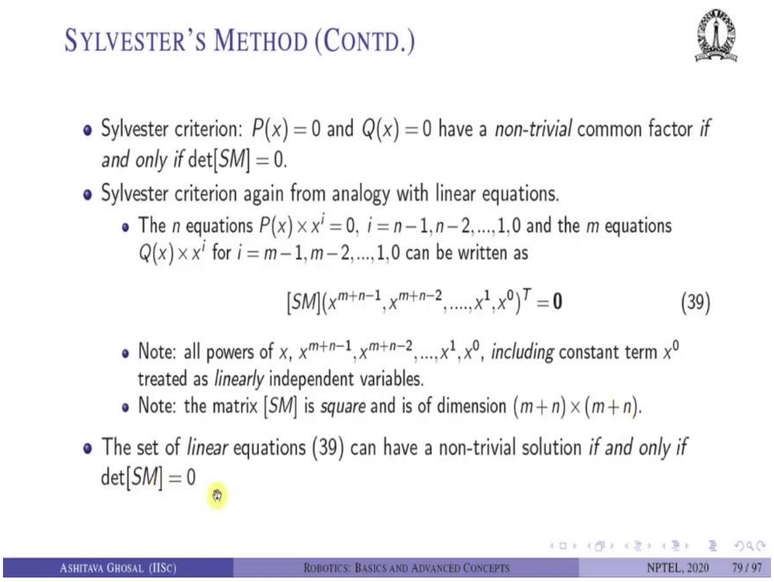

And Sylvester's criteria is that the polynomials $P(x) = 0$ and $Q(x) = 0$ have a non-trivial common factor meaning as a non-trivial solution x if and only if $\det([SM]) = 0$.

So, what is the analogy? This is basically an analogy from linear equations ok. So, Sylvester's followed this idea of what happens when we have a linear equation and then, he extended it to non-linear equation except that the column vector, the unknowns are not really independent, they are x^3, x^2, x^1, x^0 and so on, ok.

So, all powers of x, x^{m+n-1} and so on, including the constant term are treated as linearly independent variables ok. And this matrix is always of dimension $(m + n) \times (m + n)$. So, in general, whenever $\det([SM]) = 0$, we will get a non-trivial solution of, this vector x^{m+n-1}, x^{m+n-2}, all the way to x^0, ok.

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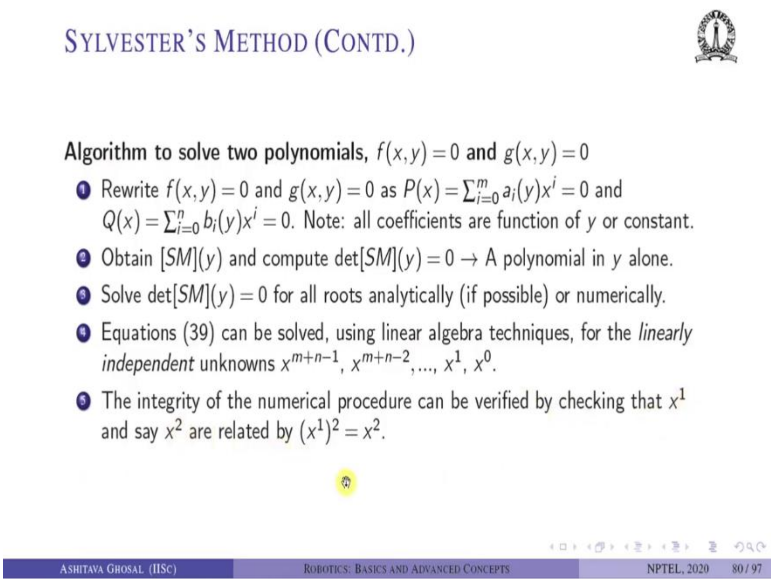

So, what is the algorithm? We can write it in steps: so, given two polynomials, $f(x,y)=0$, and $g(x,y)=0$. Rewrite these two polynomials as $\sum a_i x^i$. So, basically, I should take all the y parts in the coefficients and the other polynomial variables. So, $Q(x)$ has $\sum b_i(y)x^i=0$, ok. So, note: all the coefficients of this polynomials are functions of y or constant.

Then, obtain $[SM](y)$ exactly in the form as I showed you, you multiply $P(x)$ correct number of times with x, $Q(x)$ correct number of times with x and then, form the square matrix and then, you compute $\det[SM](y)=0$. So, this is a polynomial in y alone ok. So, in this two steps, we have eliminated x, that is why it is called theory of elimination.

So, we solve $\det[SM](y)=0$ if possible, analytically or numerically ok. So, as I have discussed last lecture, that if it is up to quartic, then we can solve analytically otherwise, you have to do numerically. And once we have y, possible solutions for this y, then this equation (39) can be solved, using standard linear algebra techniques basically, row reduction for the linearly independent unknowns x^{m+n-1} all the way to x^0, ok.

And the integrity of the numerical procedure can be verified by checking that x^1 and say x^2 are related in this form, $(x^1)^2$ is nothing but x^2.

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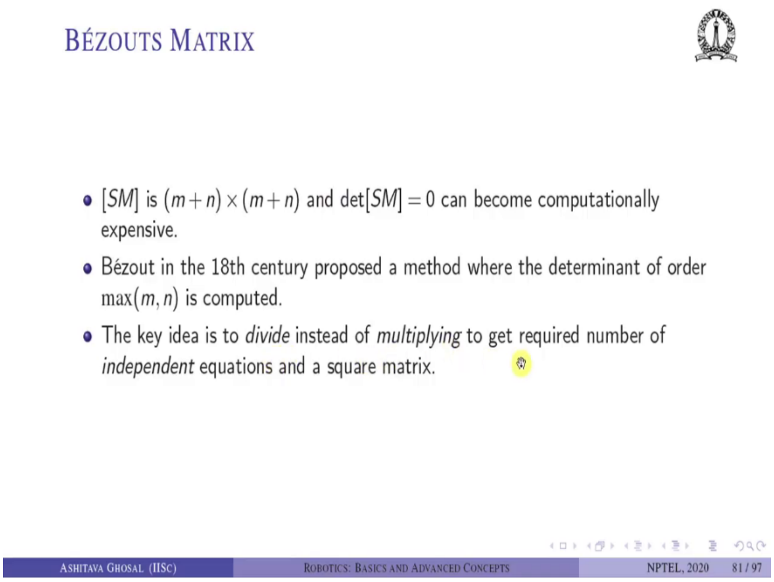

So, as we can see this Sylvester's matrix $[SM]$ is $(m+n) \times (m+n)$ and $\det[SM](y) = 0$ can become computationally quite expensive. So, for example, if $m = 5$ and $n = 6$ so, this is 11×11 matrix and determinant of an 11×11 matrix is pretty bad ok.

So, Bezout, same person as Bezout's theorem in 18th century proposed a method where the determinant of the order $\max(m, n)$ needs to be computed. We do not need to compute $(m+n) \times (m+n)$ matrix; we need to compute a matrix, which is $\max(m, n)$. So, in that example of $m = 5$ and $n = 6$, we will need to generate a matrix which is 6×6 and compute the determinant of that, ok.

So, what is the key idea in Bezout's matrix? The key idea is to divide instead of multiplying to get required number of independent equations and a square matrix, ok. So, previously, we were multiplying by x now, we have to do suitable divisions, ok.

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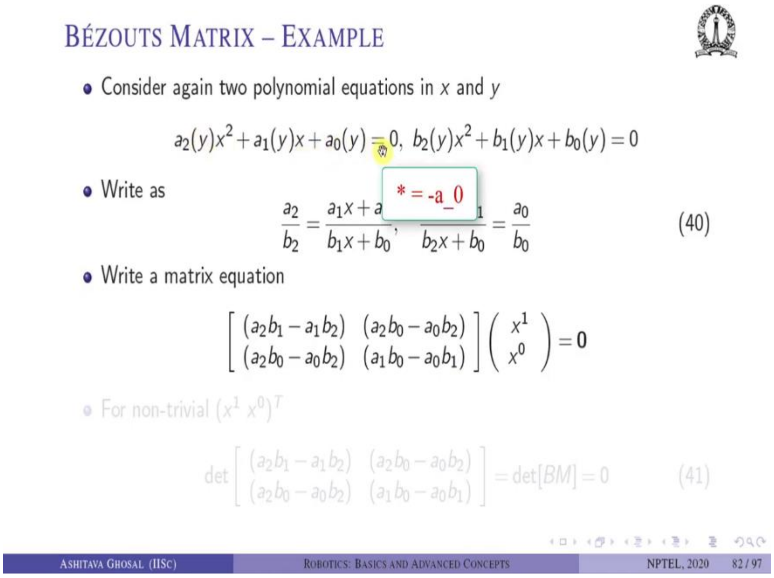

So, example: so, consider again two polynomials in x and y, same two polynomials $a_2(y)x^2 + a_1(y)x + a_0(y) = 0$ and $b_2(y)x^2 + b_1(y)x + b_0(y) = 0$. So, we write $a_2(y)x^2 = -a_1(y)x - a_0(y)$. So, basically, we take these two terms on the other side and similarly, b_2, you take these two terms on the other side.

And if $x \neq 0$, we can divide those two equations and we can get

$$\frac{a_2}{b_2} = \frac{a_1x + a_0}{b_1x + b_0}$$

We can also take $(a_2x + a_1)$ and so, basically take x common outside is equal to $-a_0$, and similarly, $(b_2x + b_1)$ with x common is equal to $-b_0$, and again divide both of them and we can get another equation in x, ok. So, basically, what have we done? By dividing suitably, we have now obtained two equations in x alone; we have managed to get rid of x^2.

And then, we can expand these two equations and write in a matrix form so, you can see it will be $\big((a_2b_1 - a_1b_2)x^1 + (a_2b_0 - a_0b_2)x^0 = 0\big)$, so that will be the first equation similarly, the second equation will be $\big((a_2b_0 - a_0b_2)x^1 + (a_1b_0 - a_0b_1)x^0 = 0\big)$.

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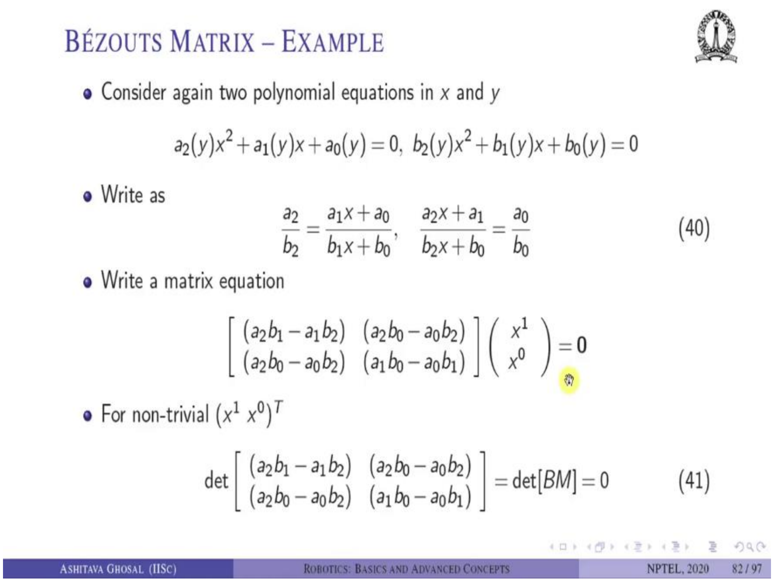

So, for non-trivial, x^1 and x^0 treated as independent variables. The determinant of this matrix must be equal to 0. So, this is the Bezout's matrix. So, we started with two polynomials, second order polynomials, quadratic polynomials and the Bezout's matrix is 2×2 ok.

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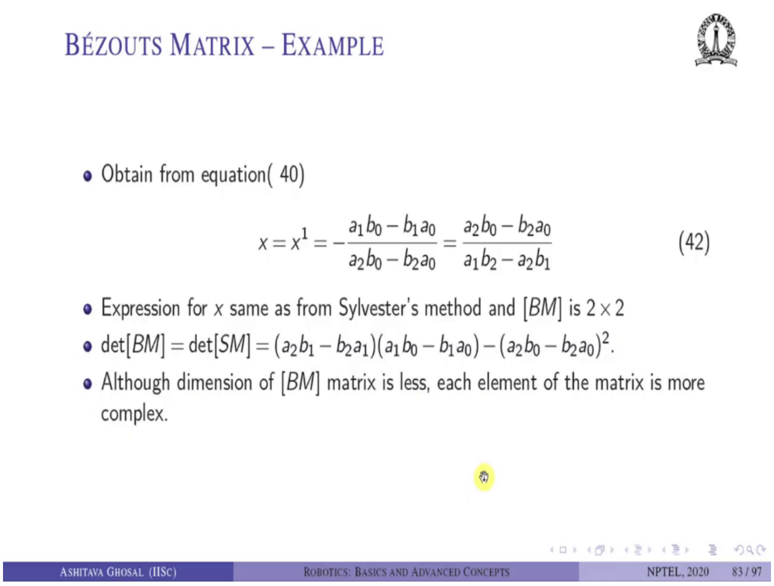

So, can we solve for x? Yes, we can go back and look at any one of these equations and solve for what is x ok. So, we can show that

$$x^1 = -\frac{a_1 b_0 - b_1 a_0}{a_2 b_0 - b_2 a_0} = \frac{a_2 b_0 - b_2 a_0}{a_1 b_2 - a_2 b_1}$$

So, the expressions for x are same as from the Sylvester's method, but the Bezout matrix is now 2×2 ok.

So, what about the determinant of the Bezout's matrix and the determinant of Sylvester's matrix? Both are exactly same, ok. So, they will both give the same expression. So, this is an expression only in y. So, we have eliminated x and what we get is some $\det[SM](\mathrm{y}) = 0$ or $\det[BM](\mathrm{y}) = 0$, ok.

So, although, the dimension of Bezout's matrix is less, it is 2×2, but each term in the matrix is more complex ok. So, remember in the Sylvester's matrix, we had the $a_2\ a_1\ a_0$ and then, we shifted left ok. So, we had a 4×4 matrix with a's and b's. Here, you have a 2×2 matrix, but each term is complicated, it is $(a_2 b_1 - a_1 b_2)$ and so on ok. So, it is not as if we are shifting the equation by left, we have to actually compute the terms. So, it is possible to do, but nevertheless it is more complex.

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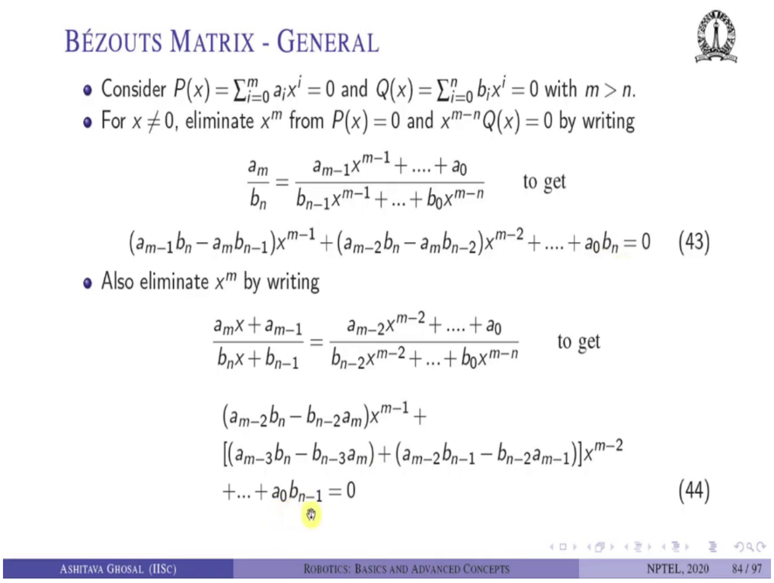

So, in general, if you are given polynomial $P(x) = \sum_{i=0}^{m} a_i x^i = 0$, m[th] degree polynomial, and $Q(x) = \sum_{i=0}^{n} b_i x^i = 0$ as an n[th] degree polynomial, with $m > n$.

So, for $x \neq 0$, eliminate x^m . So, previously, we had eliminated x^2 from $P(x)$ and x^{m-n} $Q(x)$, by rewriting it in this form, (a_m/b_n) is equal to this ok, then you can expand this and we will get an expression in x^{m-1}, x^{m-2} and all the way to x^0.

We can also eliminate x^m by writing as I had said in that previous example we can write the equations in two different ways, we can take x common and we can get $\left(\frac{a_m x + a_{m-1}}{b_n x + b_{n-1}}\right) = RHS$.

And then, we can show that we get an equation in x^{m-1} when you expand this out, which will be of the form $(a_{m-2}b_n - b_{n-2}a_m)x^{m-1}$ and so on, and the last term will be $(a_0 b_{n-1})$. So, here it was $a_0 b_n$, here it is $(a_0 b_{n-1})$ plus something else will come, ok.

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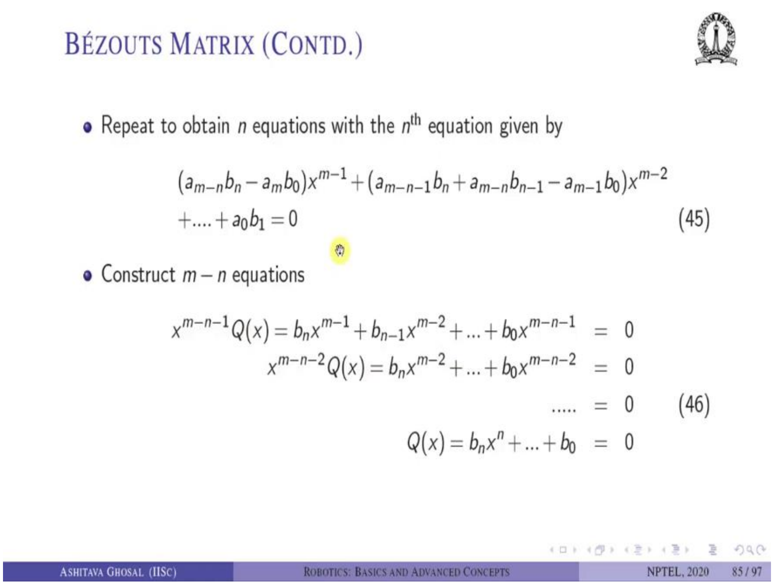

And we can repeat this to obtain n equations with the n^{th} equation given in this form. So, $(a_{m-n}b_n - a_m b_0)x^{m-1}$ likewise, some term, complicated term with a's and b's into x^{m-2} and so on. And we can also construct $(m-n)$ equations by multiplying $Q(x)$ by x^{m-n-1}. So, eventually, we have all these equations (45) and (46) and we can form the Bezout's matrix.

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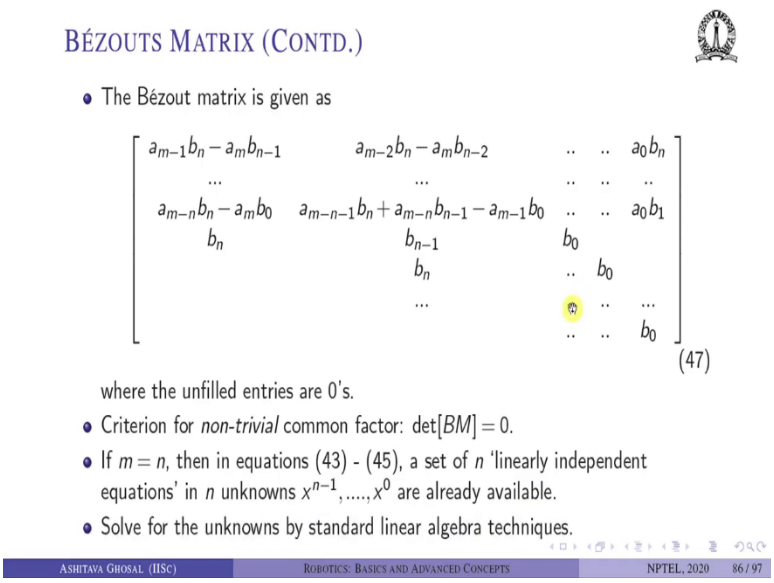

So, the Bezout's matrix consists of the first term is this, then whole bunch of terms, then last equation is a_{m-n} all the way to $a_0 b_1$ and since $m > n$, we have assumed so, we have b_n, b_{n-1} all the way to b_0 shifted left by some number of times, and all the way to b_0, the last equation is b_n all the way to b_0 so, where the unfilled entries are all 0's.

So, the criteria for non-trivial common factor between the two polynomials non-trivial x is $\det[BM] = 0$. So, if $m = n$, then in equations (43) to (45), a set of n linear independent equations in n unknowns are already available, ok. So, we do not have to do this complicated things. So, we can solve for the unknowns by standard linear algebra techniques by row reduction.

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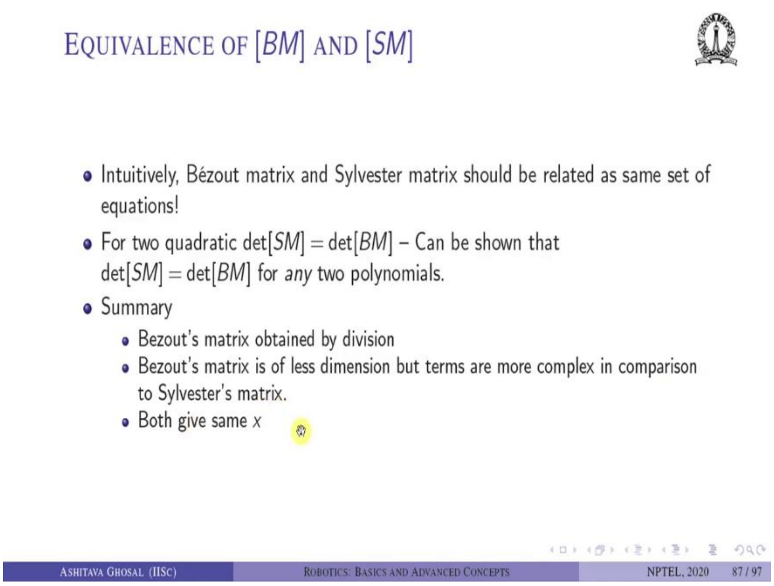

So, equivalence of $[BM]$ and $[SM]$. So, intuitively, the Bezout's matrix and the Sylvester's matrix should be related ok. We are starting from the same set of two polynomials, we are solving the same common factors (x, y) or solutions of both the two polynomials so, they must be related.

So, I showed you for the two quadratics $\det[SM] = \det[BM]$. And we can also show in general that the $\det[SM]$ and $\det[BM]$ will be equal for any two polynomials, the m^{th} and n^{th} order polynomial.

So, in summary, the Bezout's matrix is obtained by division. Sylvester's matrix was obtained by multiplying each equation. Bezout's matrix is of less dimension, but the terms are more complex in comparison to Sylvester's matrix. And both give the same x and the same eliminant $\det[SM](\mathrm{y}) = 0$ or $\det[BM](\mathrm{y}) = 0$, both the expressions are same, ok.

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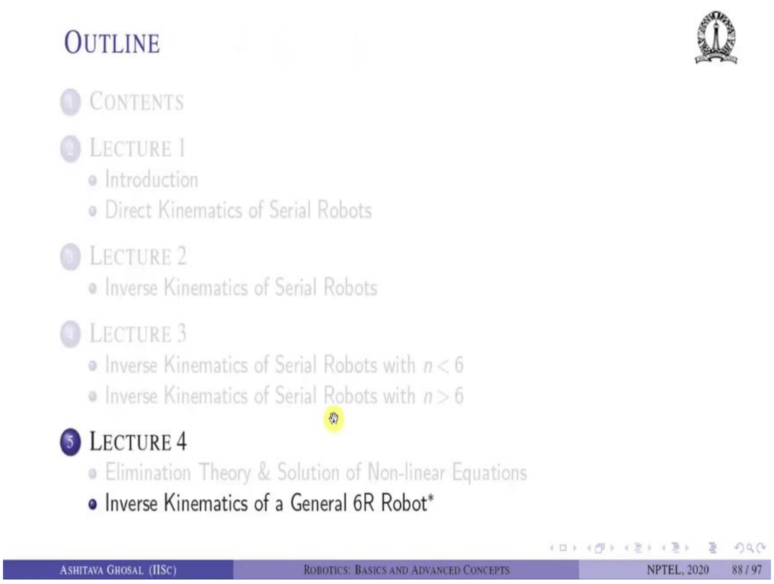

So, with this, we now come to the last item in this inverse kinematics of serial robots. So, I want to use whatever we have learned of eliminating joint variables ok, this theory of elimination whatever we have discussed to obtain the inverse kinematics of a general 6R robot ok. So, this is a fairly advanced topic. For a long time, the exact solutions to how to obtain the inverse kinematics of a general 6R robot was not known, ok.

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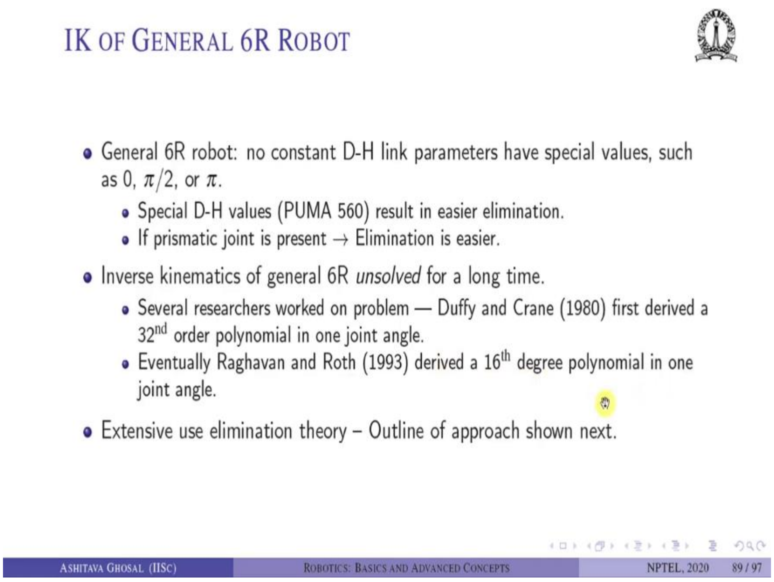

So, what is general 6R robot? So, general 6R robot means, there are no constant D-H link parameters of special values. So, remember very often, we saw α_{i-1} was 0 or a_{i-1} was 0, or d_i was 0 or a_{i-1} was $\pi/2$ so, $\cos\pi/2 = 0$ so, there are some special 0's and angles which are 0 or $\pm\pi/2$ ok. So, we do not have any of those ok.

So, if you have special values of D-H parameters such as in the case of the PUMA 560, the elimination is always much easier, ok. If there is a prismatic joint, again the elimination is much easier, ok because we do not have $\cos\theta$ and $\sin\theta$. The prismatic joint variable is d so, we do not have cosine and sine of d, right.

So, as I have mentioned, the inverse kinematics of a general 6R robot was unsolved for a long time. Several researchers have worked on the problem. Historically, Duffy and Crane first derived the 32nd order polynomial in one joint variable. So, the idea is that you do elimination from a set of equations and obtain a single equation in one of the joint variables, which you can solve numerically if required. So, Duffy and Crane in 1980's said that we can derive a 32nd order polynomial in one joint variable.

So, which means given the position and orientation of this general robot end effector, ok, we can reach that position and orientation in 32 possible ways, upper bound ok, many would be imaginary. Eventually, Raghavan and Roth in 1993 derived a 16th degree polynomial in one joint angle ok. So, this is what is known, and this is what is agreed that given a position and orientation of an end effector of a general 6 degree of freedom robot with rotary joints, there are 16 possible solutions, inverse kinematic solutions.

And they used extensively this elimination theory which we just finished, and I am going to show you the outline of the approach ok, it is very complicated, they have written many papers, it is many many steps are involved, but just to get a feel that we can solve problems like this now, that we have tools from elimination theory, I am going to give you the steps which are used by Raghavan and Roth.

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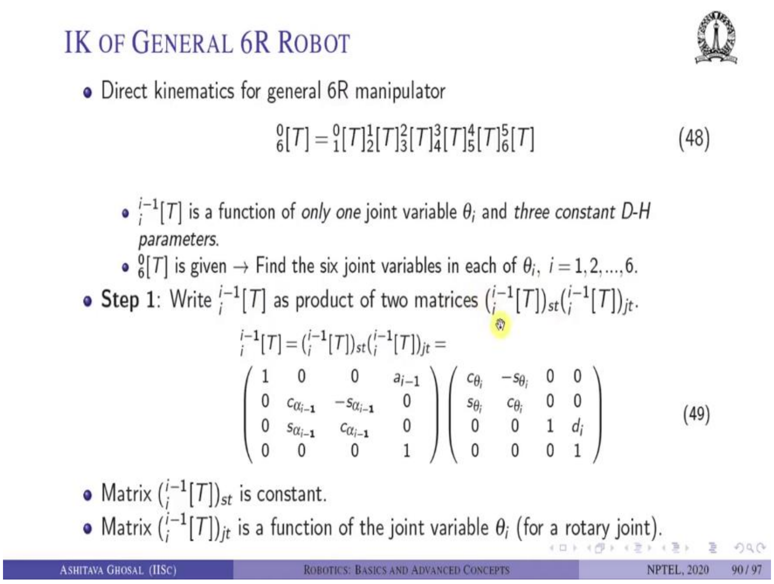

So, the first important step is that the direct kinematics for a general 6R manipulator is given by this equation, it is standard by now, that the 6^{th} coordinate system position and orientation with respect to the 0^{th} coordinate system is multiplication of 6 link transformation matrices. And each of these link transformation matrix say $(i-1)$ to i is a function of only one joint variable θ_i and three constant D-H parameters, this we know ok.

So, what is the inverse kinematics problem? The left-hand side is given, and we need to find the six joint variables in each of these transformation matrices so, in each of θ_i. First step 1, first write ${}^{i-1}_{i}[T]$, the link transformation matrix as a product of two matrices ok.

So, this is Raghavan and Roth's approach. What is the two matrices? One is something which is dependent on only the constant D-H parameters, which is called this st or structural part and then, another part which is ${}^{i-1}_{i}[T]$, which is the joint part, ok. Can we do this? Yes.

So, for example, ${}^{i-1}_{i}[T]$ the structural part × joint part so, the structural part looks like this,

$$\left({}^{i-1}_{i}[T]\right)_{st} = \begin{pmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & \cos\alpha_{i-1} & -\sin\alpha_{i-1} & 0 \\ 0 & \sin\alpha_{i-1} & \cos\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

And the joint part is

$$\left({}^{i-1}_{i}[T]\right)_{jt} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So, this second matrix contains the joint variables θ or d and the first matrix contains the twist angle and the link length ok. So, this you can see is constant whereas, this is a function of joint variable θ for rotary joint and if it was a prismatic joint, then you will have d. So, that is step 1.

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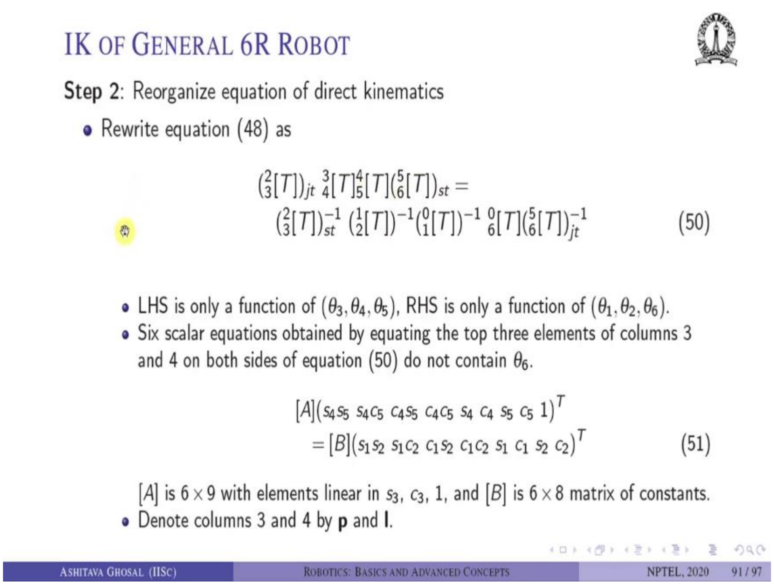

Step 2: we reorganize the equation of direct kinematics, rewrite it as $({}^{2}_{3}[T])_{jt}{}^{3}_{4}[T]{}^{4}_{5}[T]({}^{5}_{6}[T])_{jt} = ({}^{2}_{3}[T])^{-1}_{st}({}^{1}_{2}[T])^{-1}({}^{0}_{1}[T])^{-1}{}^{0}_{6}[T]({}^{5}_{6}[T])^{-1}_{jt}$, this is the same as

the direct kinematics equation. So, if you think a little bit about it, we have broken some of the matrices into the structural part and the joint part and then, rearranged.

So, it turns out that the LHS is only a function of 3, 4 and 5 so, θ_3, θ_4 and θ_5 whereas; the right-hand side is a function of θ_1, θ_2 and θ_6 ok. So, this is a link transformation matrix equation ok. So, there are six scalar equations, or six independent scalar equations in this matrix equation. Remember, the last row is 0, 0, 0, 1 so, there is nothing in it. So, out of the top 3×3 and the last 3×1 column, we can only get 3 + 3 = 6 independent equations, ok.

And this six scalar equations are obtained by equating the top three elements of column 3rd, and 4th on both sides of the equation (50) ok. So, they do not contain θ_6. So, we can rewrite as

$$[A](s_4s_5, s_4c_5, c_4s_5, c_4c_5, s_4, c_4, s_5, c_5, 1)^T = [B](s_1s_2, s_1c_2, c_1s_2, c_1c_2, s_1, c_1, s_2, c_2)^T.$$

So, we have managed to look at these equations and pick the one's which do not contain θ_6 ok. So, $[A]$ is a 6×9 matrix with elements linear in $\sin\theta_3$, $\cos\theta_3$ and 1 this is also important. So, in this matrix $[A]$, θ_3 is there, but they are containing only $\sin\theta_3$, $\cos\theta_3$ and 1, there is no square and product terms and in $[B]$ matrix, it is a 6×8 matrix of constants, ok. So, we denote the columns 3 and 4 by $\boldsymbol{p}$ and $\boldsymbol{l}$, ok.

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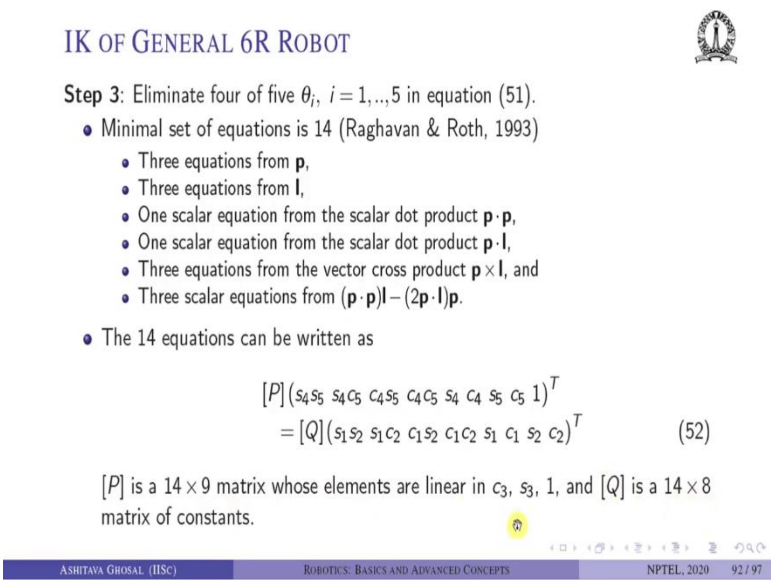

So, now, we come to the elimination. So, we eliminate four of the five variables θ_1, θ_2, θ_3, θ_4 and θ_5. So, in this set, there is θ_1, θ_2, θ_3, θ_4 and θ_5, θ_6 is not there and how do we do that? This is the contribution of Raghavan and Roth in 1993. We obtained a minimal set of 14 equations.

Three equations from $\boldsymbol{p}$, equating $\boldsymbol{p}$ from both sides, three equations from $\boldsymbol{l}$, one scalar equations which is the $\boldsymbol{p} \cdot \boldsymbol{p}$, one scalar equations from the scalar dot product $\boldsymbol{p} \cdot \boldsymbol{l}$, three equations from $\boldsymbol{p} \times \boldsymbol{l}$ and three scalar equations from this vector equation $(\boldsymbol{p} \cdot \boldsymbol{p})\boldsymbol{l} - (2\boldsymbol{p} \cdot \boldsymbol{l})\boldsymbol{p}$, ok.

So, you can see 3 + 3 + 2 + 6 = 14 equations they obtained from that matrix equation, ok. So, this 14 equations can be written as $[P] \times (s_4, s_5, s_4, c_5 \ldots$ and so on so, θ_4 and θ_5 is equal to $[Q] \times$ this. So, this $[P]$ matrix is a 14×9 matrix containing $\sin\theta_3$, $\cos\theta_3$ and 1 and $[Q]$ is a matrix of 14×8 constants.

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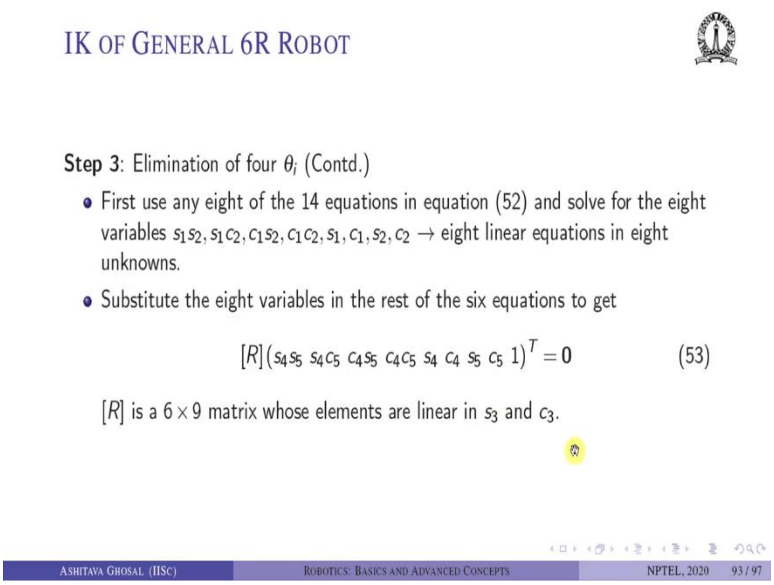

So, we eliminate four θ_i in one steps and how do we do? First, use any eight of the 14 equations in equation (52) previously, any of this from this equation, we use any eight of the 14 equation and solve for the eight variables, independent variables quote, unquote independent $s_1s_2, s_1c_2, c_1s_2, c_1c_2, s_1, c_1, s_2, c_2$ ok. So, these are eight linear equations in eight unknowns, we can always solve for them ok.

Substitute the eight variables in the rest of the six equations to get $[R]$, which is now a new matrix with this function of $\sin\theta_4$, $\sin\theta_5$, s_4, c_5 and so on to 1 ok. So, $[R]$ is a 6×9 matrix whose elements are linear in $\sin\theta_3$ and $\cos\theta_3$. They do not have any squared terms in $\sin\theta_3$ and $\cos\theta_3$.

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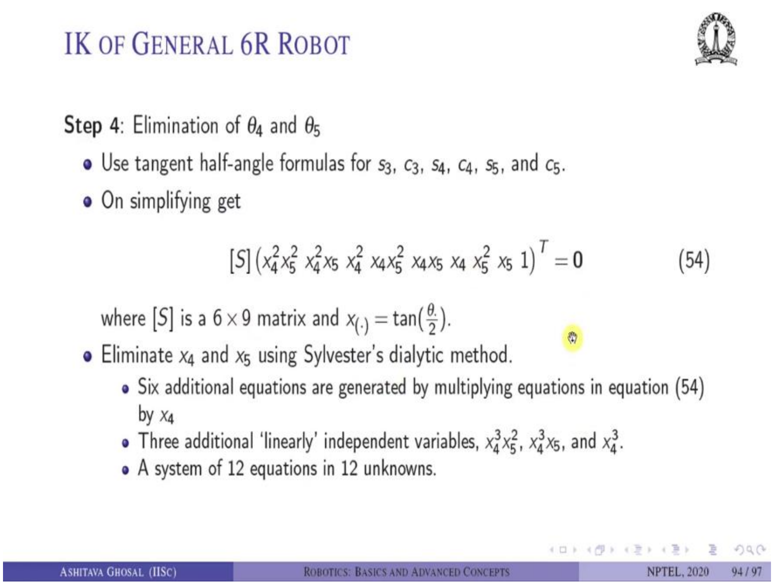

We eliminate θ_4 and θ_5. We first use tangent half-angles formulas for $\sin\theta_3$, $\cos\theta_3$, $\sin\theta_4$, $\cos\theta_4$, $\sin\theta_5$, $\cos\theta_5$. On simplification, we can write some matrix $[S] \times x_4^2$, where what is this x_4? That is $\tan(\theta_4/2)$. What is x_5? It is $\tan(\theta_5/2)$. So, we have converted the trigonometric functions, sine and cosine of θ_4 and θ_5 into polynomials. So, we are going to eliminate θ_4 and θ_5 in one step ok. So, we get this equation.

And we eliminate x_4 and x_5 using the Sylvester's dialytic method. So, basically, we have some matrix which is $(6 \times 9) \times (9 \times 1)$ equal to 0. So, we pre-shift the rows, we multiply some equations by $x_{(\cdot)}$ and so on ok. So, we generate three additional independent variables when you multiplied by x_4 and x_5 in some particular way so, we get $x_4^3 x_5^2$, $x_4^3 x_5$ and x_4^3. So, we get a system of 12 equations in 12 unknowns, ok.

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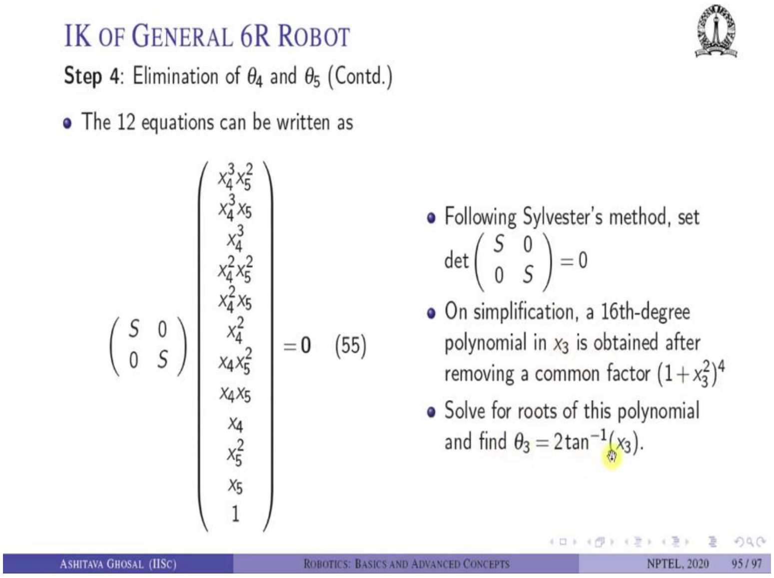

So, these 12 equations in 12 unknowns can be written as S, 0, 0, S. So, basically, they are being shifted to the left when you multiply, and we have this original 9: 1, 2, 3, 4, 5, 6, 7, 8, 9 variables in $x_4^2 x_5^2$ and so on, but then, we have multiplied, and we have generated additional variables $x_4^3 x_5^2$ and so on. So, we have the form of $[SM](y) \times x = 0$.

So, the so, following Sylvester's method, we set this determinant is equal to 0 because that is the condition such that this column vector is non-trivial ok. So, what do we have? We have $\det[S]$, which is some horribly complicated terms which contains only θ_3 and so on, $\sin\theta_3$ and $\cos\theta_3$. So, we have expression, long expression which on simplification will give us 16th-degree polynomial in $\tan(\theta_3/2)$.

So, if you do the algebra, if you do this simplification, you will actually get a 24th degree polynomial in x_3, but there are these common factors which is $(1 + x_3^2)^4$ which can be removed ok, this can never be 0.

So, we can solve this 16th-degree polynomial to obtain θ_3 ok. So, $\theta_3 = 2\tan^{-1} x_3$ which we get. So, how do we solve this 16th-degree polynomial? We have to do it numerically.

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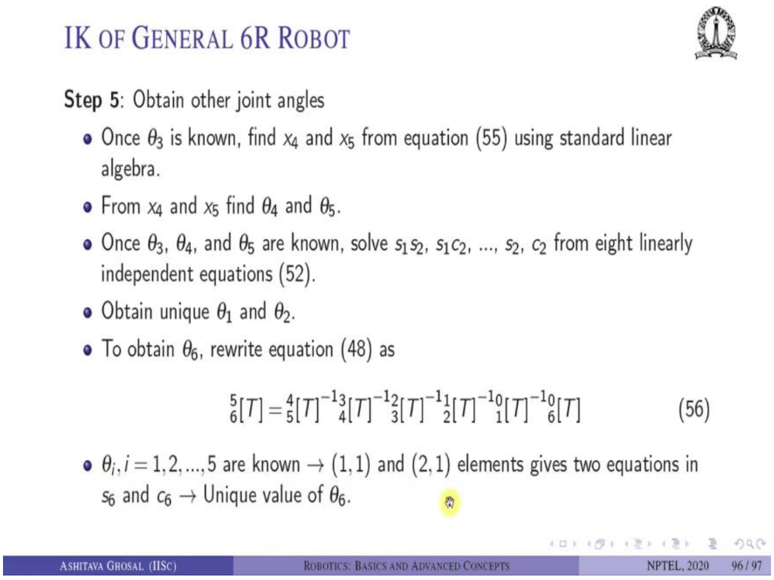

So, once θ_3 is known, we can find x_4 and x_5 from this equation, simply by row reduction so, we can find out what is x_5 and what is x_4, these two elements here, just like we found out what is x after doing elimination.

So, once we find x_4 and x_5, we find θ_4 and θ_5, by $2 \times$ tan inverse of x_4 and x_5. Once θ_3, θ_4, θ_5 are known we can go back and solve the right-hand side. Remember, the right-hand side has θ_1, θ_2 in one set of equations in (52) and we can obtain unique θ_1 and θ_2.

Finally, to obtain θ_6, we go back to the original set of equation which is $^5_6[T] = {^4_5[T]}^{-1}\ {^3_4[T]}^{-1}\ {^2_3[T]}^{-1}\ {^1_2[T]}^{-1}\ {^0_1[T]}^{-1}\ {^0_6[T]}$. So, this is given to us and each of these are functions of θ_4, θ_3, θ_2, θ_1 and so on. So, we can solve for θ_6 from this equation ok. So, these are known and from the (1,1) and (2,1) elements gives two equation in $\sin\theta_6$ and $\cos\theta_6$ and we can get a unique value of θ_6 ok. So, this is the basic algorithm.

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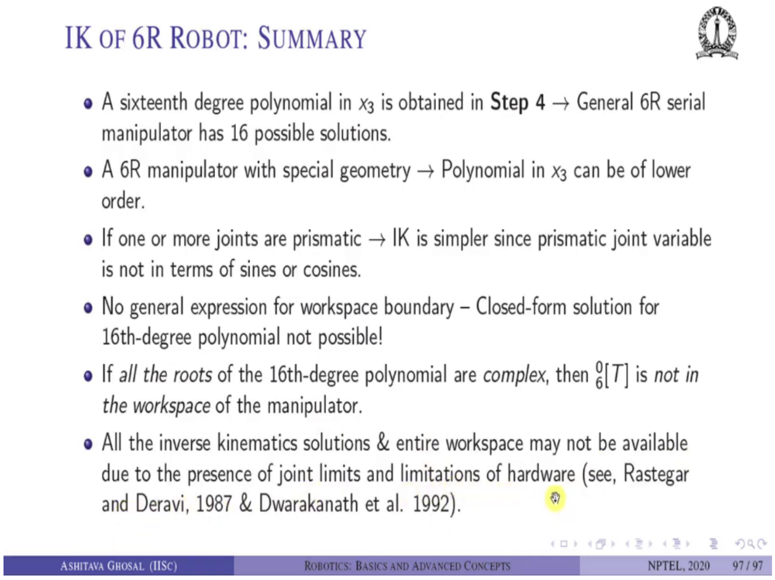

So, important thing is that we obtain the sixteen-degree polynomial in x_3 from this step. So, this is the most important step. How to obtain this matrix S, 0, 0, S into this long column vector with 12 variables, in terms of $\tan(\theta_4/2)$ and $\tan(\theta_5/2)$ and their powers ok.

So, once we know this, we find that the, whatever is in this matrix is only function of θ_3. So, hence we can obtain θ_3, ok. So, once you obtain θ_3, then we find θ_4 and θ_5, then once θ_3, θ_4 and θ_5 are known, we obtain θ_1 and θ_2 and then finally, once all the angles are known, we find θ_6, ok.

So, what is the summary? Any general 6R robot has 16 possible solutions why? Because the polynomial in θ_3 was 16^{th} degree. If you have special geometry meaning certain link lengths are 0 or some angles are 90 degrees or 0, this polynomial in x_3 can be of lower order, ok. If one or more joints are prismatic, IK is much simpler since the prismatic joint variable is not in terms of sines or cosines.

There is no general expression for workspace boundary, why? Because we do not have closed form analytic solutions for the 16^{th}-degree polynomial, we can only obtain the solutions of this 16^{th}-degree polynomial symbolic in numerically. So, how do I find the workspace? If all the roots of the 16^{th}-degree polynomials are complex, then this position and orientation is not in the workspace of the manipulator ok. So, this is an important concept.

So, I have given you a 16^{th}-degree polynomial, I go to MATLAB and find the solution of this 16^{th}-degree polynomial and see that all the roots are complex conjugates. So, then the point from which I got the 16^{th}-degree polynomial is outside the workspace.

All the inverse kinematic solutions and entire workspace may not be available due to joint limits and limitations of hardware. This is also very important concept ok. So, till now, we have not assumed that there are joint limits. So, in an actual robot ok, you cannot move the joint beyond certain limits. So, if you want to consider the joint limits, then the workspace is further reduced.

So, with this, we will stop this lecture. So, in this lecture, I showed you a way to eliminate from two polynomials $f(x, y) = 0$ and $g(x, y) = 0$. First x and then solve for y and then, ultimately solve for x again and then, I showed you that we can use this Sylvester's method or Sylvester's matrix, his idea behind eliminating and then finding the solutions for x and y in solving the inverse kinematics of the general 6R robot ok. So, with this, we will stop.