Robotics: Basics and Selected Advanced Concepts
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Lecture – 04
Inverse Kinematics of Parallel Manipulators

Welcome to this Robotics lectures on Basic and Advanced Concepts, ok. So, this is an NPTEL course on Robotics Basic and Advanced Concepts. In the last lecture we looked at mobility of parallel manipulators. In this lecture we look at the Inverse Kinematics of Parallel Manipulators ok.

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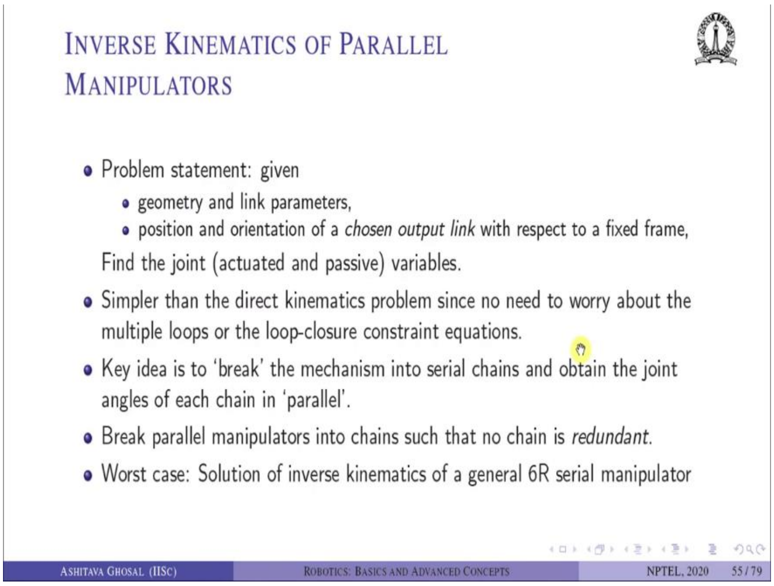

So, what is the problem statement in inverse kinematics of parallel manipulators? We are given the geometry and the link parameters and the position and orientation of a chosen output link with respect to a fixed frame and the problem is to find the joint variables. Now, in this case we have both actuated and passive joint variables.

This problem is simpler than the direct kinematics problem, since no need to worry about multiple loops or loop closure constraint equations ok. We will see that we can break up this problem into serial chains and obtain the joint angles in each chain in parallel ok. Meaning in parallel means the first chain and the second chain can be computed separately.

So, if you want to break this parallel manipulator into chains we have to make sure that no chain is redundant ok. Remember in a serial robot if you have more than 6 joints in 3D space or if you have more than 3 joints if the motion is planar the set of equations are redundant ok. So, we cannot find unique inverse kinematic solutions for a redundant chain. So, the worst case in the inverse kinematics of parallel manipulator is the solution of the inverse kinematics of a general 6R serial manipulator ok.

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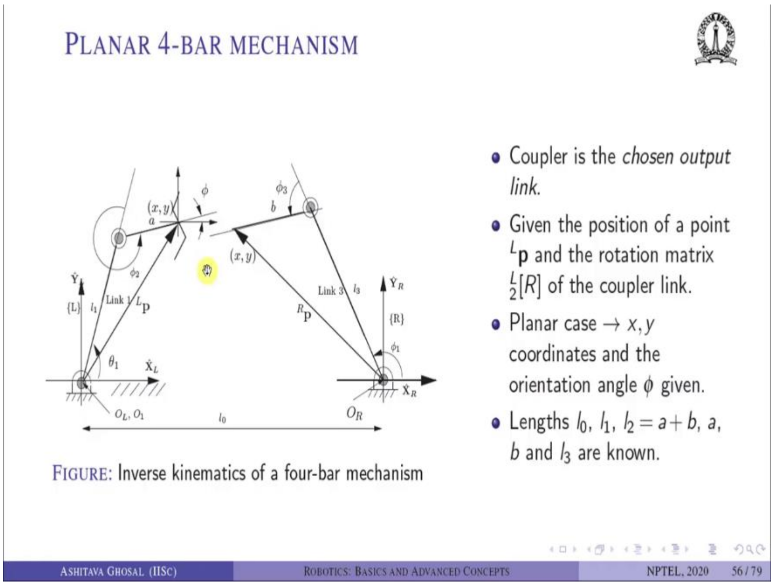

Let us go back to our very simple example of a planar 4 bar mechanism. So, we want to find the inverse kinematics of a 4 bar mechanism ok. This is in some sense the simplest possible way a problem it's also little bit cooked up nevertheless. So, we have broken this end which is the output link ok. We have broken the output link at some point a and b. The coupler is chosen as the output link and we are given the position vector of a point on the coupler link and the orientation of the coupler link ok.

With respect to let us say the $\{L\}$, which is the fixed left hand side coordinate system or the right hand side coordinate system ok. So, what do we have? So, we have broken it at this place as shown in this figure. We have a which is at the point of breaking in the coupler link at from the side it is b.

So, we are given x, y and this angle ϕ which is the orientation of this coupler link and what do we have to find out? θ_1 and ϕ_2. What else is given? l_1 is given ok, all the geometry is given l_1, l_0, l_3 and this a and b which together add up to l_2 is given.

So, to reiterate we are given the position of the point ${}^{L}\boldsymbol{p}$ and the rotation matrix ${}_{2}^{L}[R]$ of the coupler link ok. So, this is a planar case, x and y coordinates are given and the orientation angle phi is given. The link lengths l_0, l_1, $l_2 = a + b$ and l_3 are known.

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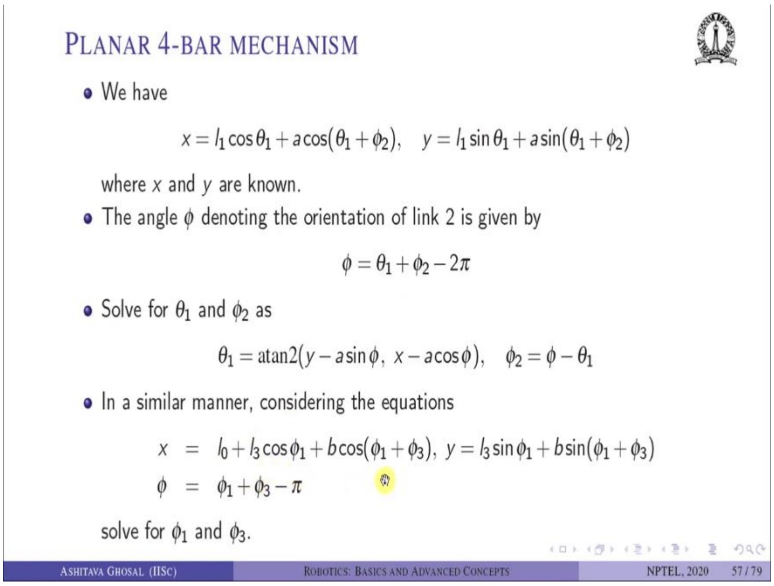

So, how do we solve this problem? It is very simple. We will look at x and y which is $x = l_1 \cos\theta_1 + a\cos(\theta_1 + \phi_2)$ ok, standard to our planar manipulator. And $y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \phi_2)$ ok. So, x and y are known. So, the angle ϕ denoted the orientation of link 2 is also given.

So, we can solve θ_1 and ϕ_2 from these two equations very simply ok, two equations in 2 unknowns. x is given, y is given θ_1 and ϕ_2 are the unknowns. So, we have done this in the lectures long time back when we looked at X, Y of some points on the planar 3R and then we did some square and adding and then we did some manipulation to show what is θ_1 and ϕ_2.

So, θ_1 is nothing but $atan2(y - a\sin\phi, x - a\cos\phi)$. So, $\theta_1 + \phi_2 = \phi + 2\pi$ and if you substitute back you will get $(x - a\cos\phi)$ and $(y - a\sin\phi)$. We can do the $atan2$ of that and we get θ_1 and ϕ_2 is nothing but $\phi - \theta_1$ ok. We can ignore this 2π business.

In a similar manner if we consider the other side we can show $x = l_0 + l_3\cos\phi_1 + b\cos(\phi_1 + \phi_3)$ and $y = l_3 \sin\phi_1 + b\sin(\phi_1 + \phi_3)$ and $\phi = \phi_1 + \phi_3 - \pi$. So, x, y and ϕ is given. So, what do we have to solve for? We have to solve for ϕ_1 and ϕ_3.

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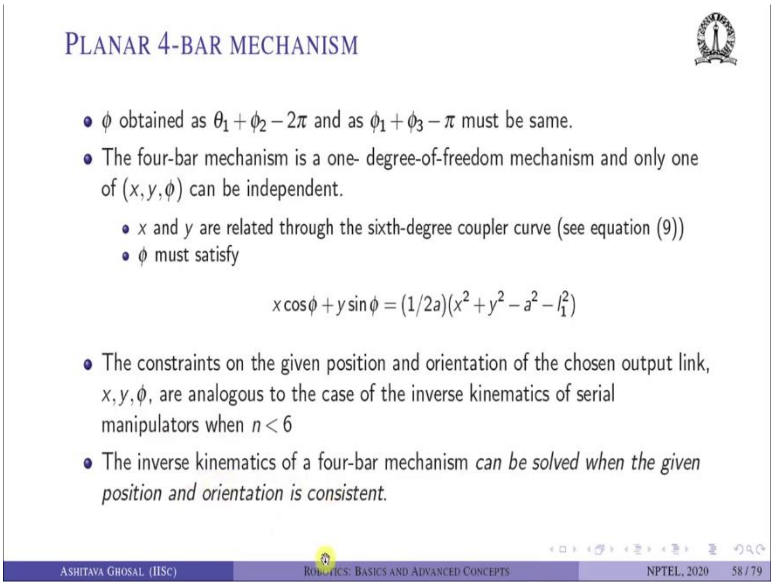

Again ϕ_1 obtained as $(\theta_1 + \phi_2 - 2\pi)$ and $(\phi_1 + \phi_3 - \pi)$ must be same ok, if you think about it. Whether you obtain from this side or from the other side both must be same. The 4 bar mechanism is a one degree of freedom mechanism and only one x, y or ϕ can be independent ok. So, we need to make sure that the x and y are related through the sixth degree coupler curve and ϕ must satisfy $(x \cos\phi + y \sin\phi)$ is given by this ok.

So, the constraints on the position and orientation of the chosen output link x, y, ϕ are analogous to the case of the inverse kinematics of serial manipulators, where$n < 6$. So, if you go back and remember in the SCARA case the motion was in 3D space.

However, the two angles of the end effector about X and Y they were 0, only the Z was allowed. So, in this case this x, y and ϕ which you give me of a point on the coupler they must satisfy the sixth degree coupler curve and also this equation ok. Otherwise, it is not a consistent problem.

We cannot solve an inverse kinematics of a point which is not on the coupler curve ok. So, the inverse kinematics of a four bar mechanism can be solved when the given position and orientation is consistent.

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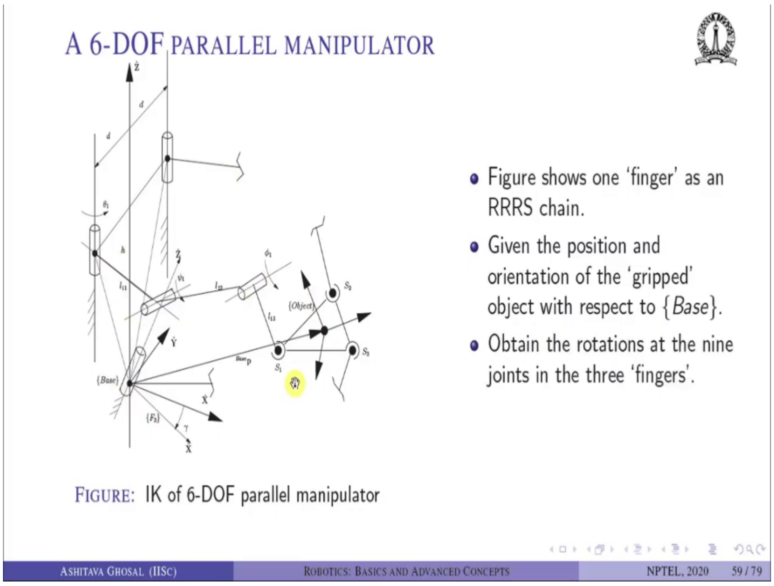

Let us look at a 6 degree of freedom parallel manipulator. So, in this case we have looked at this 3 fingered hand ripping an object. So, I am showing one finger as an R-R-R-S chain ok. So, the other two fingers are not shown. So, what is given? The position and orientation of the gripped object ok.

So, this is the object which is gripped. So, we are given some point on the gripped object. So, as we were looking at the centroid of the triangle formed by this and the X, Y and Z. X was along S_1 to S_2, Z was perpendicular to this and so on. So, the normal to the plane.

So, the orientation of this plane and the centroid of this triangle is given to you. So, what do we need to find out? We have to find out the angles θ_1 , ψ_1 and ϕ_1. Likewise we have to find our θ_2, ψ_2 and ϕ_2 and θ_3, ψ_3 and ϕ_3, we need to find out all the joint angles.

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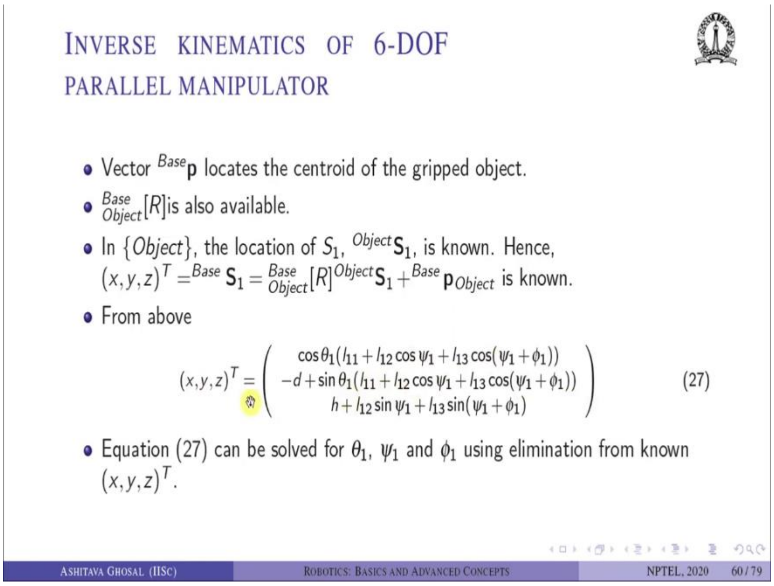

So, how do we do this? It is not very hard because we have just the position vector of a point which is let us say the centroid of the gripped object. The rotation matrix of the object with respect to base is also given.

So, we can find out that the x, y, z of the point is given by ${}^{Base}S_1$ which is the corner point where it is gripped that is nothing, but $\{Base\}$ to $\{Object\}$, $\{Object\}$ to S_1 ok. So, we are going to the centroid and coming back and this will be equal to $\{Base\}$ to $\boldsymbol{p}_{Object}$ ok. So, if you go back and see the figure.

So, this point x, y, z is nothing but a vector from here to here and from here to here and then that should be equal to this vector ok. So, I want to find out the position vector of each one of the spherical joints from the given data which is very simple we come to the given point of the centroid and then go back to this spherical joint which is exactly what is done here. And it turns out that the x, y, z is related to $\cos\theta_1$ and l_{11}, l_{12}, l_{13} and the angles $\cos\psi_1$, x coordinate and $\cos(\psi_1 + \phi_1)$.

The y coordinate is related to the sin of these angles, $\sin\theta_1$ and then $\cos\psi_1$ and $\cos(\psi_1 + \phi_1)$. And z coordinate is related to $\sin\psi_1$, $\sin(\psi_1 + \phi_1)$ ok. And then this d and h; the d is the distance between the two index finger and the middle finger and h is a distance along the vertical.

So, these are 3 equations in 3 unknowns x, y, z is known. x, y, z is known from the location of the object centroid of the object and the from the centroid to that point which is given.

And then we can find out θ_1 , ψ_1 and ϕ_1 from these 3 equations ok. It is very easy we can eliminate one at a time and obtained.

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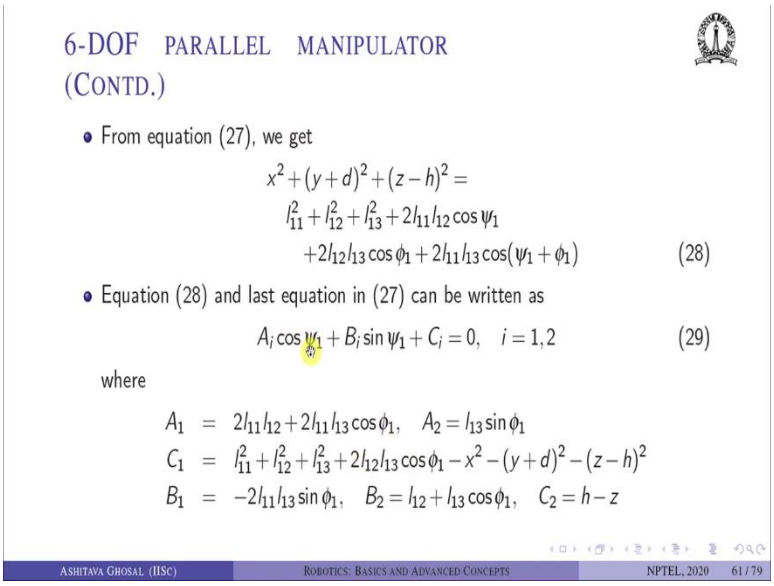

So, for example, if you do this operation $x^2 + (y + d)^2 + (z - h)^2$ we will see that we will get one equation in l_{11}^2 , l_{12}^2,l_{13}^2 and $\cos\psi_1$ and $\cos(\psi_1 + \phi_1)$. The last set component of this equation does not contain θ_1.

It contains only ψ_1 and $(\psi_1 + \phi_1)$. So, we got one equation in ψ_1 and $(\psi_1 + \phi_1)$ from this three of them and one which is the z component ok. So, we have two equations ok. The last equation can be written in only in terms of ψ_1 and ϕ_1. So, we have two equations in ϕ_1, ψ_1 and they can be written in this form.

So, $A_1\cos\psi_1 + B_1\sin\psi_1 + C_1 = 0$ and similarly i equals 2 and we can compute $A_1, B_1, C_1, A_2, B_2, C_2$. So, they will be all functions of only ϕ_1 because the ψ_1 is taken out here; thus usual trick when we learn how to do elimination of one variable from two equations.

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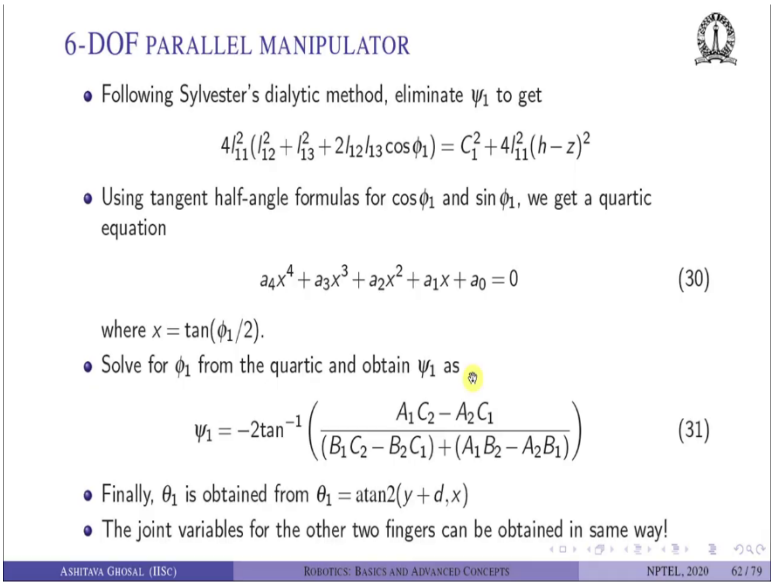

And then we use the Sylvester's dialytic method, which eliminate ψ_1 and we find out the determinant of Sylvester's matrix equal to 0. So, in this case that will lead to this equation, one single equation in ϕ_1 equal to all this constraints. So, C_1, l_{11}, z, h all these will show up ok.

So, it is a single equation in cosine ϕ_1 and we can go back and substitute the tangent half angle for $\cos\phi_1$ and $\sin\phi_1$ and we will get a quartic equation remember C_1 also had $\sin\phi_1$ sorry $\cos\phi_1$ ok. So, it looks like that there is only $\cos\phi_1$ here, but there is a $\cos^2\phi_1$ here ok.

So, when you do tangent half angle substitution $\cos\phi_1$ will become $(1 - x^2)$ divided by$(1 + x^2)$. So, square of that will give you a fourth degree or a quartic equation in $\tan\frac{\phi_1}{2}$ ok.

So, you can solve this from the ϕ_1 from this quartic and then in the process we obtained also for ψ_1. This is the standard elimination technique ok. So, when we write a_0, y, x squared and so on, when we obtained y we also obtained x in the quadratic or in the quartic.

So, ϕ_1 can be obtained by solution of this quartic equation and ψ_1 can be obtained by simply 2tan inverse of $(A_1C_2 - A_2C_1)$ and so on divided by $((B_1C_2 - B_2C_1) + (A_1B_2 - A_2B_1))$ ok. So, these contain ϕ. And finally, θ_1 can be obtained by $atan2(y + d, x)$ very simple because why.

So, θ_1 contains here something here and only one θ. So, x and y will contain θ_1. So, if you take d on that side and then you can manipulate very simply because the quantity in the bracket are same in both of them ok. So, we will get some $atan2(y + d, x)$ which is what is written here ok.

So, the joint variable for the other two fingers can be obtained in exactly the same way in parallel ok. So, we solved for the 3 angles in one finger, the other 3 angles in the other finger and the third 3 angles in the third finger can be obtained in exactly the same way.

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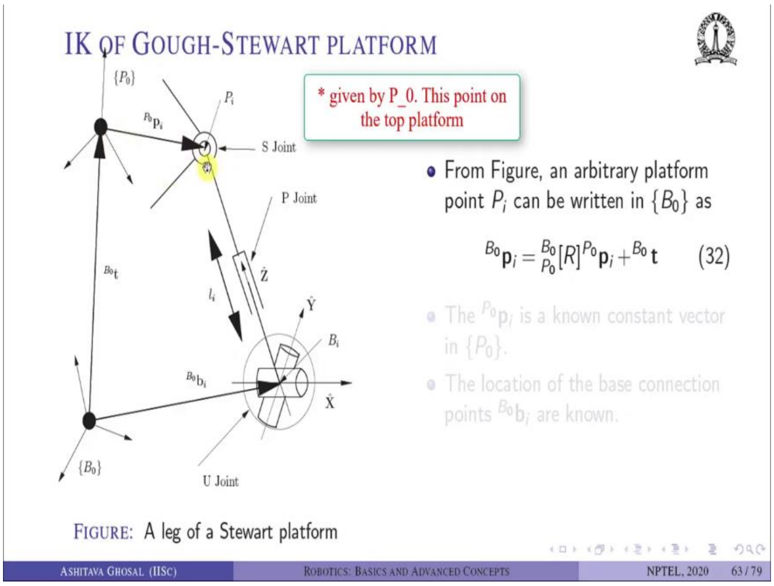

Let us look at this inverse kinematics of the Stewart-Gough platform this is a very famous platform 6 degrees of freedom Stewart platform. So, what do we have here? We are looking at one leg of the Stewart-Gough platform. So, there is a fixed coordinate system $\{B_0\}$. There is a coordinate system which is attached to the top platform which is given by $\{P_0\}$ .This point on the top platform which is an S joint is located by this vector $^{P_0}\boldsymbol{p}_i$.

A point on the base that there is a U joint is located by this vector $^{B_0}\boldsymbol{p}_i$ok. So, what you can see is this vector from fixed base to this point which is $^{B_0}\boldsymbol{p}_i$ is nothing but $^{P_0}\boldsymbol{p}_i$ which is this vector, pre multiplied by a rotation matrix which gives the rotation of the top platform with respect to the base plus this vector$^{B_0}\boldsymbol{t}$.

So, this plus this will be given by this. This is known and this is also given to us in the inverse kinematics problem. We are given the position and orientation of the top platform.

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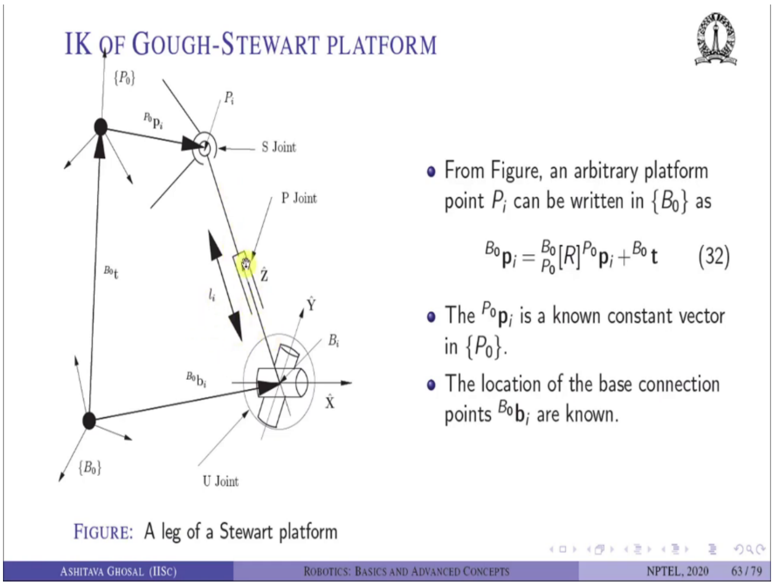

So, hence from this the location of all these $^{P_0}\boldsymbol{p}_i$, $^{B_0}\boldsymbol{p}_i$ everything is known. So, this point is known basically ok. So, the location of the spherical joint can be obtained in the fixed coordinate system. Now, as you can see this location of this spherical joint can also be written as a vector from the fixed coordinate system to the U joint which is $^{B_0}\boldsymbol{b}_i$ plus a vector along the leg which will be like l_i along some direction.

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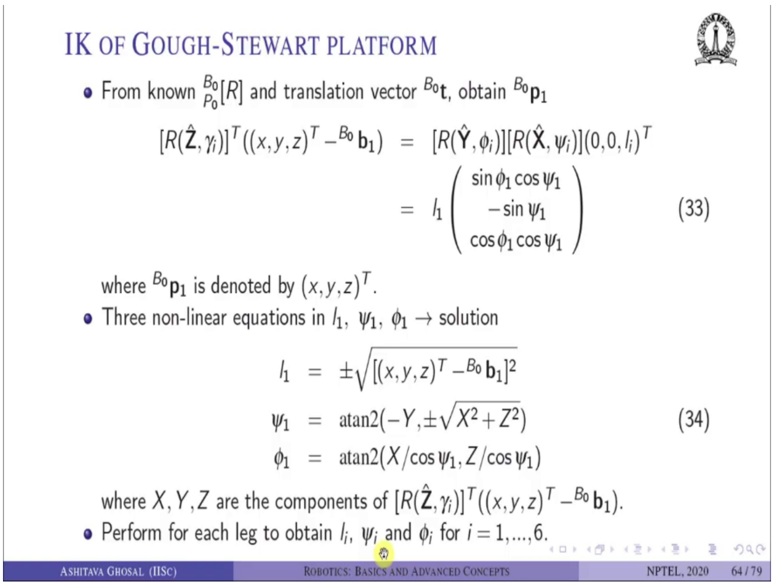

So, that is what is given here. So, this x, y, z which we have obtained from the coordinate of the S joint minus ${}^{B_0}\boldsymbol{b}_i$, will give a vector along this length ok. So, that is what is given as some rotation matrix because we need to multiply this vector and this is given as first one rotation about a the U joint another rotation that the U joint and a translation along the length ok.

So, let us go over it once more. So, this vector here can be written as this vector minus this vector. So, we know this vector from this argument that the position and orientation of the moving plate is known. So, from this vector if you subtract this fixed vector we will get a vector along this. Now, this vector is basically due to rotations two rotations at the U joint and one translation.

So, that is what is written here. If you multiply out these two rotation matrices one about Y by ϕ X by ψ and then this translation you will get l_1 the X coordinate is $(\sin\phi_1 \cos\psi_1)$, $-l_1 \sin\psi_1$ and $\cos\phi_1 \cos\psi_1$ ok. The left hand side is known because we know x, y, z, we know the location of the U joint in the fixed base and we know how to rotate this to obtain the correct direction ok.

So, we have 3 non-linear equations left hand side is equal to 3 one length which is we do not know and two angles ϕ and ψ. So, 3 equations in 3 unknowns which can be solved ok. It turns out $l_1$is nothing but$\pm\sqrt{\left[(x, y, z)^T - {}^{B}\boldsymbol{b}_1\right]^2}$. So, the length along the translation along the prismatic joint is nothing but the location of the spherical joint minus the location of the fixed base and the magnitude of that.

ψ_1 can be obtained as $atan2(-Y, \pm\sqrt{X^2 + Z^2})$ and $\phi_1 = atan2(\frac{X}{\cos\psi_1}, \frac{Z}{\cos\psi_1})$. So, and what is the x, y, z? They are nothing but the components of the left hand side. So, x, y, z are the components of this rotation matrix into this and we can do this for each leg. There are 6 legs in the Stewart platform and we can find out the translation at the prismatic joint and the rotations at the U joint or the hook joint at the base.

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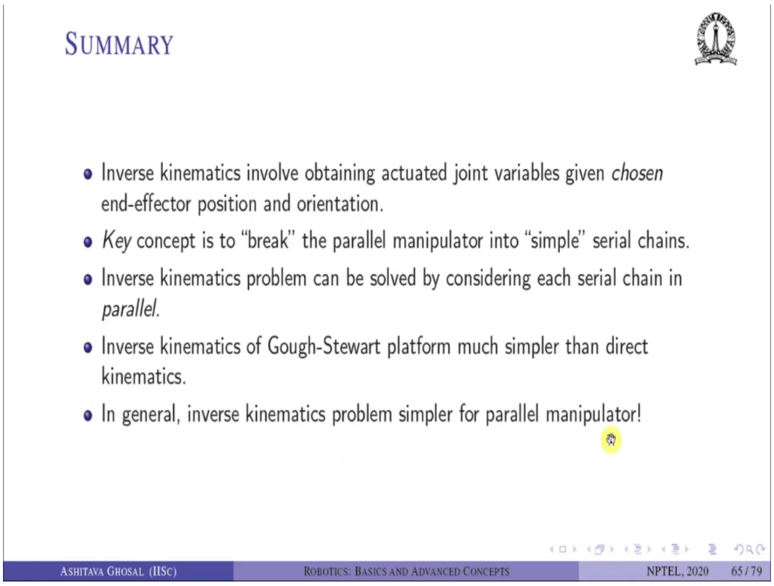

In summary the inverse kinematic involves obtaining actuated joint variables given chosen end effector position and orientation. Again the key concept is to break the parallel manipulator into simple serial chains.

The inverse kinematics problem can be solved by considering each serial chain in parallel. So, we solve each of them separately and independently that is what is meant by parallel ok. This is also one of the reason why these things are called parallel manipulators because we can solve the inverse kinematics in parallel.

So, the inverse kinematics of a Stewart-Gough platform is much simpler than the direct kinematics. In the next lecture we will look at the direct kinematics of the Stewart-Gough platform ok. This is one of the hardest problem to solve. In general inverse kinematics problems simpler for parallel manipulators ok.

In the serial manipulators the direct kinematics was very simple, inverse kinematics were harder. In the case of parallel manipulators the direct kinematics is much harder and the inverse kinematics is simple ok.

So, with this we come to the end of this lecture and the next lecture we will look at the Direct Kinematics of the Stewart Platform Manipulators.