Robotics: Basics and Selected Advanced Concepts
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Lecture – 15
Direct Kinematics of Stewart Platform Manipulators

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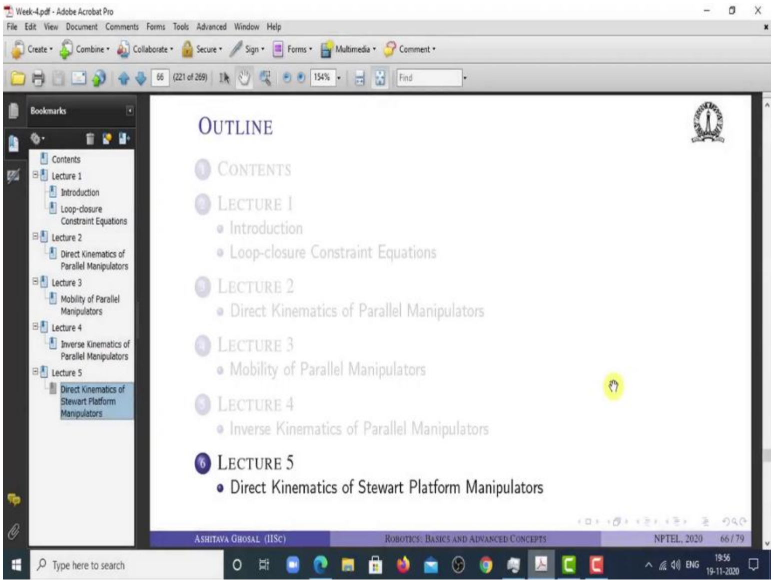

Welcome to this NPTEL lectures on Robotics: Basic and Advanced Concepts. In this last lecture in this sequence, we will look at the Direct Kinematics of Stewart Platform Manipulator.

The Stewart platform manipulator is one of the most well-known parallel manipulator parallel robot and lot of effort has gone into the direct kinematics of Stewart platform manipulators. So, we will very briefly look at the direct kinematics problem of a Stewart platform manipulator and how to go about finding the solution to the direct kinematics problem.

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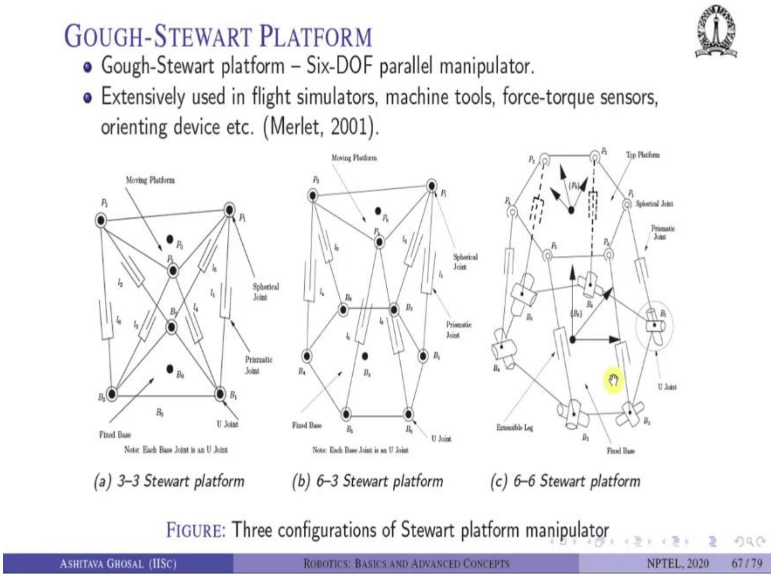

So, the Gough-Stewart platform sometimes it is known as the Gough-Stewart platform is a Six degree of freedom parallel manipulator ok. It is extensively used in flight simulators, machine tools, force-torque sensors, orienting device and it has been all these applications and many more are shown in this book by Merlet in 2001. So, there are 3 main Stewart platform manipulators, 3 main configurations.

The first one is something called as a 3-3 Stewart platform. So, ok. So, the 3-3 Stewart platform consist of a moving platform with 3 distinct points let us say P_1, P_2, P_3 and 3 distinct base points which is B_1, B_2, B_3 ok.

And the base points are connected to the moving platform points in two ways using two legs. So, for example, there is a B_3 P_3 with the sliding joint l_3, but P_3 is also connected to B_1 and the variable here is l_4.

Likewise, if you look at the point P_1, which is connected to B_2 by one leg, which is P_1 B_2 and likewise P_1. is also connected to B_1. with a sliding joint l_1. So, as you can see there are 6; l_1, l_2, l_3, l_4, l_5 and l_6 ; 6 actuator joints. The top platform points are always spherical joints, whereas the bottom platform points are actually U joints.

Here it is shown as spherical joints just for the sake of easy drawing, but they are actually P joints sorry U joints with 2 degree of freedom. The second well known configuration in a Stewart platform is the 6-3 Stewart platform ok. So, in a 6-3 Stewart platform the base has 6 distinct points ok.

So, in the previous case the base is form of a triangle. In the second case of 6-3 Stewart platform the base is a hexagon most of the time ok. So, we have B_1, B_2, B_3, B_4, B_5 and B_6; 6 base points at the fixed base, but the moving platform has 3 distinct points. So, as a result the P_1 moving platform point is connected to both B_1 and B_2 ok. So, B_1 P_1 is one such leg, B_2 P_1 is another leg, and for B_1 P_1 the sliding joint is l_1 denoted by l_1 and B_2, P_1 the sliding joint is l_2.

Likewise B_3 P_2, B_4 P_2 and with l_3 and l_4 and B_5 $P_3$and B_6 $P_3$with sliding joint or the joint variable as l_5 and l_6. Here also the top platform points are spherical joints, but the bottom platform points are Hooke joints although it is not shown as Hooke joints. In the last well known configuration is the 6-6 Stewart platform, in which case the top platform has 6 distinct points; P_1, P_2, P_3 all the way P_6 and the bottom platform points fixed base platform points are also 6 of them.

So, which is why it is called 6-6, previous one was called 6-3 and this is called 3-3. So, here also we have connection between $B_1 and\ P_1$ with a prismatic joint, similarly $B_2\ and\ P_2$ with a prismatic joint and so on, or in this figure there is a slight mistake. So, this should be actually B_2 P_2 and so on $B_3 and\ P_3$ whereas, labeling is wrong here.

It does not matter. Basically, we are having 6 points and the base platform on in a hexagon and 6 points on the top platform also in a hexagon and we connect each one of these points using a S P U chain.

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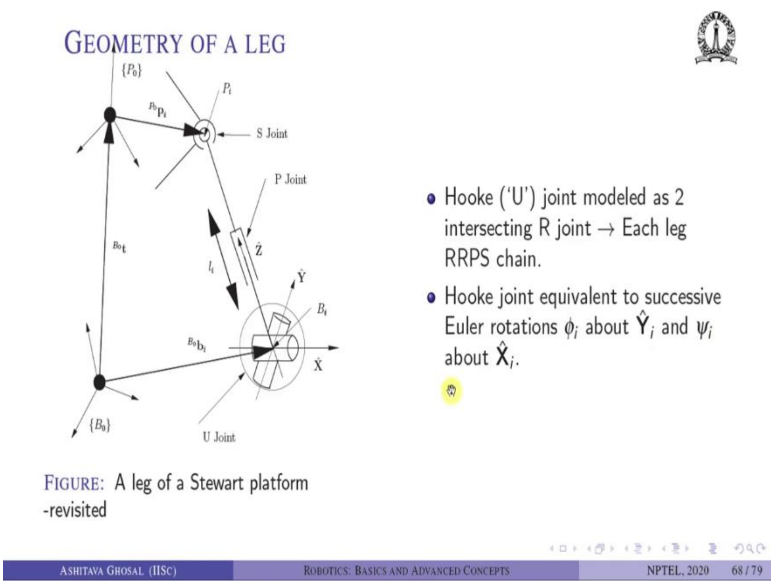

So, let us look at the geometry of one single leg. So, basically we have a fixed coordinate system in the base which is labeled as B_0. There is a moving coordinate system on the top platform which is labeled with P_0. So, we have a vector from B_0 to this P_0 as B_{0_t} and we have another vector in the moving coordinate system, which is $P_{0_{P_i}}$.

So, any one of the spherical joints is, so, P_1, P_2, P_3 and so on. Likewise there is a vector which locates the U joint at the base with this fixed vector $B_{0_{b_i}}$ as you can see it is with respect to B_0. And we have another vector which goes from this fix this U joint to the S joint and this is along this prismatic joint axis.

So, l_i is the variable for that. So, this Hooke joint is basically equivalent to two successive Euler rotations ϕ_I about Y axis and Ψ_i about X axis, and there is this spherical joint which has 3 possible degrees of freedom. So, these are the main vectors and main variables which describe the leg of a robot of a Stewart platform robot.

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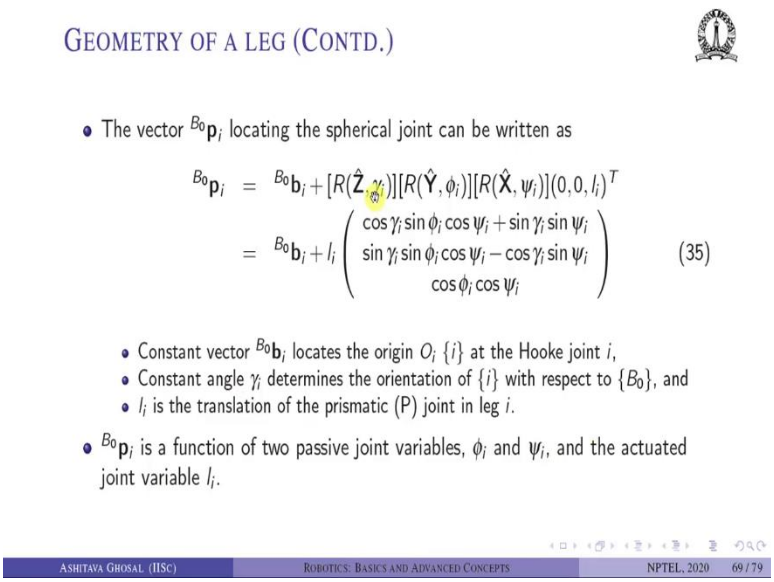

So, the vector $B_{0_{p_i}}$ ok. So, I want to find out this vector from B_0 to p_i ok. So, this can be written either a sum of these two vectors or it can be written as sum of these two vectors ok. So, I can go to the U joint and from U joint to this or I can go to the fixed platform point and go to this. So, let us start with going from the base. So, $B_{0_{p_i}}$can be written as $B_{0_{b_i}}$and as I said there are two Euler rotations about Y and X and then there is the translation along l_i. Now, this quantity rotation matrix Z about γ_i basically tells you that this point is not necessarily always along the X axis ok. So, there are 6 of these points.

So, I have to take this vector $B_{0_{b_i}}$ each vector which is a constant vector, but rotate it about the Z axis to reach the 6 points. So, basically this γ angles would be something like 60 degrees if it is a regular hexagon. So, hence this position vector of the i th spherical joint is the vector along the base plus 3 rotations and one translation.

So, out of these 3 rotations this γ_i is constant ok. So, we can expand this and rewrite it as $B_{0_{b_i}}$. So, remember this is a constant vector into l_i. $l_i$is the translation along the prismatic joint and then we have this complicated expressions of cos and sin of the 3 angles; γ, ϕ and Ψ ok.

So, if you work it out this is what you will get. So, the Z component does not have gamma you can show or you can think a little bit and you can realize that the Z component is only a function of ϕ_i , Ψ_I and l_I ok. So, the constant vector $B_{0_{b_i}}$ locates the origin of the ith coordinate system at the Hooke joint and the relationship between the base coordinate system and the Hooke joint ith coordinate system there is a rotation which is Z_i, ok.

The constant angle γ_i determines the orientation of the ith coordinate system with respect to B_0 and l_I is the translation of the prismatic joint along leg i. So, $B_{0_{P_i}}$ is actually a function of two passive variables ϕ_i and Ψ_I and the actuated joint variable l_I ok, γ_i is constant.

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DK OF 3–3 CONFIGURATION

- 6 legs are $B_1 - P_1$, $B_1 - P_3$, $B_2 - P_1$, $B_2 - P_2$, $B_3 - P_2$ and $B_3 - P_3$.
- 6 actuated and 12 passive variables $\rightarrow$ 12 constraint equations needed.
- Three constraints: Distances between P_1, P_2 and P_3 are constant (similar to 3-RPS).
- Point P_1 reached in *two ways*: 3 vector equations or 9 scalar equations.

$$
\begin{aligned}
{}^{B_0}\mathbf{b}_1 + \overrightarrow{B_1P_1} &= {}^{B_0}\mathbf{b}_2 + \overrightarrow{B_2P_1} \\
{}^{B_0}\mathbf{b}_2 + \overrightarrow{B_2P_2} &= {}^{B_0}\mathbf{b}_3 + \overrightarrow{B_3P_2} \\
{}^{B_0}\mathbf{b}_3 + \overrightarrow{B_3P_3} &= {}^{B_0}\mathbf{b}_1 + \overrightarrow{B_1P_3}
\end{aligned}
$$

- 16^{th} degree polynomial in tangent half-angle obtained after elimination (Nanua, Waldron, Murthy, 1990).

So, let us look at the direct kinematics of the 3-3 configuration. So, there are 6 legs which are connected by $B_1\ P_1$, $B_1\ P_3$, $B_2\ P_1$. As I showed you there are 3 connection points at the bottom and 3 at the top. So, there are from each connection point of the base there are two legs which are coming out and similarly to the connection point at the top there are two legs which are coming in and this is the arrangement.

So, for 6 legs each leg there are 2 passive joint variables and 1 actuated joint variables ok. So, for the 6 legs we have 6 actuated joint variables and 12 passive joint variables. So, basically we need 12 constraint equations to solve the direct kinematics. Remember in direct kinematics of parallel robots we have active joints and passive joints and in order to solve for the passive joints in terms of the active joints we need constraint equations.

So, these are 12 in number because there are 12 passive joint variables. So, the 3 constraints distances between $P_1, P_2\ and\ P_3$ are constant ok. So, they are very similar to the 3 RPS robot which we had looked at. So, between $P_1\ and, P_2$ is some distance which is fixed between $P_1\ and, P_3$ another fixed and between $P_2\ and\ P_3$ another fixed ok. So, 3 top platform points.

Secondly this P_1 can be reached in two ways ok. So, the top point can go from $B_{0_{b_1}}$, $B_1\ P_1$, or it can go to these $B_{0_{b_2}}$ and then $B_2\ P_1$ ok. Remember there are two points two vectors coming to one of the top points ok. So, this is one such equation vector equation. Likewise, we can reach the second point again in two ways and these are the. So, the two vectors must be equal and the third point must be can also be reached in two ways with two vectors.

Because there are two vectors entering one point of the top and this is given this way ok. So, there are these are vector equations. So, there are 3 of them ok. So, this is the X, Y, Z 3 components. So, there are 3 into 3; 9 scalar equations in these 3 vector equations ok. So, we have 9 here, and then we have distance between P_1, P_2, P_3 and P_3, P_1 as constant. So, 9 + 3; 12.

So, we have managed to generate 12 constraint equations and we have 12 passive joint variables and then we can now use elimination methods like the Sylvester's Dialytic method to solve for these 12 passive joint variables. And it turns out that this was solved in 1990 by these 3 researchers. They obtained a 16th degree polynomial in tangent half angle of one of the passive joint variables after elimination.

So, they use the Sylvester's method, generated a single monomial with one joint variable, one passive joint variable and it was a 16 degree polynomial. So, which means what? There are 16 possible solutions of that passive joint variable ok. And this solves the direct kinematics of the 3-3 configuration. Why? Because once one of the passive joint variables is known then by back substitution, we can find all the other 11 joint variables ok.

And hence the even 6 actuated joint variables we have solved for the 12 passive joint variables and hence we know the complete configuration of the 3-3 Stewart platform robot.

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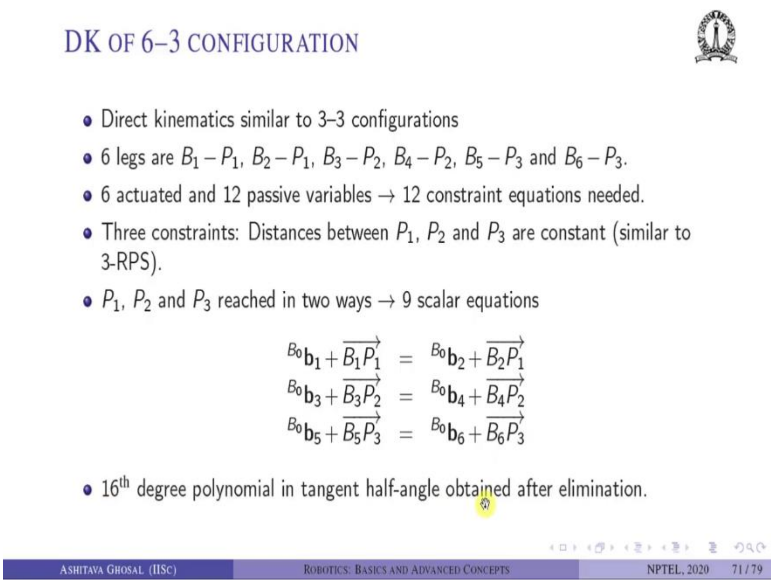

Now, let us look at the direct kinematics of the 6-3 configuration. So, in the 6-3 there were 6 connection points at the base and 3 connection points at the top. So, the direct kinematics is sort of similar to the 3-3 configuration ok. So, we have 6 legs; $B_1\ P_1$, $B_2\ P_1$, $B_3\ P_2$ and so on.

Again we have 6 actuated and 12 passive joint variables and again to solve the direct kinematics problem we need to obtain 12 constraint equations. So, the 3 constraints are exactly the same as earlier. We have distance between $P_1, P_2\ and\ \ P_3$ are constant, similar to 3 RPS or similar to the 3-3 configuration.

Also P_1, P_2, P_3 can be reached in two ways because there are 3 connection points on the top, but there are 6 two leg vectors, which is reaching one of the connection points of the top. So, again we can write $B_{0_{b_1}}$ plus $B_1\ P_1$ this vector should be equal to $B_{0_{b_2}}$ plus $B_2\ P_1$ ok. Likewise, $B_{0_{b_3}}$, $B_3\ P_2$, $B_{0_{b_4}}$, $B_4\ P_2$ and $B_{0_{b_5}}$, $B_5\ P_3$, $B_{0_{b_6}}$, $B_6\ P_3$ ok.

So, we have these 3 vector equations, each of them has X, Y and Z components. So, we have 9 scalar equations here and again we have 3 of them earlier constant distance between two prismatic, two spherical joints and hence we have the correct number of passive joints and the number of constraint equations.

So, again this was solved and we obtained again a 16th degree polynomial in the tangent half angle and this was also obtained after elimination using Sylvester's method. So, this also completes a problem for the 6-3 configuration. Remember once we find one of the passive joint variable using elimination theory of elimination we can back substitute and solve for all other passive joints the actuated 6 actuated joints are already known. So, we know completely the configuration of the 6-3 Stewart platform manipulator, ok.

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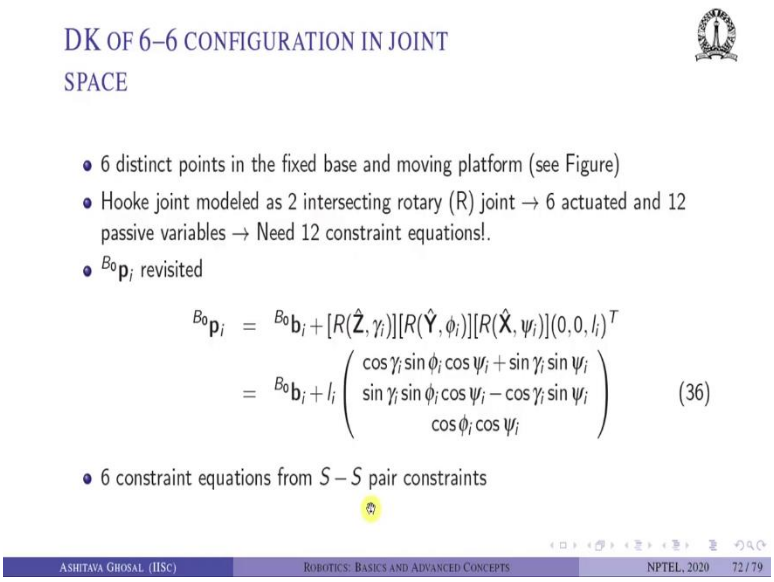

Let us finally, look at the direct kinematics of the 6-6 configuration in joint space. The reason I am saying joint space we will see there is also another way to obtain the direct kinematics of the 6-6 configuration in Cartesian space. So, we have in the 6-6 configuration we have 6 distinct points in the fixed base and moving platform ok.

I showed you the figure earlier, but we remember there is a very well known you know parallel robot with 6 connection points of the base and 6 connection points of the top. So, at the base there are Hooke joints which has modeled as 2 intersecting rotary joints. So, basically again 2 successive rotations.

So, there are 6 actuated variables and 12 passive variables again. So, 6 into 2 for the Hooke joints and 6 actuated joint variables. So, you have 12 passive variables in all and 6 actuated variables. So, we need 12 constraint equation again to solve the direct kinematics problem.

So, let us go back and see this point P which is on the top platform where this spherical joint is. So, that can again be written as go along the base via distance $B_{0_{b_i}}$ plus then these 3 rotations as I said about Y, X and Z. So, this is a constant and followed by a translation of along the z axis of the prismatic joint.

So, we can again simplify this. So, there is a constant term $B_{0_{b_i}}$and then l_i into this cos and sin of you know various angles. So, this I have we had discussed before. So, I can locate the spherical joint on the top platform in terms of l_i and these 3 angles γ_i, ϕ_i, Ψ_I sort of clear, it is not very hard to imagine. Out of which γ_i is constant and the angles in the Hooke joint ϕ and Ψ are need to be solved for. So, we have 6 constraint equations from the S-S pair. So, there are 6 spherical joints on the top. So, between 1 and 2 is a constant distance between 2 and 3 there is a constant distance 3 and 4 and so on.

So, there can we can derive 6 constraint equations constant distance constraint equations from the 6 S-S pairs. We are looking for 12 of 12 constraint equations to solve for the 12 passive joint variables. So, 6 are already known.

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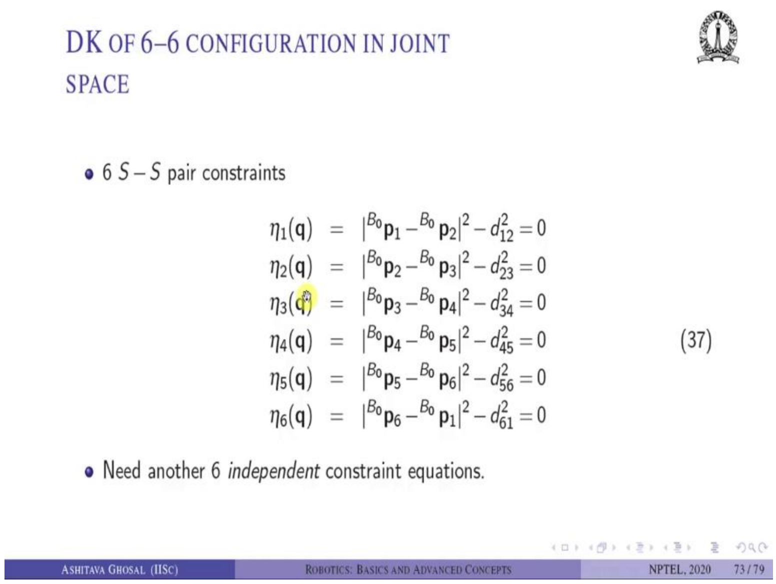

We can also so, these are the 6 S-S pair constraints. So, P_1 *to* P_2 the distance is d_{12}^2, P_2 *and* P_3 is d_{23}^2. So, these are distances. So, we have 6 such constraint equations containing l's and the two Hooke joint variables ok. So, the left hand side contains all the passive and active joint variables. So, we need another 6 independent constraint equations.

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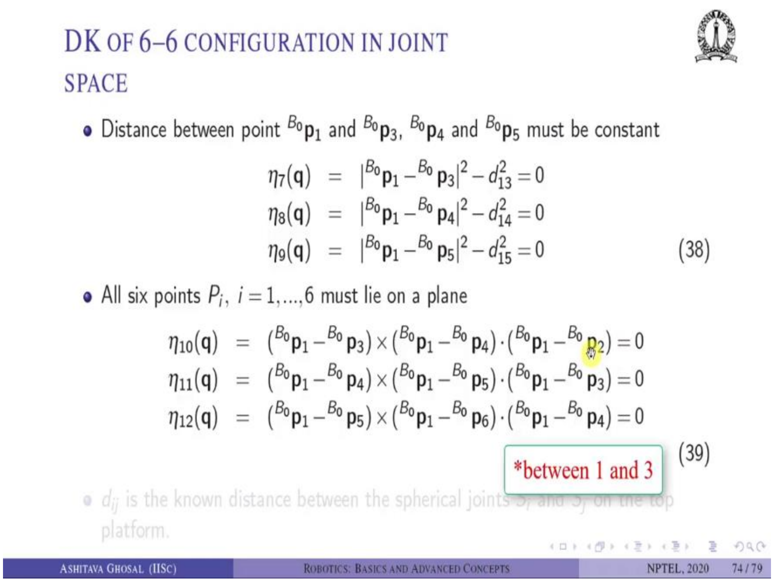

So, we can also look at the distance between P_1 *and* P_3 ok, P_1 *and* P_4 and P_1 *and* P_5 ok. So, on the top platform. So, they also must be constant ok. So, between P_1 *and* P_3 is d_{13}^2 between P_1 *and* P_4 is d_{14}^2 and between P_1 *and* P_5 it is d_{15}^2 ok. We have already taken into 1 to 6 previously. So, we can get 3 additional constraint equations if we consider the points 1 and 3, 1 and 4 and 1 and 5. So, skipping basically two in between.

The third set of constraint equation is that all these 6 points P_i must lie on a plane ok. So, the vector from P_1 *and* P_3 and P_1 *and* P_4 they form cross product of these two vectors from the normal to the plane and this should be perpendicular to the vector between P_1 *to* P_2 . So, you can think of you take 3 points 1, 2 and 3 and so, sorry 1, 2, 3 and 4.

So, between 1 and 2 there is a vector between 1 and 4 there is a vector and between 1 and 2 there is another vector all this would lie in a plane. Likewise we can go that the vector from P_1 *to* P_4 one vector $P_1 - P_4$ cross product with $P_1 - P_5$ dot product with $P_1 - P_3$ should be equal to 0, right.

So, these 3 equation tells you that these 6 points are lie in a plane. So, I have 3 dot product equations. So, these are scalar equations. These are 3 again scalar equations. So, we have 6 additional scalar equations ok.

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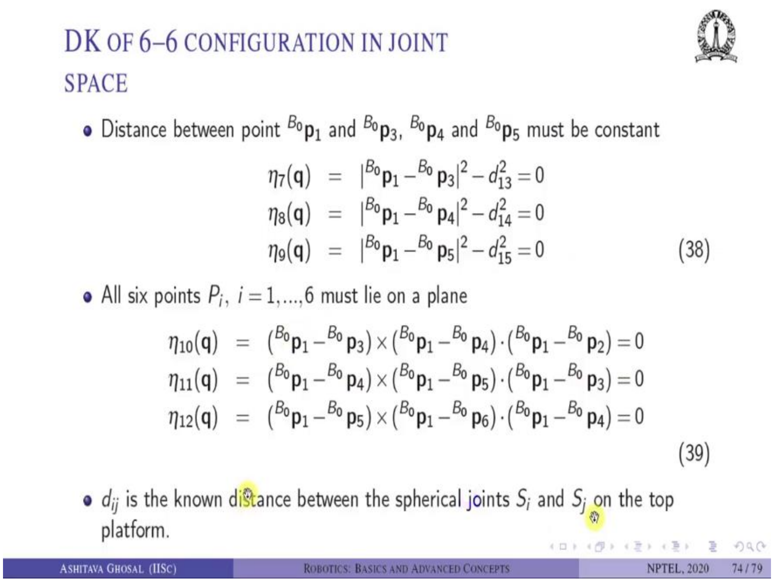

So, the distance d_{ij} is known between the spherical joint S_i and S_j on the top platform. So, we have now 6 plus 6; 12 constraint equations required.

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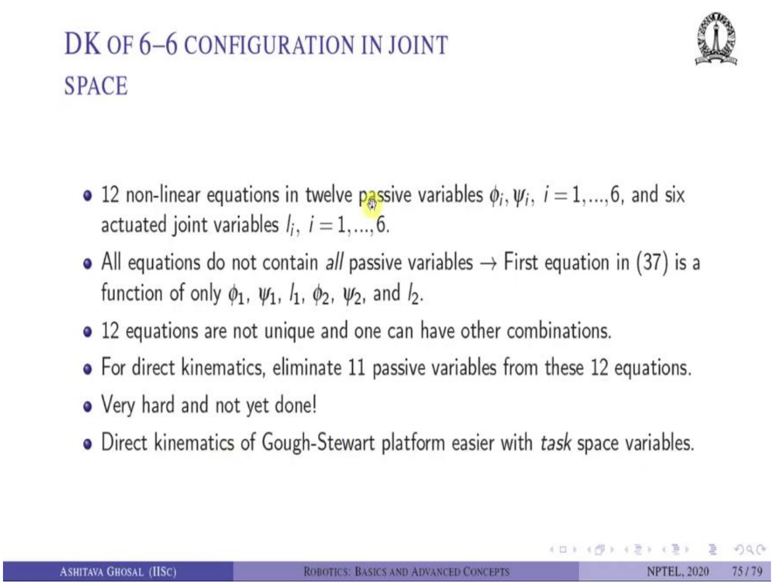

So, we have 12 non-linear equations in twelve passive variables ϕ_i, Ψ_i; i equals 1 to 6 and 6 actuated joints variables l_i, i equals 1 to 6 ok. Now, all equations do not contain all the joint variables ok. So, for example, the distance between first spherical and the second spherical joint will not contain say let us say l_3, l_4 and all other passive variables from the other legs. It will contain only the variables coming from the first and the second leg.

So, as I said it is a function of only ϕ_I, Ψ_I , l_1, ϕ_2, Ψ_2 and l_2. The other important thing is all these 12 equations are not unique ok. Why is it not unique? We had looked at the distance between 1 and 3, 1 and 4 and so on ok. I could have also taken the distance between 2 and 4 and 2 and 5 and so on. So, there could be other possible combinations.

So, nevertheless we can take this 12 constraint equations and we can hope to eliminate 11 of these passive joint variables from this 12 equation at least theoretically and get one single equation in one of the passive joint variables. Unfortunately, this is very hard and not yet done to the best of my knowledge ok. It is not available in literature. Unlike the 3-3 and the 6-3 we cannot solve the direct kinematics easily in the joint space using joint space variables.

Because we need to eliminate 11 variables from 12 constraint equation it is not very simple ok. So, the direct kinematics of Gough-Stewart platform is much easier when we use task space variable. So, joint space variable at least we can formulate the problem, but it is sort of hard to eliminate 11 from variables from 12 equations. So, let us look at the direct kinematics of Gough-Stewart platform with task space variables ok.

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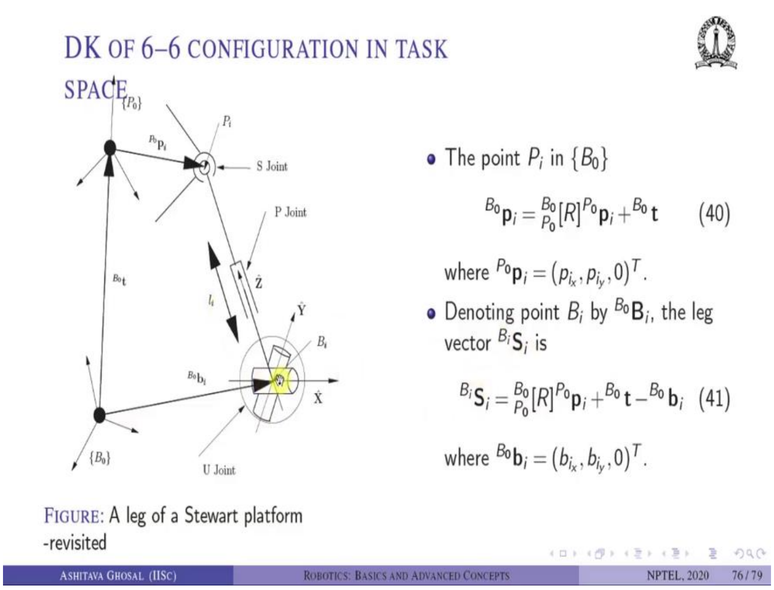

So, what is the problem here? Again let us consider a leg of the Stewart platform. So, this point P_i $B_{0_{p_i}}$ in the fixed reference coordinate system can also be written as a vector B_{0_t} to the center of the top platform or where the origin of the coordinate system in the moving platform is located plus a vector from P_0 to P_i ok.

Now, this vector is in the moving coordinate system. So, we need to pre multiply by a rotation matrix ${}^{\boldsymbol{B}_0}_{P_0}R$ and then this is $P_{0_{p_i}}$ plus this vector. So, these 6 points lie in a plane. So, basically this$P_{0_{p_i}}$ will have only X and Y coordinates locally ok. So, what do we have we can locate this vector from the translation vector to the origin of the moving coordinate system plus a vector in the moving coordinate system ok.

So, what is given? The position ok; we need to for somehow obtain the relationship between this t and rotation matrix this is what are the important things rest of it is constant ok. $P_{0_{p_i}}$is some constant number with two components. So, we denote the point B_i by $B_{0_{B_i}}$ fixed base. So, and the leg vector by $B_{0_{S_i}}$. So, from B_i to S_i this is the leg vector.

So, now, we can show that this $B_{i_{S_i}}$. So, this leg vector from the center of the Hooke joint to the spherical joint is given by ${}^{\boldsymbol{B}_0}_{P_0}R$ this vector, this vector plus this vector minus this vector. So, this vector is nothing, but this plus this, which is this minus this $B_{0_{b_i}}$ and suitably we can see that all these vectors are in the same coordinate system and this vector which locates the intersection of the two Hooke in the Hooke joint is also a X and Y component only because this is also lying in a plane.

So, this is the starting point of the direct kinematics analysis of a 6-6 Stewart platform. So, I need to know this vector $B_{i_{S_i}}$ in terms of this vector, this vector and this vector ok.

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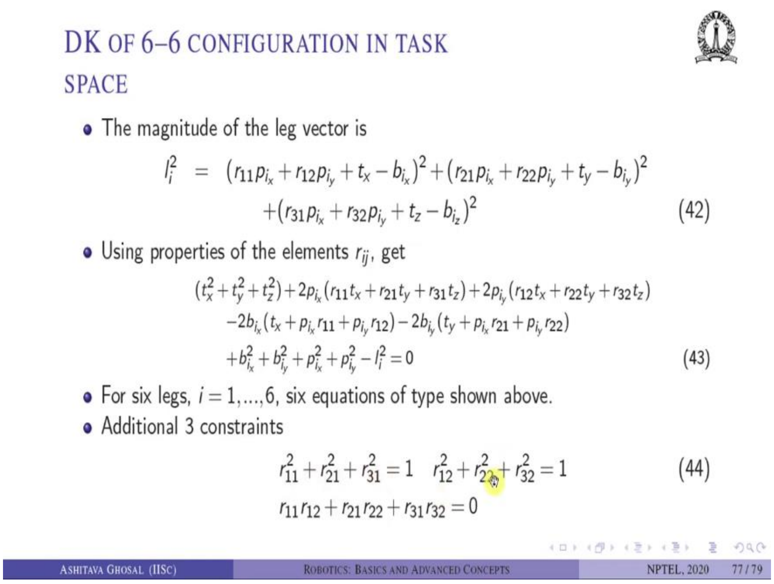

So, this magnitude of this leg vector if I do the square and addition of these 3 components it is a, it is a vector with 3 components X, Y and Z. So, the magnitude of this leg vector will be l_i^2. And if you expand it you can see that it contains terms like the elements of the rotation matrix r_{11} , r_{12} and it contains the position vector p_{i_x}, p_{i_y}, it also contains the translation vector t_x t_y t_z and so on and it also contains b_{i_x} the location of the base point.

So, using the properties of the elements of this rotation matrix this can be simplified and rewritten in this form. This is important. What are the properties of the rotation matrix? That the first column is a unit vector and the first and the second column is orthogonal and so on.

Second column is unit vector, third column is unit vector and it is a orthonormal matrix, the rotation matrix. So, if you use the properties of these elements r_{ij}, you can simplify and what you will get is a term which is like this; $t_x^2 + t_y^2 + t_z^2$.

Then something into t_x t_y t_z into r_{ij} into p_{i_x} then something into p_{i_y} into again t_x t_y t_z with some other r_{ij} and something into $t_x + p_{i_x} r_{11} + p_{i_y} r_{12}$ only t_x multiplied by b_x and minus $2b_{i_y}\left(t_y + p_{i_x} r_{21} + p_{i_y} r_{22}\right)$ and finally, one more term which is $b_{i_x}^2 + b_{i_y}^2 + p_{i_x}^2 + p_{i_y}^2 - l_i^2$, ok.

So, for each of these six legs we will get six equations of this type. So, we have six equations of this type. We also have 3 constraint equations which are that the first column of the rotation matrix is unity; it is a unit vector ok. Then r_{12}^2, r_{22}^2, r_{32}^2 is also 1 and then these 2 vectors are perpendicular 2 columns are perpendicular.

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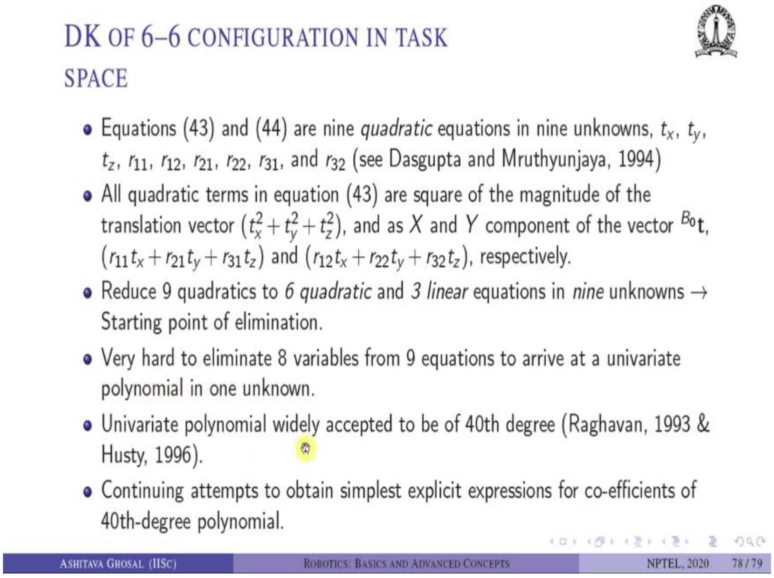

So, equation 43 and 44, what are equations 43 and 44? So, this is 43. So, there are 6 of these and 44 there are 3 of these ok. So, there are nine such equations we will get nine quadratic equations in unknowns t_x t_y t_z and the elements of the rotation matrix r_{11} , r_{12}, r_{21} , r_{22}, r_{31} , r_{32}, this was developed by Dasgupta and Mruthyunjaya in 1994,

So, all the quadratic terms in 43 are square of the magnitude of the translation vector ok. So, we have $t_x^2 + t_y^2 + t_z^2$ and as X and Y components of the vector B_{0_t} which is like this, $r_{11}t_x + r_{21}t_y + r_{31}t_z$ like this and so on. So, we can reduce this 9 quadratic into 6 quadratic and 3 linear equations in nine unknowns ok.

So, instead of taking all these 9 quadratic equations we can manipulate them we can looked at it and see that it can be reduced to 6 quadratic in 3 linear equations and this is the starting point of the elimination procedure. So, very hard to eliminate 8 variables from 9 equations to arrive at a univariate polynomial.

Remember in the joint space we have 12 constraints and we needed to eliminate 11 variables, 11 passive joint variables, whereas, in the Cartesian space we have only 9 quadratics not 12 constraint equations ok. So, this work has been going on for a while. So, it is sort of hard to eliminate 8 variables from 9 equations better than 11 from 12 equations in joint space ok. However, it has been done ok.

People have shown that this univariate polynomial is a 40th degree authors like Raghavan and Husty in 1996. So, I can obtain a single polynomial in one of these variables ok, out of this 9 equations and 9 variables and obtain a 40th degree polynomial.

So, people are still working on it to you know once in a while papers will come out which say that this are the simpler expressions or explicit expressions of the coefficient of the 40th degree polynomial, but it is now accepted that this will be a 40th degree polynomial.

What does it mean? That there are 40 possible configurations. If I give you the 6 translations of the 6 joints the Stewart platform can be assembled in 40 possible configuration maximum ok. Many solutions could be imaginary and in that case it will be lower than that, but the upper limit is forty possible configurations.

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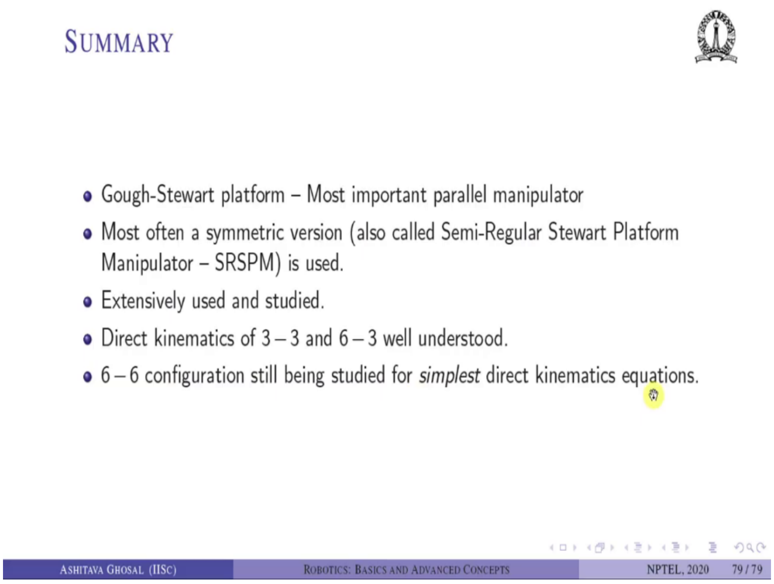

So, in summary the Gough-Stewart platform is one of the most important parallel manipulators it is used extensively in many applications. Most often a symmetric version is used ok. It is also sometimes called as an SRSPM. So, basically symmetric configuration is sort of similar to the 3-3 configuration ok.

But, the top point is not coincident, but it is slightly spaced out ok. So, if you draw a line from the origin of the moving coordinate system you know plate in the top, so, the two points are closed to each other by separated by some two angle 2 alpha. Let us say. So, this SRSPM has been widely studied and extensively used ok.

So, the direct kinematics of 3-3 and 6-3 are very well understood, 16 degree polynomial people have solved it. And the 6-6 configuration is still being studied for simplest direct kinematics equations. So, we know it is a 40th degree polynomial , but people still think that the coefficients of the polynomial could be simplified and so, we need to know what are the simplest possible kinematical equations ok.

So, with this I will stop. In the next week we will. So, we have finished the kinematics of parallel robots.

Thank you.