Robotics: Basics and Selected Advanced Concepts
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Lecture - 19
Velocity and Static Analysis of Manipulators

Welcome to this NPTEL course on Robotics, Basics and Advanced Concepts. In the next five lectures, we look at the velocity and static analysis of manipulators.

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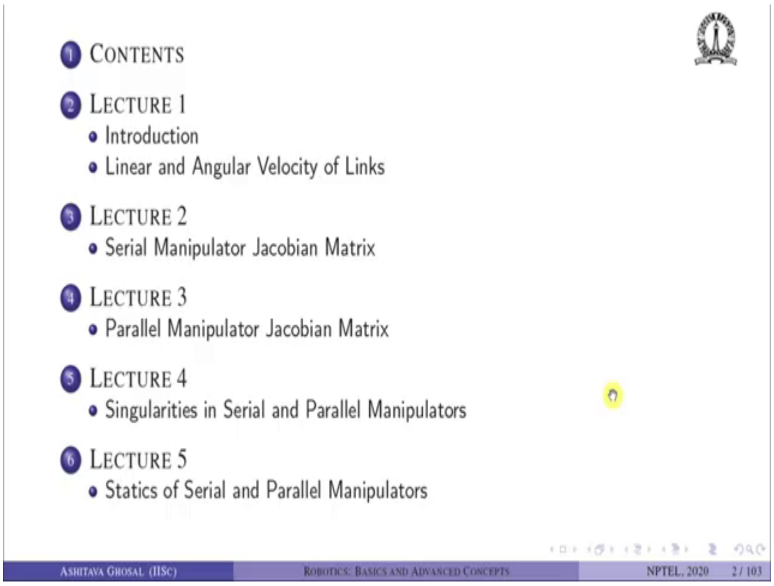

With the first lecture, we will introduce the topic of velocity analysis, then we will look at Linear and Angular Velocity of Links. In the second lecture, we look at this very important concept called Serial Manipulator Jacobian Matrix.

In the third lecture, we look at Parallel Manipulator Jacobian Matrix. In the fourth lecture, we look at the Singularities in Serial and Parallel Manipulators. And in the last lecture with the sequence, we look at the Statics of Serial and Parallel Manipulators.

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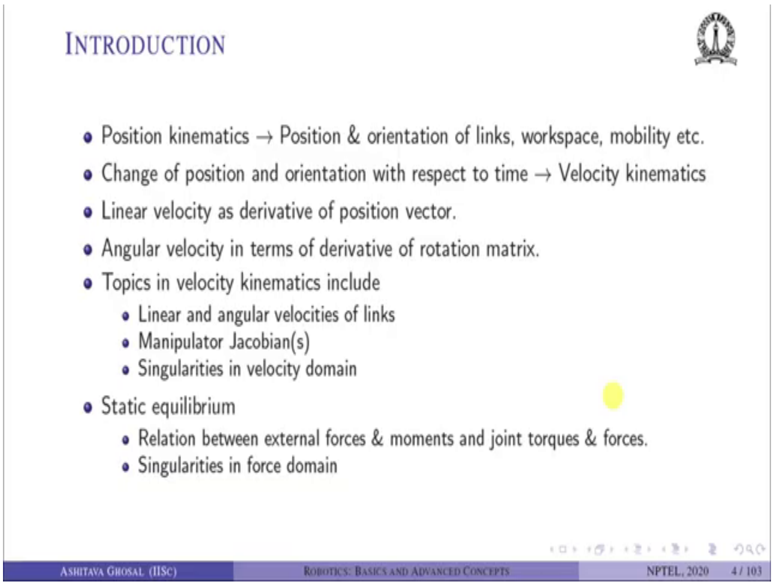

So, let us continue in introduction. So, basically till now we have looked at position kinematics. What is position kinematics? We have looked at position and orientation of links, the concept of workspace, the concept of mobility and so on. Now, we will look at the change of position and orientation with respect to time.

So, this is also called velocity kinematics. So, basically we want to find the linear velocity as a derivative of position vector, and you also want to look at the angular velocity of a rigid body or a link in terms of derivative of a rotation matrix. So, the topics in velocity kinematics include linear and angular velocity of links in a robot, the manipulator Jacobians – there are a few of them, and singularities in velocity domain.

In static equilibrium, we look at the relationship between external forces and moments, and the joint torques and forces. And we will look at singularities in the force domain.

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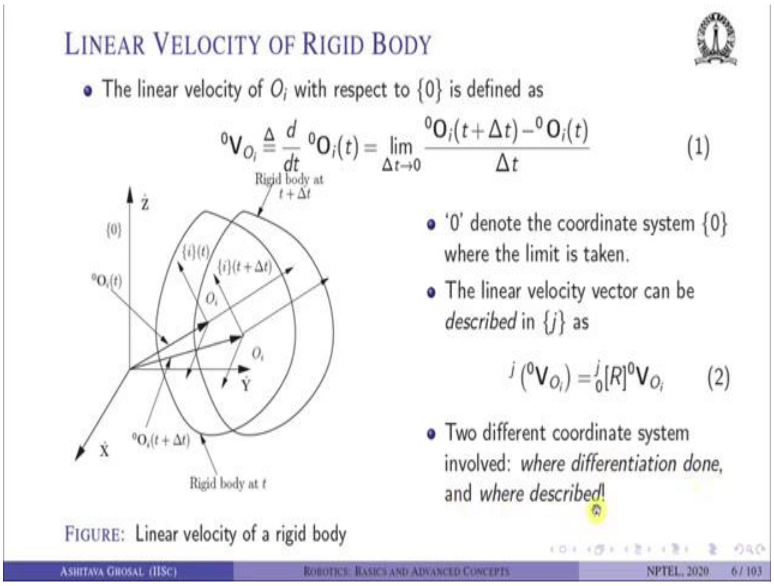

So, in lecture 1, we are going to start with the linear and angular velocity of links. So, the linear velocity of a point O_i which is the origin of a coordinate system fixed to the rigid body $\{i\}$ ok, can be defined as the derivative of the position vector. So, from a fixed reference coordinate system $\{0\}$ ok, labeled as $\{0\}$. So, this is the position vector ${}^0O_i(t)$. And as you can see I have drawn the X, Y and Z-axis at some instant of time t, and then the X, Y and Z-axis at drawn at an instant of time $(t + \Delta t)$.

So, what you can see is this axis at parallel ok. So, there is no change in the orientation of this rigid body. So, the rigid body at t is this potato looking shape; and the rigid body at $(t + \Delta t)$ is a translated version of the same potato looking shape ok.

So, let us go back to the very basic definition of a derivative. So, the derivative is nothing, but $\left(\lim_{\Delta t \to 0} \frac{{}^0O_i(t+\Delta t) - {}^0O_i(t)}{\Delta t}\right)$. So, this is the most basic definition of the derivative of a vector. So, this is what we will call that is the velocity of the point O_i in the rigid body $\{i\}$ ok.

So, the '0' denotes the coordinate system reference coordinate system $\{0\}$, where the limit is taken. Why we need to worry about this? Because this is a subtraction of two vectors, and we need to make sure that the vectors are in the same coordinate system when we subtract two vectors ok. And then we have to say that the velocity is with is with respect to that coordinate system; in this case the $\{0\}$ coordinate system.

The linear velocity can also be described in any other coordinate system you take any other vector, if we pre multiply this linear velocity vector with the rotation matrix ${}^{j}_{0}[R]$, so that we will denote it as ${}^{j}({}^{0}\boldsymbol{V}_{O_i})$ with in the jth coordinate system. So, sometimes this 0 will be omitted, and we will just write ${}^{j}\boldsymbol{V}_{O_i}$.

So, basically what it means is the linear velocity is first computed in the {0} coordinate system, but it is described in an another coordinate system {j} ok. So, this two ideas that the derivative is taken in one coordinate system, but it can be written in another coordinate system its very useful ok. So, just to repeat, a two different coordinate systems involved one where the differentiation is done, and one where the vector is described.

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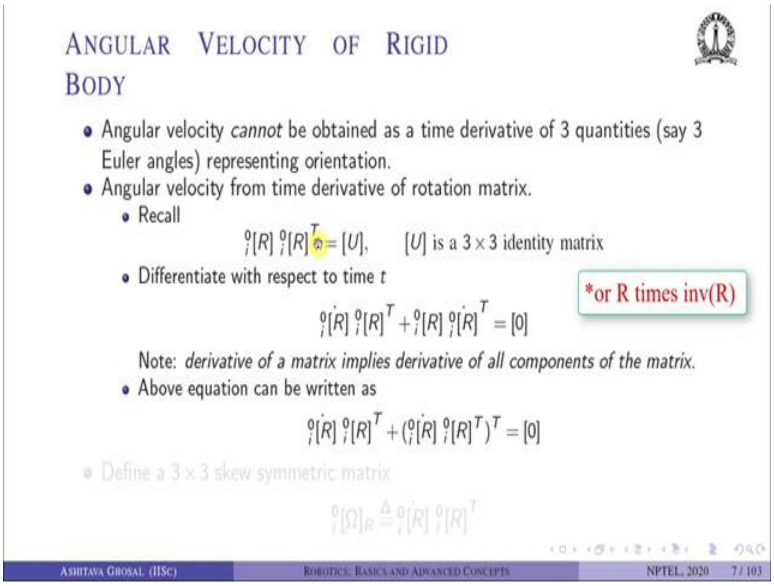

The angular velocity of rigid body, however, cannot be described or obtained as a time derivative of 3 quantities ok. So, we had shown that a rotation matrix or the orientation of a rigid body can be described by 3 Euler angles ok. However, the angular velocity cannot be a straight forward derivative of those 3 Euler angles ok. Unlike the position vector, because the position vector as x, y and z; and the velocity vector is nothing but dx/dt, dy/dt, and dz/dt.

So, the angular velocity from the time derivative of rotation matrix requires a few steps. So, let us start. So, remember that ${}^{0}_{i}[R]{}^{0}_{i}[R]^{T}$ is identity ok. Or ${}^{0}_{i}[R]{}^{0}_{i}[R]^{-1}$, inverse is same as transpose is identity matrix. So, let us differentiate this matrix equation with respect to time. So, we use chain rule. And we write ${}^0_i[\dot{R}]{}^0_i[R]^T + {}^0_i[R]{}^0_i[\dot{R}]^T = [0]$. The derivative of identity is 0 ok.

So, what do you mean by derivative of a matrix? It is derivative of each and every term in the matrix it is implies the derivative of components of some matrix. So, this above equation can be written as $\left({}^0_i[\dot{R}]{}^0_i[R]^T + \left({}^0_i[\dot{R}]{}^0_i[R]^T\right)^T = [0]\right)$. So, this follows the rules of matrix multiplication and transpose. And this will be equal to [0].

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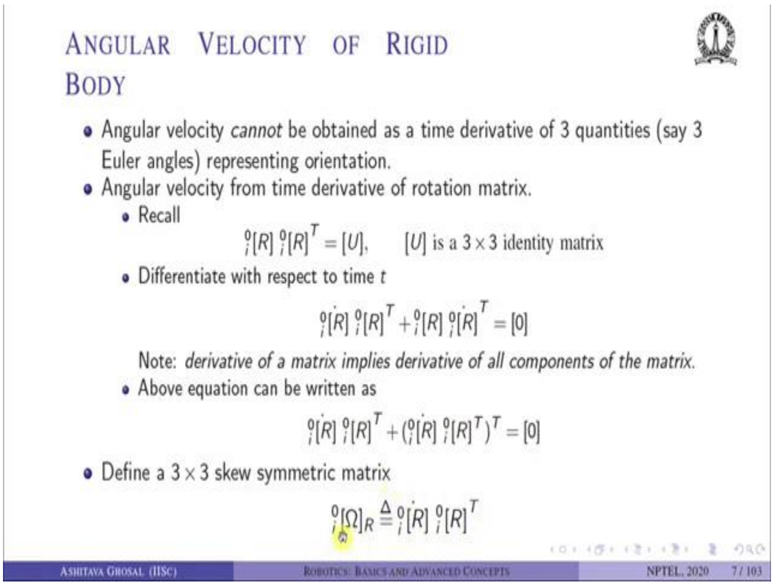

So, now what you can see is that this matrix ${}^0_i[\dot{R}]{}^0_i[R]^T$ is skew-symmetric. Why, because this plus its transpose is equal to [0] ok. So, we denote this skew-symmetric matrix by ${}^0_i[\Omega]_R$. The 0 here implies that it is with respect to the $\{0\}$ coordinate system; i here implies that it was the matrix corresponding to the ith rigid body or the ith coordinate system; and R here comes from the fact that we started with the right multiplication.

So, we took ${}^0_i[R]{}^0_i[R]^{-1}$, ${}^0_i[R]{}^0_i[R]^T$ is identity ok. So, we started with the right multiplication. Later on we will see that we put a started with the left multiplication also.

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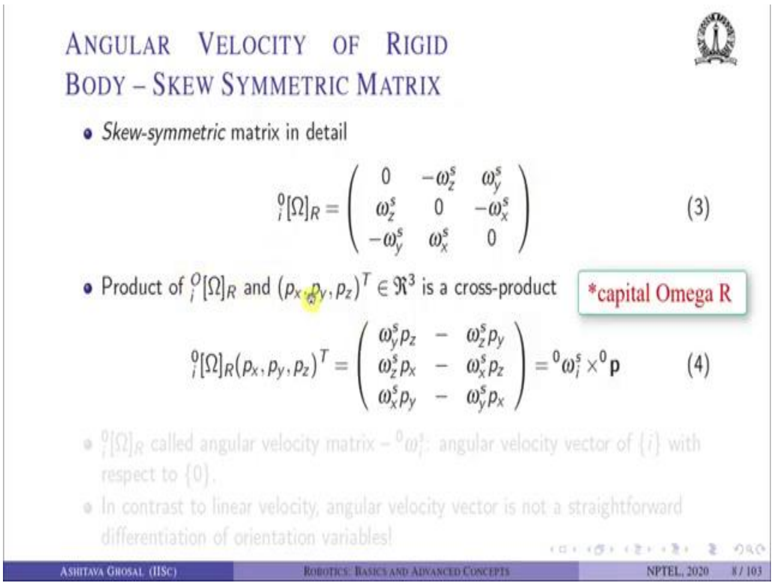

So, the skew-symmetric matrix in detail can be written as this matrix. So, the diagonals are 0. And the (1,2) term is $-\omega_z^s$, (1,3) term is ω_y^s, and the (2,3) term is $-\omega_x^s$. And why do we write it like this? Because the product of this matrix ${}^{0}_{i}[\Omega]_R (p_x, p_y, p_z)^T$ is a cross product in 3D space ok. So, ${}^{0}_{i}[\Omega]_R (p_x, p_y, p_z)^T$ can be written like this. And this is ${}^{0}\omega_i^s \times {}^{0}\boldsymbol{p}$.

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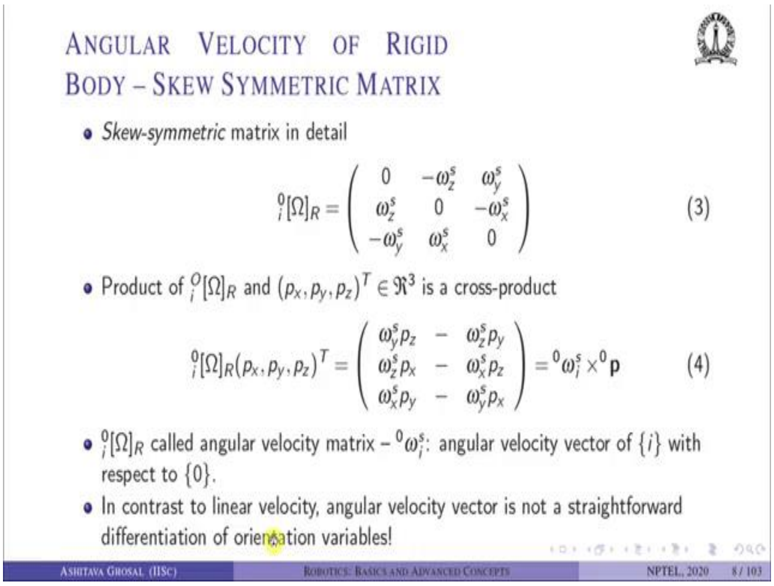

So, let us see what we are trying to say. So, ω_x^s, ω_y^s, ω_z^s are the components of the angular velocity vector extracted from the skew-symmetric matrix ok. The s here stands for something called as a space fixed angular velocity vector which we will look at in more detail later ok. So, the components of ${}^0\omega_i^s$ are the angular velocity components of coordinate system {i} with respect to {0} ok.

So, as you can see in contrast to the linear velocity, angular velocity vector is not a straight forward differentiation of the orientation variables. We do not even have the orientation variables anywhere.

We have started from a rotation matrix with 9 components. We have done ${}^0_i[\dot{R}]{}^0_i[R]^T$, showed that ${}^0_i[\dot{R}]{}^0_i[R]^T$ is a skew-symmetric matrix. And from the skew-symmetric matrix, we have recovered or extracted components ω_x^s, ω_y^s, and ω_z^s.

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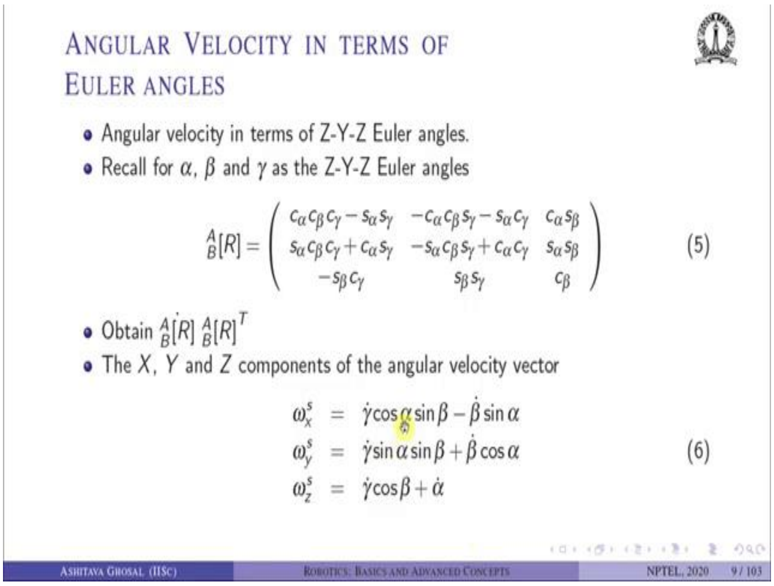

So, let us take an example. So, suppose we want to find the angular velocity in terms of Z-Y-Z Euler angles ok. Remember in a previous lecture, we had looked at this Z-Y-Z Euler angles. So, basically it consists of a rotation about the Z-axis by an angle α, about the Y-axis by an angle β, and again rotation of γ about the Z-axis.

So, the rotation matrix in product of three rotation matrixes about Z, Y, and Z, and eventually we will get this complicated rotation matrix ok. It consists of $c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma$ and so on. So, the R(3,3) term is $\cos\beta$– angle β ok.

So, R(3,2) term is $\sin\beta\sin\gamma$ and so on ok. So, this we have done already it is just a recapitulation. We can find out ${}^{A}_{B}[\dot{R}]{}^{A}_{B}[R]^{T}$ because ${}^{A}_{B}[R]$ is there we can take the derivative of each term post multiply by ${}^{A}_{B}[R]^{T}$, this matrix simplify use the fact all the trigonometric identities that we know off.

And eventually we can extract ω_x^s, ω_y^s, ω_z^s ok. And it turns out it is related to $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$ that is of course there because derivatives are there, but it also depends on $\cos\alpha$, $\sin\beta$, $\sin\alpha$ and so on.

So, for example,

$$\omega_x^s = \dot{\gamma}\cos\alpha\sin\beta - \dot{\beta}\sin\alpha$$

$$\omega_y^s = \dot{\gamma}\sin\alpha\sin\beta + \dot{\beta}\cos\alpha$$

$$\omega_z^s = \dot{\gamma}\cos\beta + \dot{\alpha}$$

So, most important thing is that ω_x^s, ω_y^s, ω_z^s are not directly related to $\dot{\alpha}$, $\dot{\beta}$ and $\dot{\gamma}$, but it also contains cosine and sin of the angles ok.

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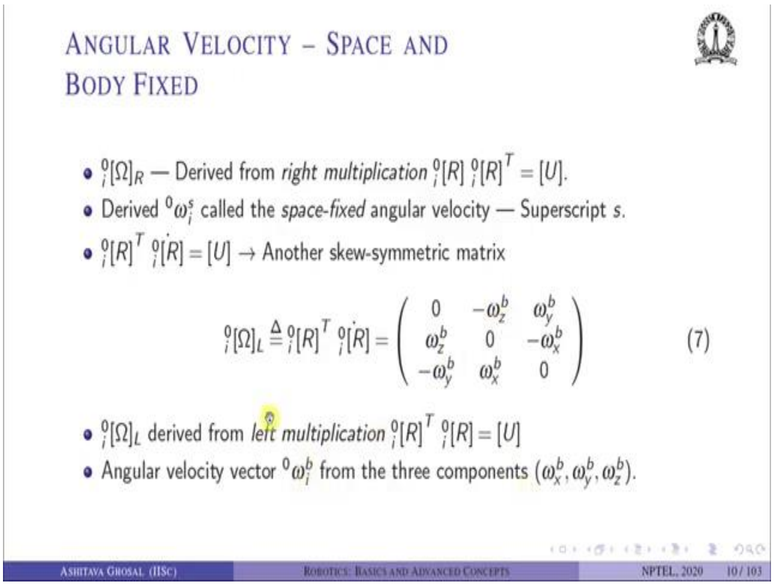

So, we have derived this ${}^{0}_{i}[\Omega]_R$ from the right multiplication. Right multiplication meaning we started with ${}^{0}_{i}[R]{}^{0}_{i}[R]^{T}$ is identity ok. From this, we derived an angular velocity vector as I said we have used a superscript s. So, this is called as the space fixed angular velocity vector ok.

We see some geometrical interpretation of space fixed angular velocity vector little later. We could have also started with left multiplication which is ${}^0_i[R]^T\,{}^0_i[R]$ is identity ok. And then we could have derived a skew-symmetric matrix which is ${}^0_i[R]^T\,{}^0_i[\dot{R}]$ ok. This, should be [0].

So, this is another skew-symmetric matrix which is ${}^0_i[R]^T\,{}^0_i[\dot{R}]$ ok. And then you could have extracted an angular velocity vector ω_x^b, ω_y^{sb}, ω_z^b, but now with the super script b ok. So, this is the left multiplication ok. And it is basically started from ${}^0_i[R]^T\,{}^0_i[R] = [U]$. So, just ignore this equal to $[U]$. So, this is a skew-symmetric matrix ${}^0_i[R]^T\,{}^0_i[\dot{R}]$. And it came from ${}^0_i[R]^T\,{}^0_i[R]$ which is identity again.

So, the angular velocity vector ω_x^b, ω_y^b, ω_z^b with the superscript b of the rigid body i with respect to the 0$^{\text{th}}$ coordinate system can be obtained from this ${}^0_i[R]^T\,{}^0_i[\dot{R}]$ ok.

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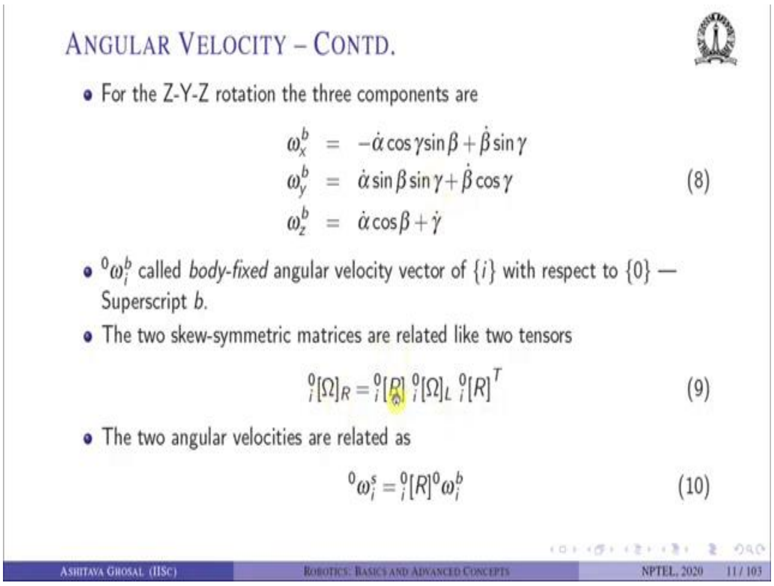

So, let us compute the x, y, z components of the angular velocity vector with starting from ${}^0_i[R]^T\,{}^0_i[\dot{R}]$ ok. So,

$$\omega_x^b = -\dot{\alpha}\cos\gamma\sin\beta + \dot{\beta}\sin\gamma$$

$$\omega_y^b = \dot{\alpha}\sin\gamma\sin\beta + \dot{\beta}\cos\gamma$$

$$\omega_z^b = \dot{\alpha}\cos\beta + \dot{\gamma}$$

So, what you can see here is that the ω_x^b, ω_y^b and ω_z^b which we have derived from this ${}_i^0[R]^T\,{}_i^0[\dot{R}]$ is very much different from when we did the other ${}_i^0[\dot{R}]\,{}_i^0[R]^T$ ok.

So, for example, the ω_z^b component is $(\dot{\gamma}\cos\beta + \dot{\alpha})$, whereas in when we did this right multiplication, we have left multiplication ${}_i^0[R]^T\,{}_i^0[\dot{R}]$, then we get $(\dot{\alpha}\cos\beta + \dot{\gamma})$. So, this vector ${}^0\omega_i^b$ of rigid body i with respect to $\{0\}$ coordinate system is all the body fixed angular velocity of i^{th} rigid body with respect to the 0th rigid body. So, it has a super script b.

The two skew-symmetric matrix which is ${}_i^0[\Omega]_R$ and omega ${}_i^0[\Omega]_L$, they are related like any other tensor. So, ${}_i^0[\Omega]_R = {}_i^0[R]\,{}_i^0[\Omega]_L\,{}_i^0[R]^T$. And you can also show that the angular velocities are related as follows. That this fixed angular velocity vector is related to the body fixed angular velocity vector pre multiplied by a rotation matrix which describes the rigid body i with respect to the 0th coordinate system.

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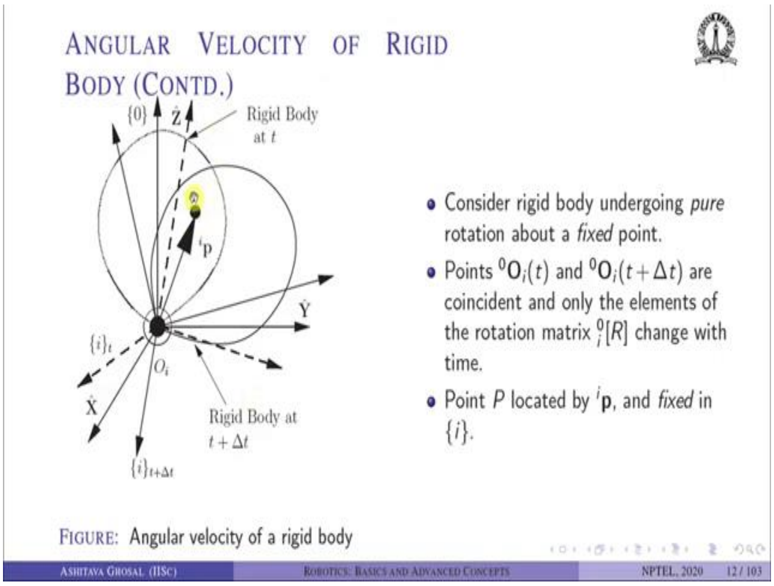

So, now let us look at a little bit of a geometrical interpretation of this rigid body, and the two different concepts of angular velocity vector. So, what do we have? We have a rigid body at time t and we have a rigid body at time $(t + \Delta t)$. So, at time t, we have X, Y and

Z. At time $(t + \Delta t)$, the axis have changed ok. The origin is at the same place because we are only interested in the orientation of the rigid body. So, we will assume that the rigid bodies undergoing pure rotation.

So, the points $O_i(t)$ and $O_i(t + \Delta t)$ are coincidents and only the elements of the rotation matrix change with time ok. So, let us consider a point P located in the i^{th} coordinate system and fixed in $\{i\}$. This is important. So, this point is rigid is fixed in the rigid body i.

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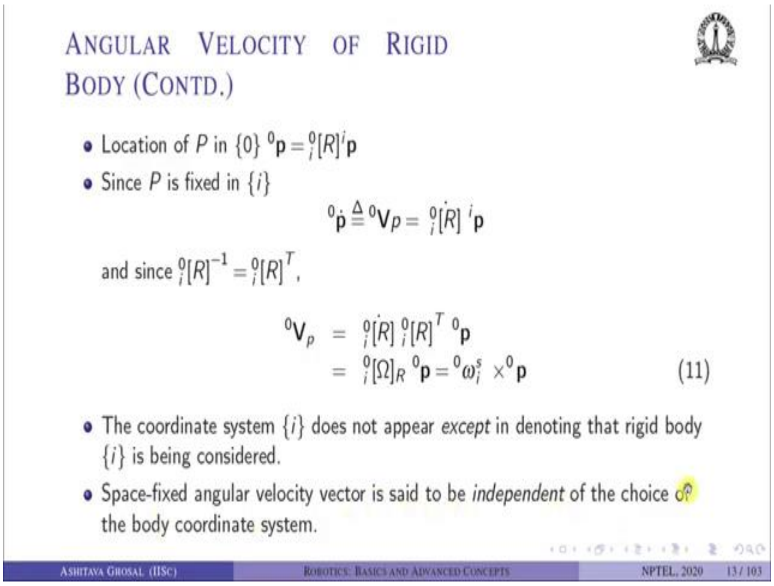

So, the location of P in the 0^{th} coordinate system can be written as ${}^0\boldsymbol{p}$ which is nothing but ${}^0_i[R]\ {}^i\boldsymbol{p}$. So, just like any other vector, we transform from the i^{th} coordinate system to the 0^{th} coordinate system.

So, since P is fixed in {i}, the derivative of this vector ${}^0\dot{\boldsymbol{p}}$ which is nothing but the derivative ${}^0\boldsymbol{V}_p$ in our notation is nothing but ${}^0_i[\dot{R}]\ {}^i\boldsymbol{p}$. Because by chain rule, the term ${}^0_i[R]\dfrac{d(\ {}^i\boldsymbol{p})}{dt} = 0$, then and since ${}^0_i[R]^{-1}$ is same as ${}^0_i[R]^T$, we can write the velocity of the point as ${}^0_i[\dot{R}]{}^0_i[R]^T\ {}^0\boldsymbol{p}$.

So, basically ${}^i\boldsymbol{p}$ can be written as ${}^0_i[R]^T\ {}^0\boldsymbol{p}$. So, think of it. This is inverse which is same as transpose, and we can convert ${}^i\boldsymbol{p}$ into ${}^0\boldsymbol{p}$ matrix ${}^0_i[R]$ pre-multiplied by that.

Now, this ${}^0_i[\dot{R}]{}^0_i[R]^T$ is nothing but this ${}^0_i[\Omega]_R$ ok. So, what do I have? So, velocity of this point P in the 0^{th} coordinate system is given by ${}^0_i[\dot{R}]{}^0_i[R]^T$ which is ${}^0_i[\Omega]_R\ {}^0\boldsymbol{p}$. And this we know is same as the skew-symmetric matrix, so which is nothing but ${}^0\omega_i^s \times {}^0\boldsymbol{p}$.

So, what have we derived that the velocity of the point in the 0th coordinate system is nothing but the space fixed angular velocity vector $\times\ {}^0\boldsymbol{p}$ ok. So, remember the coordinate system $\{i\}$ does not appear except in denoting the rigid body i that is being considered ok.

There is no this side is 0, this is, ${}^0\boldsymbol{p}$ everywhere it is only 0. This i is just to ensure that there is a rigid body i we are talking about. So, the space fixed angular velocity vector is said to be independent of the choice of the body coordinate system. So, we do not really care what is the i^{th} coordinate system.

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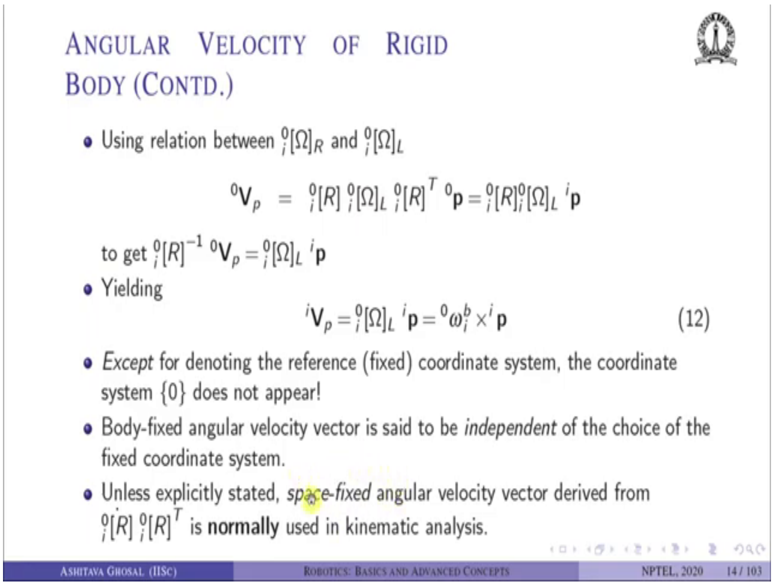

Using the relationship that is ${}^0_i[\Omega]_R$ and ${}^0_i[\Omega]_L$ is given by this tensor transformation. We can also rewrite that ${}^0\boldsymbol{V}_p$ the velocity of point P in the 0^{th} coordinate system is ${}^0_i[R]{}^0_i[\Omega]_L{}^0_i[R]^T\ {}^0\boldsymbol{p}$. So, this we can simplify these two portions can be written as ${}^i\boldsymbol{p}$. So, we have ${}^0_i[R]{}^0_i[\Omega]_L\ {}^o\boldsymbol{p}$ ok.

Now, we can pre-multiply by this ${}^0_i[R]^{-1}\ {}^0\boldsymbol{p}$, so we will get ${}^0_i[\Omega]_L\ {}^i\boldsymbol{p}$ ok. So, ${}^0_i[R]{}^0_i[R]^{-1}$ is identity. And then we can rewrite this as ${}^i\boldsymbol{V}_p$, because what are we doing we are converting this 0 vector into the velocity vector in the 0^{th} coordinate system to the i^{th}

coordinate system. So, $^0\boldsymbol{V}_p$ is described in the i^{th} coordinate system which is nothing but $^i\boldsymbol{V}_p$. And on the right hand side, you have $^0_i[\Omega]_L\ ^i\boldsymbol{p}$, so which is nothing but the cross product of the body fixed angular velocity vector, $^0\omega_i^b$ and $^i\boldsymbol{p}$ ok.

So, let us go over it once more. So, $^0\boldsymbol{V}_p$ pre-multiplied by $^0_i[R]^{-1}$ which is nothing but $^i_0[R]$. So, which is nothing but $^0\boldsymbol{V}_p$ described in the i^{th} coordinate system. Remember we had two ways of describing a linear velocity vector; one where the velocity was done, and one where it was described. So, this is described now in the i^{th} coordinate system. So, this is $^i\boldsymbol{V}_p$, which is equal to $^0\omega_i^b \times\ ^i\boldsymbol{p}$, and where $^0\omega_i^b$ is the body fixed angular velocity vector.

So, again except for denoting the reference or the fixed coordinate system which is $\{0\}$, the coordinate system $\{0\}$ does not appear anywhere in this equation. So, hence the body fixed angular velocity vector is said to be independent of the choice of the fixed coordinate system ok. Unless explicitly stated and in some problems we will be using the body fixed angular velocity vector. Most of the time the space-fixed angular velocity vector derived from $^0_i\dot{[R]}\,^0_i[R]^T$ will be used in kinematic analysis ok.

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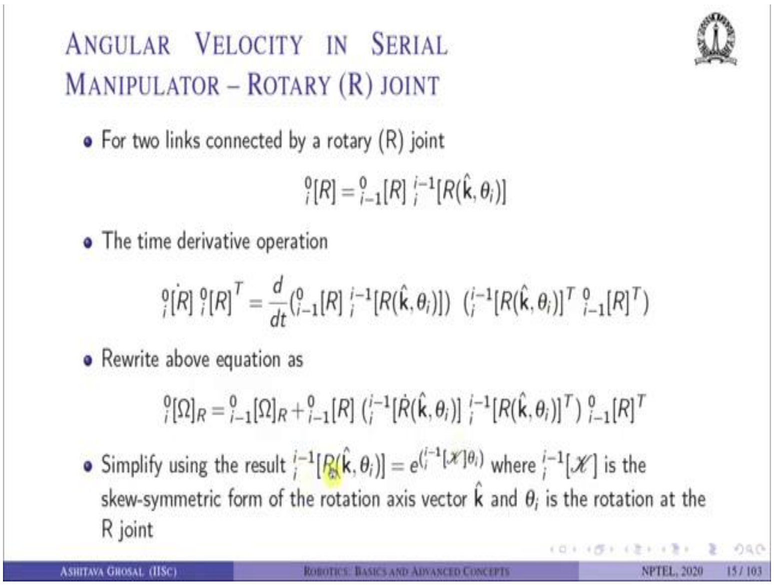

Let us continue. How do we find the angular velocity in a serial manipulated with rotary joints? So, for two links connected by a rotary joint R joint, remember we had derived the constrained equation for a rotary joint. So, the i^{th} rigid body with respect to 0 the rotation matrix is the rotation matrix with the, i minus $1th$ previous one and multiplied by a rotation about the k-axis which is the rotary joint axis divide by θ_i ok.

So, the time derivative of this ${}^{0}_{i}[\dot{R}]{}^{0}_{i}[R]^T$ can be rewritten in this form. So, basically what we are doing is, we are going to do the time derivative of this side into transpose of this. And the transpose of this is $(AB)^T = B^T A^T$. We can rewrite the above equation this side is ${}^{0}_{i}[\Omega]_R$, this side is ${}^{0}_{i-1}[\Omega]_R$.

And then we have this long term which is nothing but the rotation matrix of $i-1$ rigid body with respect to the 0$^{\text{th}}$ coordinate system, then multiplied by some $[\dot{R}]$ matrix, where $\hat{k}$, θ_i is there, and then there is a, $[R]^T$, and then there is a ${}^{0}_{i-1}[R]^T$.

It follows from the above equation. And if we simplify and we write ${}^{0}_{i-1}[R(\hat{k}, \theta_i)] = e^{({}^{i-1}_{i}[\kappa]\theta_i)}$, this is a well known formula ok. So, we can express e to the power a skew-symmetric matrix times θ_i as the rotation matrix ok.

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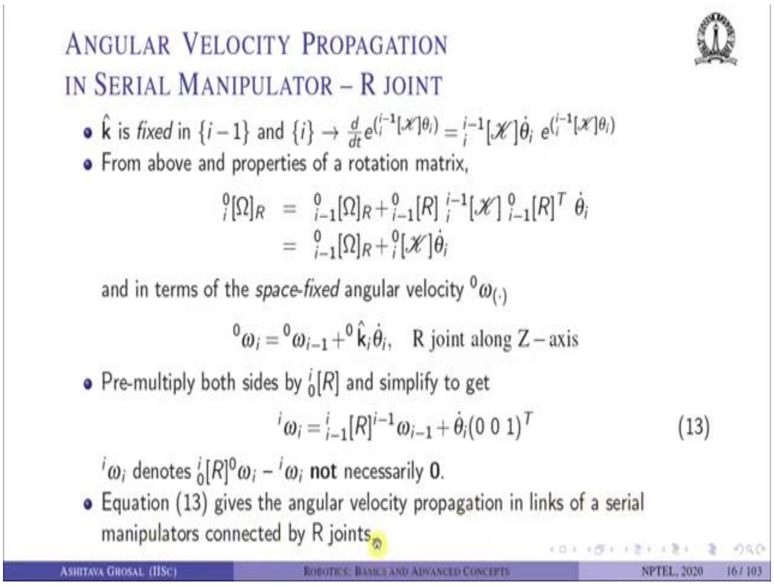

So, if you use this result and simplify and assumed that $\hat{k}$ is fixed in $(i-1)$ and i, so d/dt that will be ${}^{i-1}_{i}[\kappa]\dot{\theta}_i e^{({}^{i-1}_{i}[\kappa]\theta_i)}$. So, derivative of $e^{({}^{i-1}_{i}[\kappa]\theta_i)}$ is ${}^{i-1}_{i}[\kappa]\dot{\theta}_i e^{({}^{i-1}_{i}[\kappa]\theta_i)}$ just like derivative of e^x.

So, from the above and properties of the rotation matrix, we can show that the ${}^0_i[\Omega]_R$ is nothing but ${}_{i-1}^{\;\;0}[\Omega]_R + {}_{i-1}^{\;\;0}[R]\,{}^{i-1}_{\;\;i}[\kappa]\,{}_{i-1}^{\;\;0}[R]^T\dot{\theta}_i$ which is nothing but ${}_{i-1}^{\;\;0}[\Omega]_R + {}^0_i[\kappa]\dot{\theta}_i$.

So, here $[\kappa]$ is the skew-symmetric matrix ok. And in terms of vectors space fixed angular velocity vectors, we can write the ${}^0\omega_i = {}^0\omega_{i-1} + {}^0\hat{k}_i\dot{\theta}_i$. So, I have intentionally done this, but it is obvious that if there are two rigid bodies connected by a rotary joint which is along the Z-axis or the $\hat{k}$ axis, so the angular velocity at the i^{th} rigid body will be equal to the angular velocity of the $(i-1)^{\text{th}}$ rigid body times $\dot{\theta}_i$ along the $\hat{k}$-axis ok.

The whole idea was that although this is an obvious result, we know that and you can intuitively see what is happening when two rigid bodies are connected by rotary joint ok. We can prove it mathematically. We are not going to do this all the time, but just to show that it can be done ok. We can also pre-multiply the above equation by ${}^i_0[R]$ ok. So, ${}^i_0[R]$ basically means that you want to express this vector ${}^0\omega_i$ in the i^{th} coordinate system ok.

So, pre-multiply by ${}^i_0[R]$ will give you ${}^i\omega_i$ ok. Remember ${}^i\omega_i$ is not 0. What is the meaning of ${}^i\omega_i$? It is the angular velocity of the rigid body i derived in the 0th coordinate system, but expressed in the i^{th} coordinate system ok. So, ${}^i\omega_i$ will be ${}_{i-1}^{\;\;i}[R]\,{}^{i-1}\omega_{i-1} + \dot{\theta}_i(0,0,1)^T$, because the $\hat{k}$ axis in its own coordinate system will be $(0,0,1)^T$ ok. So, ${}^i\omega_i$ denotes ${}^0_i[R]\;{}^0\omega_i$ which is not necessarily 0 that is important.

So, equation (13) gives the angular velocity propagation in links of a serial manipulated connected by R joints ok. But what is the meaning of this? That suppose I start from some i = 1 ok. So, I know ${}^1\omega_1$ pre-multiplied by a rotation matrix ${}^1_0[R]$. And then if you add to it, the $\dot{\theta}$ which is happening at the first joint, we will get the angular velocity of the first link ok.

So, i =1, ${}^0\omega_0$ ok, which is basically fixed the base is not moving if you add to it the $\dot{\theta}$ when you get the angular velocity of the first link. How about the angular velocity of the second link? You put i = 2, then add to it $\dot{\theta}_2$, and you will get ${}^2\omega_2$ ok. So, we can go from i = 1, 2, 3, all the way to the last link ok.

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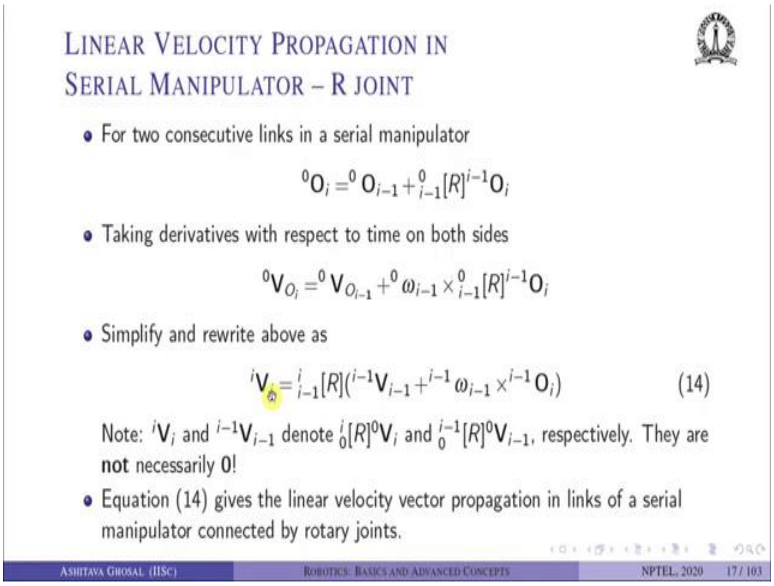

Now, let us look at the linear velocity propagation in a serial manipulator. And again we will consider R joint. So, if there are two consecutive links in a serial manipulator, the origin of the i^{th} link can be written as the origin of the $(i-1)^{\text{th}}$ link thus the origin of the i^{th} link with respect to the, $(i-1)^{\text{th}}$ coordinate system pre multiplied by a rotation matrix ok.

So, we can take the derivative of a position vector that is straightforward which is not like the angular velocity and rotation matrix business. So, the derivative with respect to time on both sides if you take, we will get the linear velocity of the origin of the i^{th} link will be same as the linear velocity of the origin of the $(i-1)^{\text{th}}$ link plus $^0\omega_{i-1} \times {}_{i-1}^{0}[R]^{i-1}O_i$. This formula is very well known except we have written it in a more formal way.

So, except this R is the vector from $(i-1)$ to i ok. So, we need to convert it back to the 0^{th} coordinate system before we can do cross product. So, we can simplify and rewrite by pre multiplying by ${}_0^i[R]$. So, $^i\boldsymbol{V}_i$ can be written as ${}^{i-1}_{i}[R]({}^{i-1}\boldsymbol{V}_{1-1} + {}^{i-1}\omega_{i-1} \times {}^{i-1}O_i)$. So, again note that, this is not 0. So, $^i\boldsymbol{V}_i$ is nothing but the linear velocity of the i^{th} link obtained in the 0^{th} coordinate system, but pre multiplied by a rotation matrix ok.

So, this equation gives the propagation of linear velocity vector from one link to the next link. So, if I know the linear velocity of the, $(i-1)^{\text{th}}$ link, from this, I can find the linear velocity of the i^{th} link ok. This we have considered when there are connected by rotary joints.

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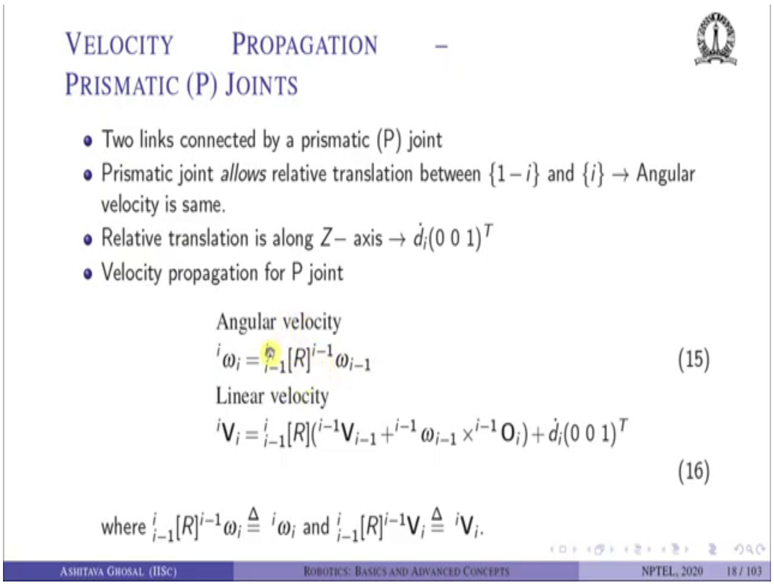

We can also find the velocity propagation when the two links are connected by prismatic joints. So, the prismatic joint allows translation between $(i-1)$ and i ok, the angular velocities are same, so there are two links and there is a prismatic joint in between the angular velocity if the i^{th} link and the $(i-1)^{\text{th}}$ link same. That is however, the relative translation from the Z-axis given by $\dot{d}_\iota(0{,}0{,}1)^T$.

So, the velocity propagation for a prismatic joint from one link to the next, the angular velocities are same ok. It is just written, so that we write it as ${}^{i}_{i-1}[R]^{i-1}\omega_{i-1}$, but basically the angular velocities are same in both in with respect to the fixed coordinate system. And the linear velocity is nothing but that the linear velocity V plus some ω plus some $\dot{d}_\iota$. And again we have pre multiplied by rotation matrix to write in this nice compact manner ok.

So, what is ${}^{i}_{i-1}[R]^{i-1}\omega_1$? This is ${}^{i}\omega_i$. Again it is not necessarily equal to 0. Similarly, ${}^{i}\boldsymbol{V}_i$, again it is not necessarily equal to 0.

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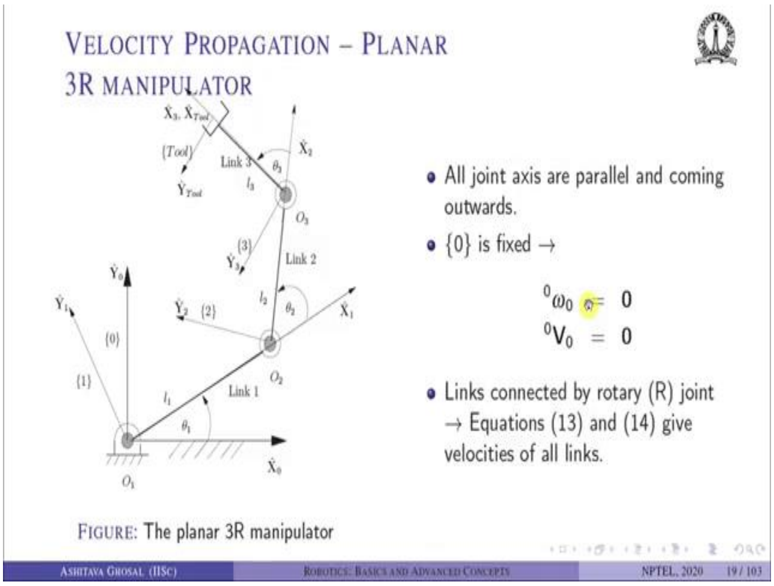

Let us look at an example, velocity propagation in planar 3R manipulator. So, again we have this well known planar 3R manipulator that are three rotary joints – 1, 2, and 3, there are three links with l_1, l_2 and then l_3 which is the link lengths of the tool or the end effector coordinate system ok. So, all joints are parallel and coming out. So, angles are θ_1, θ_2, and θ_3.

So, hence {0} is a fixed coordinate system, ${}^0\omega_0$ = 0, and ${}^0\boldsymbol{V}_0 = \boldsymbol{0}$. Links are connected by rotary joints ok. So, hence we have to use equation (13) and (14) ok. What is (13) and (14)? These two equations. (14) is this – how the linear velocity propagates; and (13) is this – how the angular velocity propagate.

So, rotary joint means previous angular velocity plus $\dot{\theta}_i$ will give you the next angular velocity, of course, added properly. And linear velocity, previous linear velocity plus $\omega \times [R]O$ will give you the next two linear velocity. So, we are going to use these two ideas or these two formulas ok.

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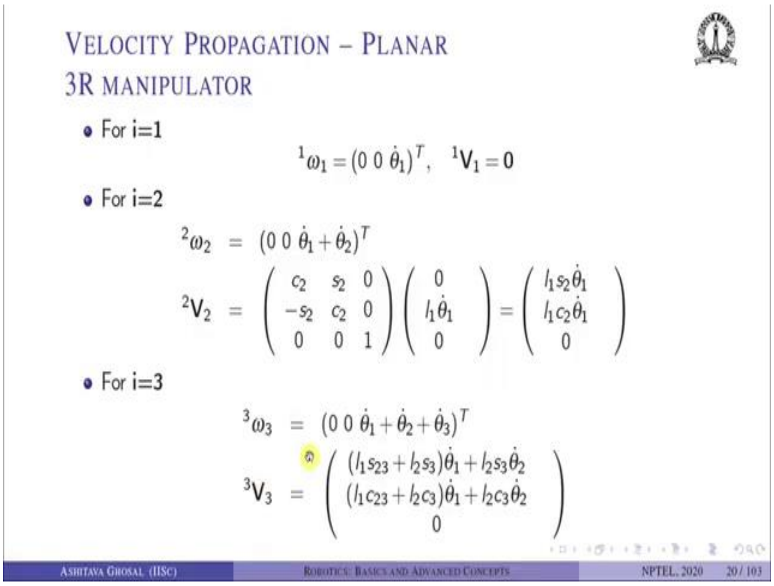

So, for $i = 1$, $^0\omega_0$ was 0. So, $^1\omega_1 = (0,0,\dot{\theta}_1)^T$ first term is 0 and $^1\boldsymbol{V}_1 = 0$. Because, why, we can see the formula for $i = 2$, $^2\omega_2 = (0,0,\dot{\theta}_1 + \dot{\theta}_2)^T$; and $\dot{\theta}_2$ will be some

$$\begin{pmatrix} \cos\theta_2 & \sin\theta_2 & 0 \\ -\sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ l_1\dot{\theta}_1 \\ 0 \end{pmatrix}$$

So, hence you will get $(l_1 s_2 \dot{\theta}_1, l_1 c_2 \dot{\theta}_1, 0)^T$. And for $i = 3$, the angular velocity will be sum of the $\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\theta}_3$; and $^3\boldsymbol{V}_3 = \big((l_1 s_{23} + l_2 s_3)\dot{\theta}_1 + l_2 s_3 \dot{\theta}_2, (l_1 c_{23} + l_2 c_3)\dot{\theta}_1 + l_2 c_3 \dot{\theta}_2, 0\ \big)^T$.

This is the planar example. So, as I have set several times now the $^2\boldsymbol{V}_2$ is not 0 ok; $^1\boldsymbol{V}_1$ turns out to be 0, but $^2\boldsymbol{V}_2$ is not 0, $^3\boldsymbol{V}_3$ is not 0 ok.

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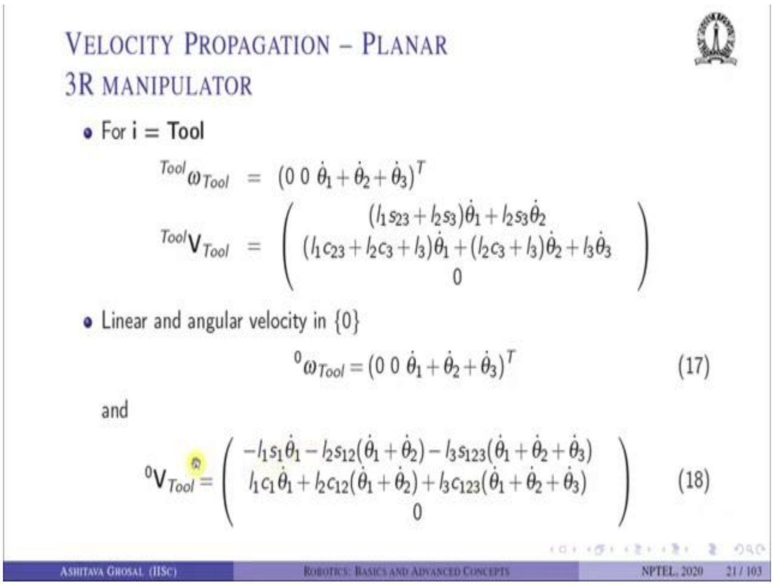

And finally, the angular velocity of the tool described in the {Tool} coordinate system is $\left(0,0,\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3\right)^T$. The velocity of the tool the end effector described in the end effector coordinate system – tool coordinate system is given by $(l_1 s_{23} + l_2 s_3)\dot{\theta}_1$ and so on. Again the z component is 0.

Now, we can rewrite these angular velocity of the tool and the linear velocity of the tool in the 0th coordinate system. How, which has pre multiplied by a rotation matrix ${}_{Tool}^{0}[R]$ and ${}_{Tool}^{0}[R]$ here also. So, if you pre multiply, you will get the angular velocity vector is nothing but theta $\left(0,0,\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3\right)^T$.

The linear velocity is $\left(-l_1 s_1 \dot{\theta}_1 - l_2 s_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right) - l_3 s_{123}\left(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3\right)\right)$ and so on, and the y component will contain $\cos\theta_1$, $\cos(\theta_1 + \theta_2)$ and $\cos(\theta_1 + \theta_2 + \theta_3)$.

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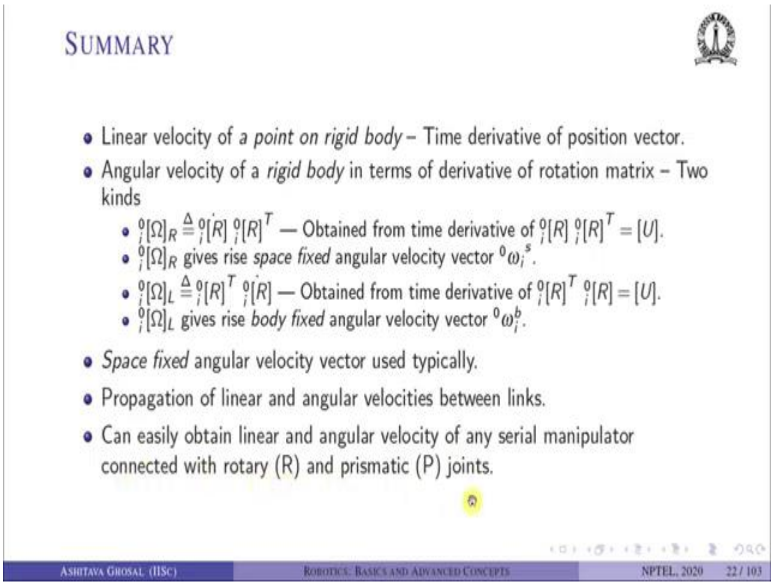

So, does it make sense? Yes. Why? Because if you remember that x-component of the position vector ok, let us go back and see. So, the x-component here will be $(l_1 \cos\theta_1 + l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3))$.

So, if you take the derivative of the x-component, you will get $(-l_1 \sin\theta_1\, \dot{\theta}_1), \left(-l_2 \sin(\theta_1+\theta_2)\,(\dot{\theta}_1+\dot{\theta}_2)\right)$ and so on ok. So, the position vector we know we can and we can easily take the derivatives, and we can see that this is what you will get ok. So, this is an example where we can very easily go back to what we know from our previous vector algebra and derivatives of a vector, and find out the linear velocity of the Tool.

How about the angular velocity? Is that obvious? Yes. So, the first joint is rotating by θ_1; second joint is rotating by relative rotation is θ_2; third joint relative rotation is θ_3. So, the total rotation of the first three joints the end effector orientation is $(\theta_1+\theta_2+\theta_3)$, and the angular velocity will be $(\dot{\theta}_1+\dot{\theta}_1+\dot{\theta}_3)$. Remember this is because this is the planar mechanism, all the joint axis are coming out.

So, in summary the linear velocity of a point on the rigid body is nothing but the time derivative of the position vector. The angular velocity of a rigid body in terms of the derivative of a rotation matrix, there are two of them. One is this ${}^0_i[\dot{R}]\,{}^0_i[R]^T$ which is called ${}^0_i[\Omega]_R$ ok. So, it is obtained from the time derivative of ${}^0_i[R]\,{}^0_i[R]^T$ equal to identity.

So, this ${}^0_i[\dot{R}]\,{}^0_i[R]^T$ gives rise to space fixed angular velocity vector ${}^0\omega_i^s$ with superscript s. We can also have ${}^0_i[R]^T\,{}^0_i[\dot{R}]$ which is the left multiplication ok. And we can do ${}^0_i[R]^T\,{}^0_i[R]$ is identity. And this ${}^0_i[\Omega]_L$ gives rise to what is called as the body fixed angular velocity vector ok.

So, this is why there is a superscript b ok. We use space fixed angular velocity vector most of the time ok. They are related. Actually angular velocity vector which is space fixed ${}^0\omega_i^s$ it can be related to ${}^0\omega_i^b$ by rotation matrixes. Likewise, the skew-symmetric matrix ${}^0_i[\Omega]_R$, and the skew-symmetric matrix ${}^0_i[\Omega]_L$ are related like a tensor transformation. So, it is ${}^0_i[R]\,{}^0_i[\Omega]_L\,{}^0_i[R]^T$.

Then I showed you how to obtain the propagation of linear and angular velocities between links connected either by a rotary joint or a prismatic joint ok. We can easily obtain the linear and angular velocity of any serial manipulator link connected with rotary and prismatic joints ok.

So, if you want the angular velocity of the third link, we can go from 0, 1, 2, and then 3 ok. So, any link with respect to any other link can be obtained by simply using these propagation equations. With this, we will come to a stop in this lecture. In the next lecture, we look at serial manipulated Jacobian matrix.