Robotics: Basics and Selected Advanced Concepts
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Lecture - 20
Serial Manipulator Jacobian Matrix

Welcome to this NPTEL lectures on Robotics, Basic and Advanced Concepts. In the last lecture, we had looked at how to obtain the linear and angular velocity of the links of a robot, and then I had shown you how to propagate the linear and angular velocity of the links of a robot from a fixed base all the way to the end effector. And finally, I had given an example of a planar 3 degree of freedom robot and showed you how to obtain the linear and angular velocity of each link, ok.

In this lecture, we will look at a very important concept in velocity kinematics of robots called the Jacobian matrix. And now this lecture, we look at the Serial Manipulator Jacobian matrix. Next, we look at the parallel manipulator Jacobian matrix.

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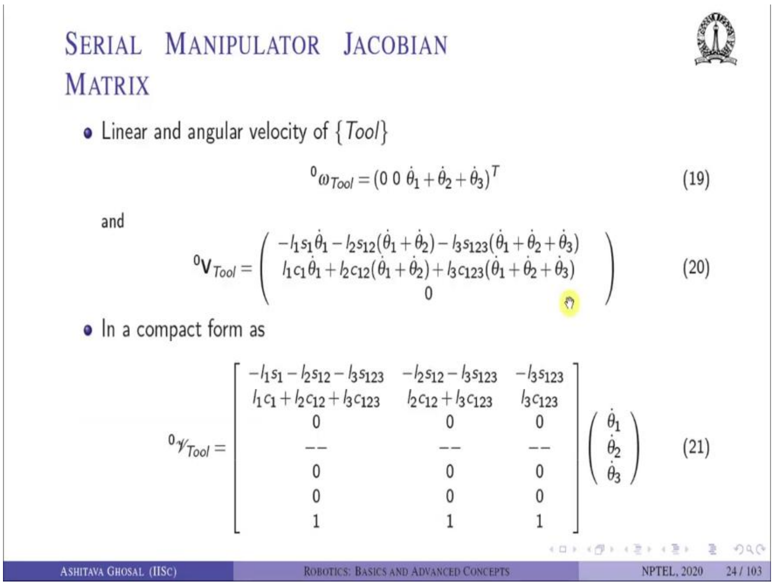

So, a serial manipulator Jacobian matrix to start, we first obtain the linear and angular velocity of the end effector of the robot. So, for example, in the planar 3R robot which we had discussed last time, I showed you that the angular velocity of the {Tool} coordinate system can be written in as sum of $\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\theta}_3$ the z component, and the x and y components were 0.

Likewise, the linear velocity of the end effector or the origin of the Tool coordinate system could be written in terms of again $\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\theta}_3$. And so, for example, the x component is $(-l_1 \sin\theta_1 \dot{\theta}_1 - l_2 \sin(\theta_1 + \theta_2)\left(\dot{\theta}_1 + \dot{\theta}_2\right) - l_3 \sin(\theta_1 + \theta_2 + \theta_3)(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3))$.

So, this was obtained and similarly the y component contains cosine of θ_1, $(\theta_1 + \theta_2)$, and $(\theta_1 + \theta_2 + \theta_3)$ and the z component was 0. So, this was obtained from the velocity propagation formula. The second set of quantities for the linear velocity of the Tool could also be obtained from simple derivative of the position vector.

Now, we can write these two expressions of angular velocity and linear velocity in a compact form. So, the first we write the linear velocity components, ok $(-l_1 s_1 - l_2 s_{12} - l_3 s_{123})\dot{\theta}_1$. Then, the second term is $(-l_2 s_{12} - l_3 s_{123})\dot{\theta}_2$ and third term is $(-l_3 s_{123})\dot{\theta}_3$.

So, basically we take all the terms containing $\dot{\theta}_1$, all the terms containing $\dot{\theta}_2$ and all the terms containing $\dot{\theta}_3$ and write it as the matrix form, ok. So, the second row of this matrix can be written as $(l_1 c_1 + l_2 c_{12} + l_3 c_{123})\dot{\theta}_1$, $(l_2 c_{12} + l_3 c_{123})\dot{\theta}_2$ and $(l_3 c_{123})\dot{\theta}_3$, and the z component is 0.

Likewise, from the angular velocity, we can write that the x component of the angular velocity is 0, the y component is 0, and the z component is $(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)$. So, these, 6 scalar equations from these two vector equations can be written in this matrix form.

So, the column vector here is $\left(\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3\right)^T$ multiplied by something which is a function of the link lengths l_1, l_2, l_3 and sin and cosine of θ_1, θ_2 and θ_3. And the left side here contains the velocity vector, linear velocity vector and the angular velocity vector.

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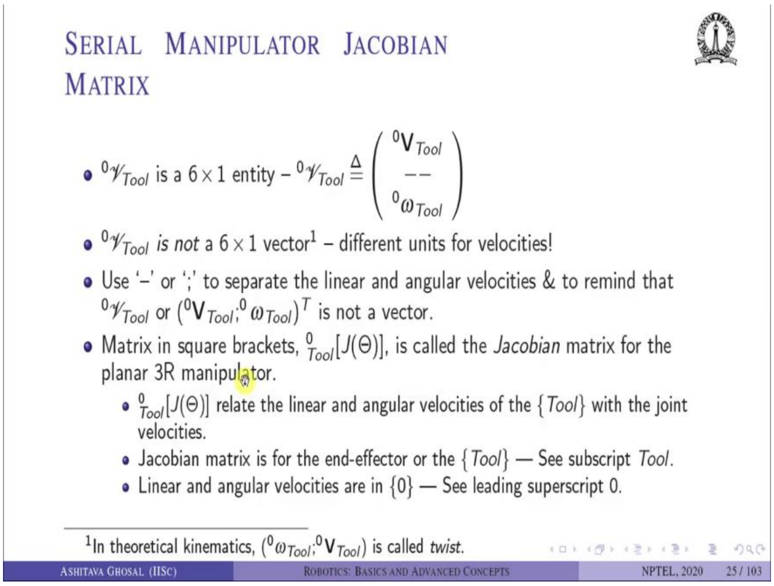

So, this left hand side quantity is a 6×1 entity. The top 3×1 part is the linear velocity of the Tool and the bottom 3×1 is the angular velocity of the Tool, ok. So, a few observations. This $^0\mathcal{V}_{Tool}$ is not actually a 6×1 vector because the units are different.

This is like meters per second and this is like radians per second. So, we will use these dash lines to separate the linear and angular velocity and to remind that the $^0\mathcal{V}_{Tool}$ or this combination of linear and angular velocity is not a vector in the strict sense of the world.

The matrix in the square bracket is called the Jacobian matrix for the planar 3R manipulator, ok. So, that this is the introduction to the Jacobian matrix. So, what does this Jacobian matrix do? It relates the linear and angular velocity of the Tool with the joint velocities. So, joint velocities are $\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\theta}_3$, the left hand side is the linear velocity of the Tool and the angular velocity of the Tool, ok.

The other important thing in this Jacobian matrix is it is the Jacobian matrix for the end effector or the Tool. So, that is why we have a leading subscript tool. It is also important to note that this Jacobian matrix is with respect to the fixed coordinate system.

Why? Because the linear and angular velocities are written with respect to the fixed coordinate system and that is the reason there is a leading superscript 0, ok. So, if I wanted to find the Jacobian matrix for some other link or in some other coordinate system we have to go back and change quite a few things.

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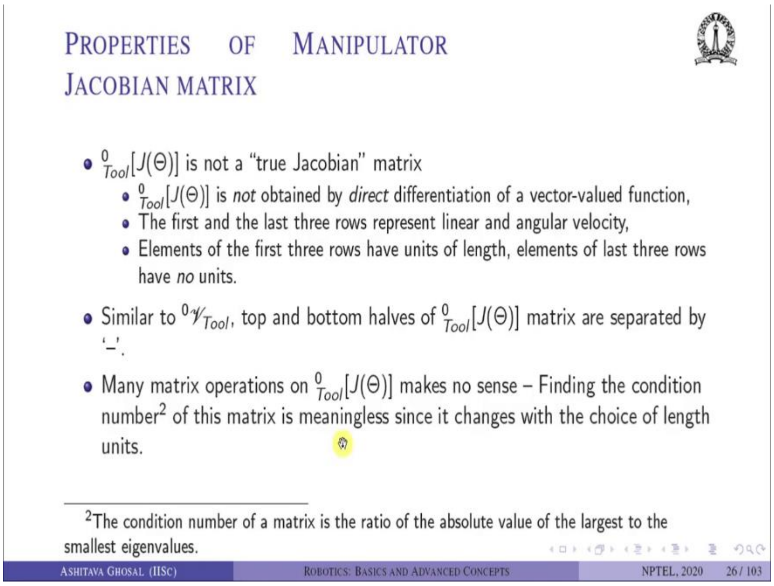

Just to remind that this ${}_{Tool}^{0}[J(\Theta)]$ is not a true Jacobian matrix. Why? Because it was not obtained as a direct differentiation of a vector valued function, ok. So, there are if you go back and see calculus, if you have a vector valued function, the first derivative of the vector valued function with respect to the independent parameters is the Jacobian matrix.

However in this case it is not so, ok. Here the first and the last three rows represent linear and angular velocity, ok. So, the first three rows have units of length, the last three rows have no units. So, the linear velocity is meters per second, ok. So, $\dot{\theta}$'s are radians per second, so the quantity inside the matrix is meters first three rows. And the second thing is angular velocity is radians per second, we already have $\dot{\theta}$'s which are radians per second, so the quantity inside the matrix is unit less, ok.

So, likewise similar to ${}^{0}\mathcal{V}_{Tool}$, we will put the top and bottom halves of this Jacobian matrix separated by this dash line, just to remind us that it is not a proper matrix. So, and many matrix operations on this Jacobian makes no sense, because it is not strictly a matrix. So, for example, finding the condition number of this matrix is meaningless, since it changes with the choice of the length units.

So, what is the condition number? It is the ratio of the absolute value of the largest to the smallest eigenvalue, ok. So, if we find the eigenvalues of this matrix, the largest divided by the smallest, ok. So, it would have units of length and depending on whether you are writing the linear velocity as meters per second or centimeters per second the eigenvalues will be different numerically and the condition number will be different. So, it does not make any sense to do some matrix operations, ok.

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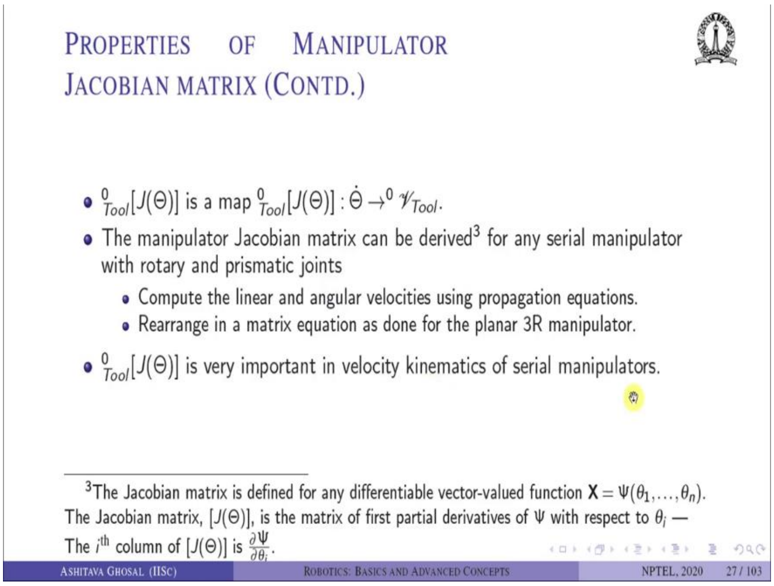

So, what can we do with this or why do we use this Jacobian matrix? So, the best way to think of the Jacobian matrix is a map. So, what does it do? It will take $\dot{\theta}$'s which are the joint rates to the linear and angular velocity of the end effector, ok. The Jacobian matrix can be derived for any serial manipulator with rotary and prismatic joints, ok. Why?

Because in general the Jacobian matrix is defined for any differentiable vector valued function $\boldsymbol{X} = \psi(\theta_1, \ldots, \theta_n)$. And this Jacobian matrix is the partial of the first derivatives of ψ with respect to θ_i. So, the i^{th} column of this Jacobian matrix is $(d\psi/d\theta_i)$. So, in this case it is not so straightforward, but you can think of it this way that the Jacobian matrix is the derivative of the position vector and then the Jacobian matrix the bottom part comes from the angular velocity vector, ok.

So, we can compute the linear and angular velocity using propagation equations and we can always rearrange in a matrix equation as was done for the planar 3R manipulator. So, we can always obtain the Jacobian matrix for any serial robot with rotary and prismatic joints. This Jacobian matrix is very important in velocity kinematics of serial manipulators. We will see that in a little while.

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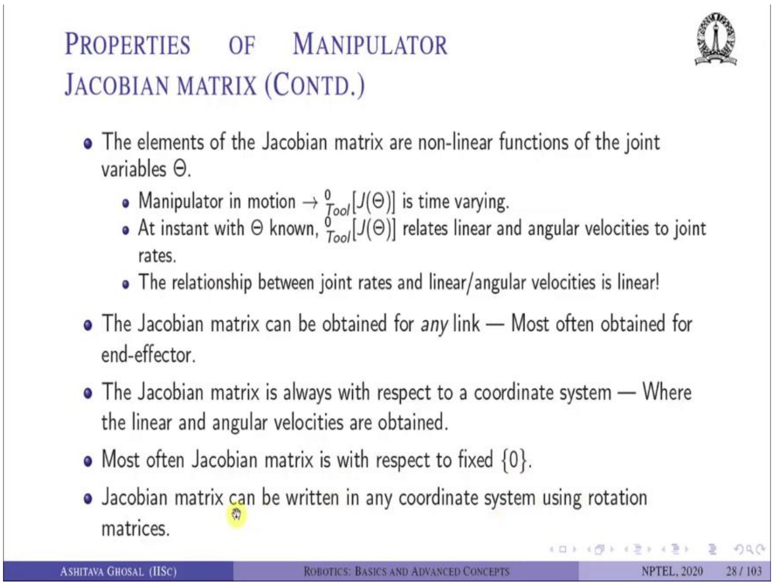

The elements of the Jacobian matrix are non-linear functions of the joint variables. Remember it contains $\cos\theta_1$, $\cos(\theta_1+\theta_2)$ and so on $\sin\theta_1$ and so on. So, if the manipulator is in motion, ok if θ_1, θ_2 and θ_3 the joint variables are changing with time this Jacobian matrix will also change with time, ok.

Any instant if the thetas are known the Jacobian matrix relates linear and angular velocity to the joint rates and this relationship is linear, ok because $^{0}\mathcal{V}_{Tool} = {}^{0}_{Tool}[J(\Theta)]\dot{\Theta}$. So, at any instant if $^{0}_{Tool}[J(\Theta)]$ is fixed $^{0}\mathcal{V}_{Tool}$ and $\dot{\Theta}$ are linearly related, ok. So, the Jacobian matrix can be obtained for any link. Why? Because we can obtain the linear and angular velocity of any link. Most of the time we would be interested in the end effector, linear and angular velocity and hence the Jacobian matrix of the end effector.

The Jacobian matrix is always with respect to a coordinate system, ok. Why? Because the linear and angular velocities are written with respect to a coordinate system. So, most of the time the Jacobian matrix is with respect to the fixed coordinate system or the 0 coordinate system. But the Jacobian matrix can be written in any coordinate system using rotation matrices.

So, if I want the linear velocity in some other coordinate system we can pre-multiply by a rotation matrix. If I want the angular velocity in another coordinate system we can pre-multiply it by a rotation matrix and again rearrange to obtain the Jacobian matrix.

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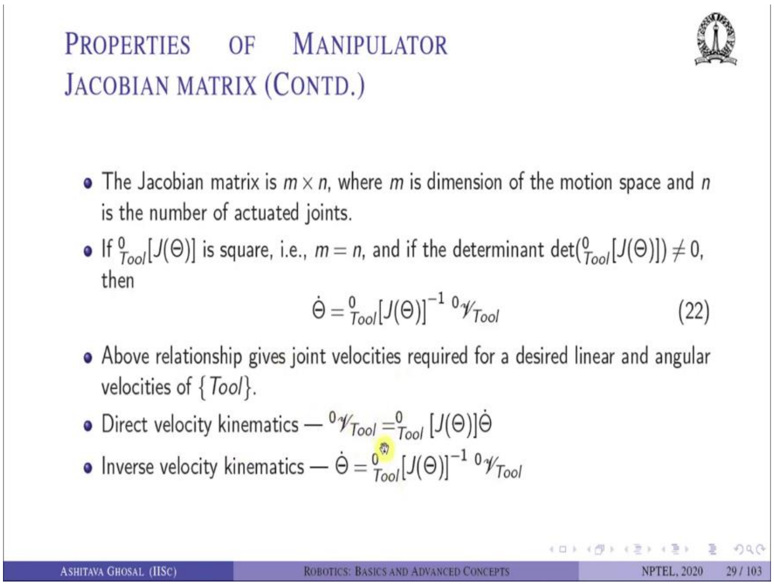

Most of the time the Jacobian matrix is $m \times n$, where m is the dimension of the motion space and n is the number of actuated joints, ok. So, in the case of planar 3R there were 3 joints $\theta_1, \theta_2, \theta_3$. So, there are 3 columns, ok. And it is moving in generally in that example some of the components were 0, but in general it could be moving in 3D space, so there are 6 rows.

If the Jacobian matrix is square and $m = n$ and if the determinant of ${}_{Tool}^{0}[J(\Theta)]$ is not equal to 0, then we can invert the relationship. So, initially we have ${}^0\mathcal{V}_{Tool} = {}_{Tool}^{0}[J(\Theta)]\dot{\Theta}$, but if it is $m = n$ and determinant of the Jacobian is not 0, then we can write $\dot{\Theta} = {}_{Tool}^{0}[J(\Theta)]^{-1}\ {}^0\mathcal{V}_{Tool}$.

So, the above relationship gives joint velocities required for a desired linear and angular velocity of the Tool. So, if I want the linear and angular velocity as let us say 1 meter per second and 1 radian per second, along certain directions I can use this expression to obtain the $\dot{\theta}$'s which I need to give at the joints.

So, the direct velocity kinematics is basically this equation which is the linear and angular velocity of the Tool, ${}^0\mathcal{V}_{Tool} = {}_{Tool}^{0}[J(\Theta)]\dot{\Theta}$ and the inverse velocity kinematics is basically $\dot{\Theta} = {}_{Tool}^{0}[J(\Theta)]^{-1}\ {}^0\mathcal{V}_{Tool}$. These are the two basic expressions in velocity kinematics. And we will take a look when m is not equal to n little later.

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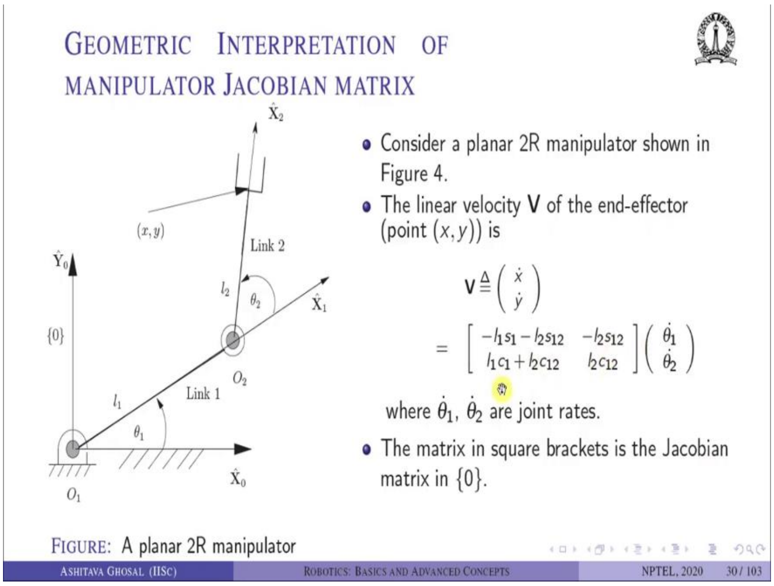

So, now let us take a look at what is the geometric interpretation of the manipulator Jacobian matrix. And this we will do through one example. So, let us consider a planar 2R manipulator as shown in this figure, ok. So, we have a rotary joint here, another rotary joint here, this is link 1, link 2 and then this point (x, y), which is at the middle of the so, called schematic parallel geographer.

So, we can write x and y in terms of θ_1 and θ_2, ok. So, we have done this before. So, x is nothing but $(l_1 c_1 + l_2 c_{12})$ and y is nothing but $(l_1 s_1 + l_2 s_{12})$. So, the linear velocity, $\boldsymbol{V}$ of the end effector can be obtained by simply taking the derivative of x and y with respect to time, ok.

So, $\dot{x}$ and $\dot{y}$ is the linear velocity of this end effector and that we can see is clearly given by $\left((-l_1 s_1 - l_2 s_{12})\dot{\theta}_1 + (-l_2 s_{12})\dot{\theta}_2\right)$. And the y component contains c_1 and c_2, ok. Just the straightforward differentiation of the x and y position vector and $\dot{\theta}_1$ and $\dot{\theta}_2$ are the joint rates.

So, in this example, the square bracket here containing minus $\begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$. So, this 2×2 matrix is the Jacobian matrix in the 0^{th} coordinate system.

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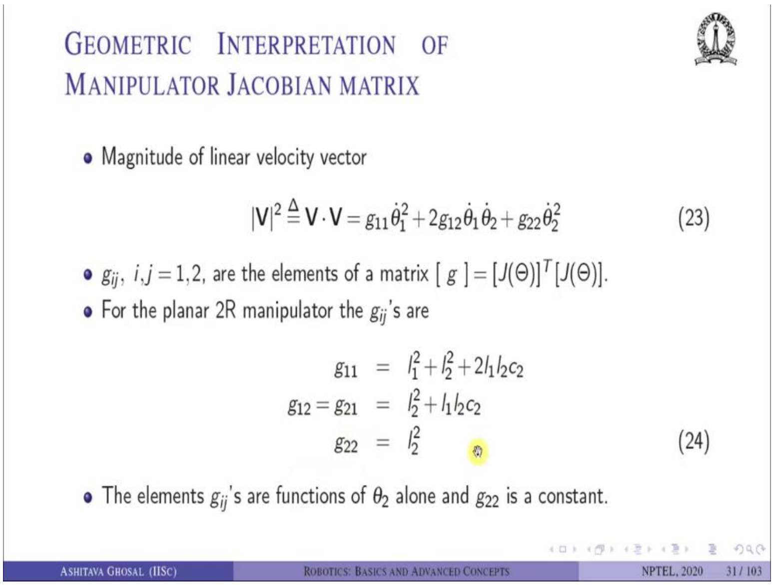

So, let us try to find the magnitude of this linear velocity vector, so which is given by the product of this vector with itself and we can write this dot product as something like $(g_{11}\dot{\theta}_1^2 + 2g_{12}\dot{\theta}_1\dot{\theta}_2 + g_{22}\dot{\theta}_2^2)$. So, this g_{ij} there are two, i and j are both 1 comma 2 are the elements of this matrix $[J(\Theta)]^T[J(\Theta)]$, ok.

So, for the planar 2R manipulator the g_{ij}'s are very simple. It can be computed. So, $g_{11} = l_1^2 + l_2^2 + 2l_1l_2c_2$ and $g_{12} = g_{21} = l_2^2 + l_1l_2c_2$ and $g_{22} = l_2^2$. So, it is a symmetric matrix. So, the elements of g_{ij} are functions of θ_2, ok. However, g_{22} in this example is constant.

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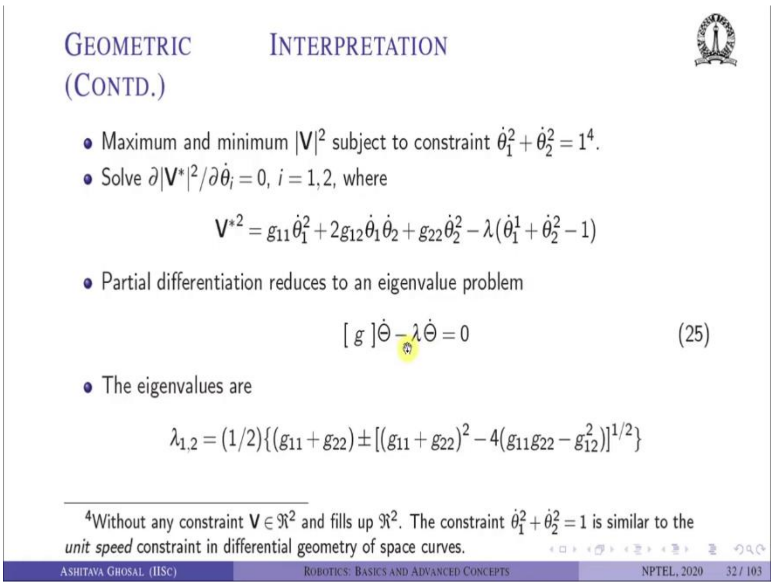

Let us try to find the maximum and minimum of this velocity square subject to a constraint $\dot{\theta}_1^2 + \dot{\theta}_2^2 = 1$. Why do we need to put a constraint? If I do not put any constraint then the velocity vector which is $(\dot{x}, \dot{y})$ is a function of $\dot{\theta}_1$ and $\dot{\theta}_2$ and I can have arbitrary velocity vector as I change $\dot{\theta}_1$ and $\dot{\theta}_2$. So, it basically fills up the two-dimensional space.

The constraint $\dot{\theta}_1^2 + \dot{\theta}_2^2 = 1$ basically is similar to a unit speed constraint in differential geometry of space curve. So, basically it is, we want to find the maximum or minimum subject to an L_2 norm on the joint rates. How do we find that?

We define a new function $\boldsymbol{V^*}^2$ which is this $g_{11}\dot{\theta}_1^2 + 2g_{12}\dot{\theta}_1\dot{\theta}_2 + g_{22}\dot{\theta}_2^2$ and so on and we put in this constraint using a Lagrange multiplier. So, $-\lambda\left(\dot{\theta}_1^2 + \dot{\theta}_2^2 - 1\right)$.

And what do we do? We solve the partial derivative $\partial|\boldsymbol{V}^*|^2/\partial\dot{\theta}_i = 0$. So, if you take the partial derivatives, we can see that this reduces to an eigenvalue problem which is $[g]\dot{\Theta} - \lambda\dot{\Theta} = 0$. These λ's are the same Lagrange multipliers we have introduced. And we can find, ok the eigenvalues of this eigenvalue problem, it can be found in closed form because $[g]$ is 2×2 it is a quadratic, determinant of this would be a quadratic function in λ.

And we can solve for this λ's. So, the λ_1 and λ_2 are some function of g_{ij}'s, ok. So, for example, it is $(1/2)\{(g_{11} + g_{22}) \pm \sqrt{(g_{11} + g_{22})^2 - 4(g_{11}g_{22} - g_{12}^2)}\}$. So, just the roots of the quadratic polynomial which you will get from expanding the determinant of $[g] - \lambda I = 0$.

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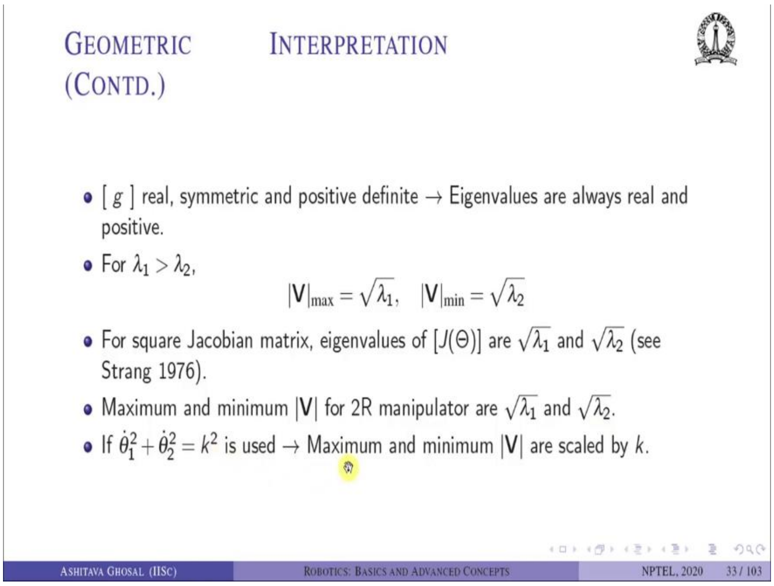

$[g]$ is real, symmetric and positive definite. Why? Because $[g] = [J(\Theta)]^T[J(\Theta)]$, ok. So, it is real and symmetric. It is also positive definite because it gives the square of the velocity, ok. So, the eigenvalues are always real and positive. If you assume that the two eigenvalues λ_1 and λ_2 are related by $\lambda_1 > \lambda_2$, then you can show that the maximum velocity magnitude is $\sqrt{\lambda_1}$ and the minimum velocity magnitude is $\sqrt{\lambda_2}$, ok.

So, for square Jacobian matrix eigenvalues of $[J(\Theta)]$ are $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$ this is well known from linear algebra, ok. So, the maximum and minimum velocity vector magnitude for the 2R manipulators are $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$. So, if $\dot{\theta}_1^2 + \dot{\theta}_2^2 = k^2$, not one then the maximum and minimum $|\boldsymbol{V}|$ are scaled by k.

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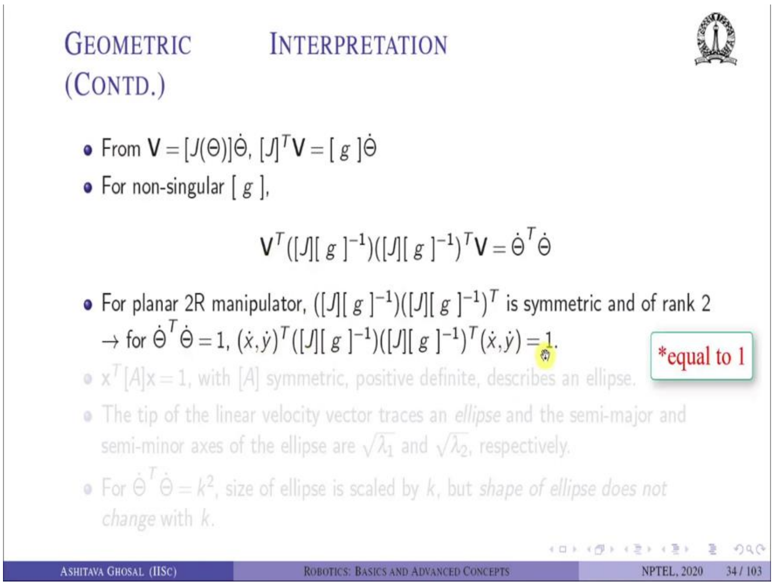

So, from $\boldsymbol{V} = [J(\Theta)]\dot{\Theta}$, I could write $[J(\Theta)]^T\boldsymbol{V} = [g]\dot{\Theta}$. So, for nonsingular $[g]$ we can rewrite this expression as $\boldsymbol{V}^T([J][g]^{-1})([J][g]^{-1})^T\boldsymbol{V} = \dot{\Theta}^T\dot{\Theta}$. So, the right hand side is equal to 1, the left hand side $\boldsymbol{V} = (\dot{x},\dot{y})^T$. So, $(\dot{x},\dot{y})^T$ times sum matrix times $(\dot{x},\dot{y})$ equal to 0, ok. So, this is of the form $x^T A x = 0$.

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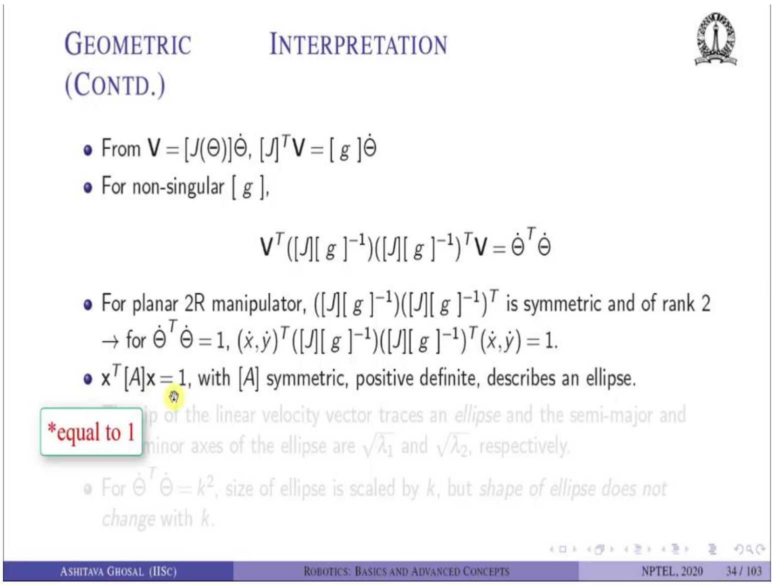

So, with A symmetric positive definite, this $x^T A x$ describe an ellipse. So, what is x here? This is $(\dot{x},\dot{y})^T$. And what is $(\dot{x},\dot{y})^T$? That is the velocity vector.

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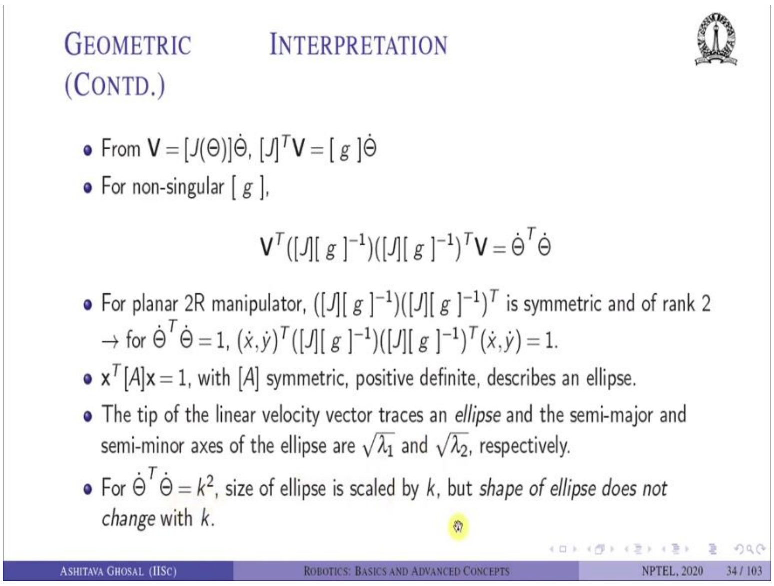

So, what we can say is that the tip of the linear velocity vector traces and ellipse and the semi major and semi minor axis of the ellipse are $\sqrt{\lambda_1}$ and $\sqrt{2}$, ok. So, these are the minimum and maximum magnitudes of the velocity vector.

So, for $\dot{\Theta}^T\dot{\Theta} = k^2$, the size of the ellipse is scaled by k, ok. So, instead of 1, if this was k^2 then we will get a larger ellipse. But the shape of the ellipse does not change, ok. The minimum and maximum velocity magnitudes will change, but the direction in which they happen the major and minor axis of the ellipse do not change, ok.

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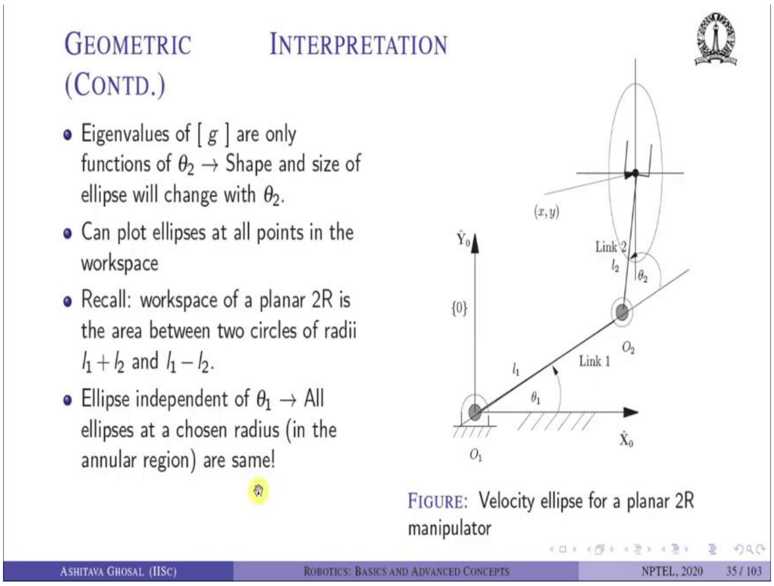

So, let us continue. So, the eigenvalues of $[g]$ are only functions of θ_2, because remember g_{ij} contains only θ_2. So, what does this mean? That the shape and size of the ellipse will change with θ_2 only, ok and we can plot ellipses at all points in the workspace.

So, we pick a value of θ_1, θ_2, so which gives the point (x, y), and I can plot the ellipse at that point. So, for example here for a particular value of θ_2 and θ_1, ok important is θ_2 I can find this ellipse it may look like this at this point, ok.

Now, let us go back and recall for this 2R manipulator the workspace lies between two circles of radii $(l_1 + l_2)$ and $(l_1 - l_2)$. The maximum is $(l_1 + l_2)$ and the minimum is $(l_1 - l_2)$. The ellipse is independent of θ_1, ok. So, all ellipse at the chosen radii in the annular region are same, ok. So, if you think about it, it does not depend on θ_1. So, if I am at some point and then I rotate θ_1, so basically I trace a circle in this annular region the ellipses which I draw will look the same.

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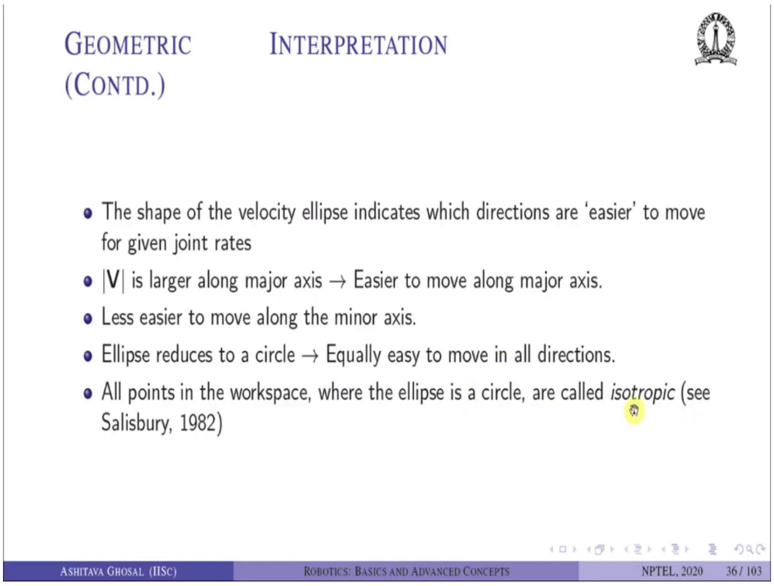

The shape of the velocity ellipse indicates which directions are easier to move for a given joint rates, ok. So, is this true? Yes, because the magnitude of the velocity vector is larger along the major axis and smaller along the minor axis. So, you can think of it that if I give you $\dot{\theta}_1$ and $\dot{\theta}_2$, I can move more easily along the major axis than along the minor axis, ok. That is what is mentioned here.

If the ellipse reduces to a circle, we can move equally easily in all the directions, ok. All points in the workspace where the ellipse is a circle are called isotropic, ok. Isotropic is word we use in many areas. So, it basically means it is same in all directions. So, in this case the magnitude of the velocity vector in same in all directions and it was coined by Salisbury in 1982.

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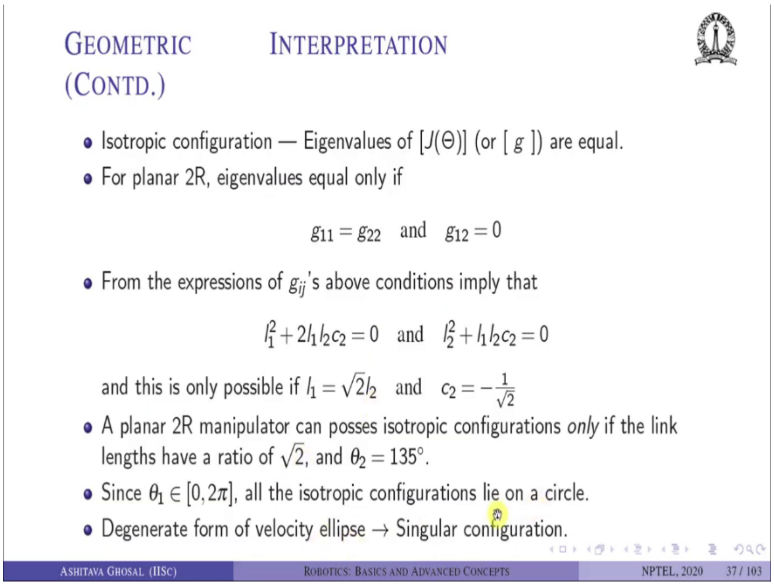

So, isotropic configuration basically the eigenvalues of $[J(\Theta)]$ or $[g]$ are equal, ok. So, for planar 2R manipulator eigenvalues are equal only if $g_{11} = g_{22}$ and $g_{12} = 0$. So, we can go back and see the expression for the eigenvalues which we derived for this case, ok. It is a quadratic equation. So, it is like $\sqrt{b^2 - 4ac}$, so that $\sqrt{b^2 - 4ac} = 0$.

So, from the expression of g_{ij}'s and using this above condition we can show that it leads to two expressions, one is $l_1^2 + 2l_1 l_2 c_2 = 0$, and $l_2^2 + l_1 l_2 c_2 = 0$, ok. And this is only possible if you have $l_1 = \sqrt{2} l_2$.

So, we can think of it one is $l_1^2 + 2l_1 l_2 c_2 = 0$, and the other one is $l_2^2 + l_1 l_2 c_2 = 0$, ok. So, $l_1 = 2l_2^2$, ok. So, $l_1 = \sqrt{2} l_2$. And $c_2 = -1/\sqrt{2}$. So, this is some angle.

What is this angle? $\theta_2 = 135$ degrees. So, a planar 2R manipulator can possess isotropic configuration only if the link lengths have a ratio of $\sqrt{2}$ and $\theta_2 = 135$ degrees. So, since θ_1 can change between 0 and 2π all the isotropic configurations lie on a circle, ok.

Remember if the link lengths are not related by this relationship $l_1 = \sqrt{2} l_2$, then we do not have any isotropic configuration. The degenerate form of the velocity ellipse is something called as the singular configuration and we will look at singular configurations later.

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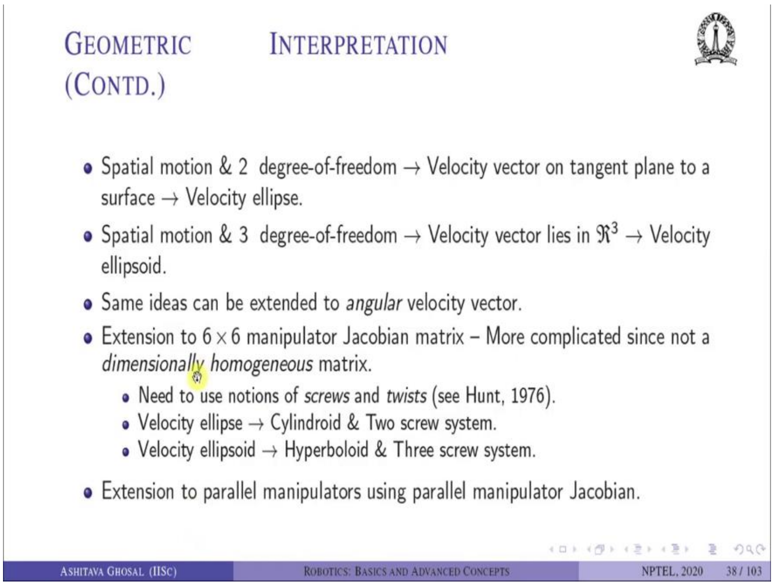

Let us continue. If you have spatial motion and 2 degrees of freedom the velocity vector lies on a tangent plane to the surface, ok. So, 2 degrees of freedom motion the end point lies in a surface in 3D space and the velocity vector will be tangent to that surface at that point. And on that tangent we again have a velocity ellipse. If it is spatial motion and 3 degrees of freedom the velocity vector lies in 3D space, ok.

And you can see that the velocity will have 3 components x, y, z and the tip of the velocity vector will describe the ellipsoid in 3D space, ok. The same ideas can be extended to angular velocity vector. So, you can think of the angular velocity vector lying on a plane and we have an ellipse in 3D space it will be on the left side.

The extension to 6×6 manipulator Jacobian matrix because in general for a 6 degree of freedom robot will have a 6×6 Jacobian matrix. It is much more complicated, since the Jacobian matrix is not dimensionally homogeneous matrix. Again, we have some part linear velocity with units of meters per second and some parts which are radians per second, ok. If you want to analyze both of them together we need to use this notions of screws and twists, ok.

It has been done by several theoretical kinematics researchers; for example, Hunt. So, the velocity ellipse which we get for planar 2R will extend and be called something as a cylindroid and two screw system. The velocity ellipsoid can be extended to a hyperboloid and a three screw system. So, we are not going to discuss this, but those of you who are interested can look up this book by Hunt.

And we can extend this notion of velocity ellipsoids to parallel manipulator using parallel manipulator Jacobian, ok. So, what is the geometric interpretation of the Jacobian? It tells you that there are certain directions where the tip of the velocity vector can be larger than in the perpendicular direction. It is easier to move along certain directions than in the perpendicular direction and this is true for plane and in 3D space, ok.

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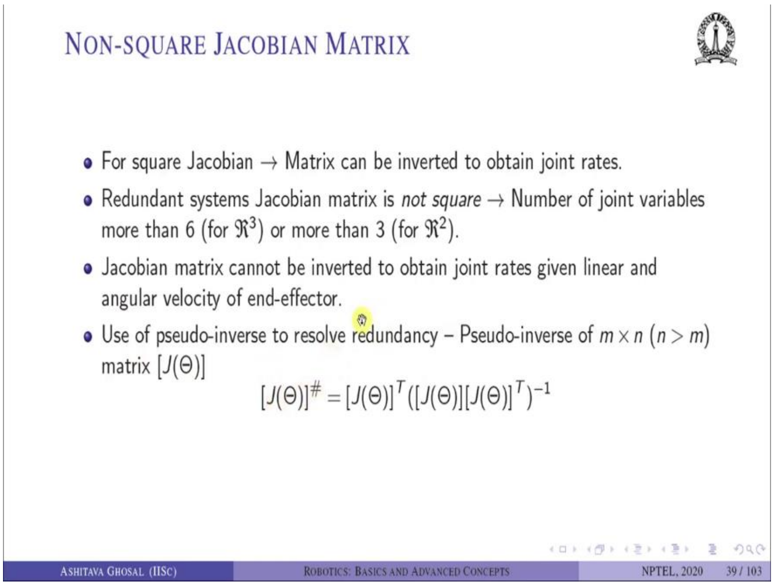

So, for square Jacobian matrix can be inverted to obtain joint rates. What happens if the Jacobian is non-square? Ok. And this happens in redundant systems, the Jacobian matrix is non-square because the number of joint variables will be more than 6, ok. So, it will have $6 \times n$, suppose I have n joint variables the number of rows are still 6 because they represent the linear and angular velocity of the end effector. But if I have 7 joints, so it will the Jacobian matrix will be 6×7, ok. It is non-square.

The Jacobian matrix cannot be inverted to obtain joint rates given linear and angular velocity of the end effector, because we have more unknowns than variables. In such a case we can do what is called as a pseudo inverse, to resolve this redundancy. So, the pseudo inverse of an $m \times n$ matrix, where $n > m$, is given by this formula and this $[J(\Theta)]^{\#}$ denotes the pseudo inverse of Jacobian matrix.

It is given by $[J(\Theta)]^T([J(\Theta)][J(\Theta)]^T)^1$. So, what you can see is if this is $m \times n$, J transpose will be $n \times m$, so this product together is $m \times m$ and we and it is a square matrix and we can find the inverse, ok. So, we can pre-multiply by the Jacobian matrix to make it consistent, ok. So, this is the formula of the pseudo inverse.

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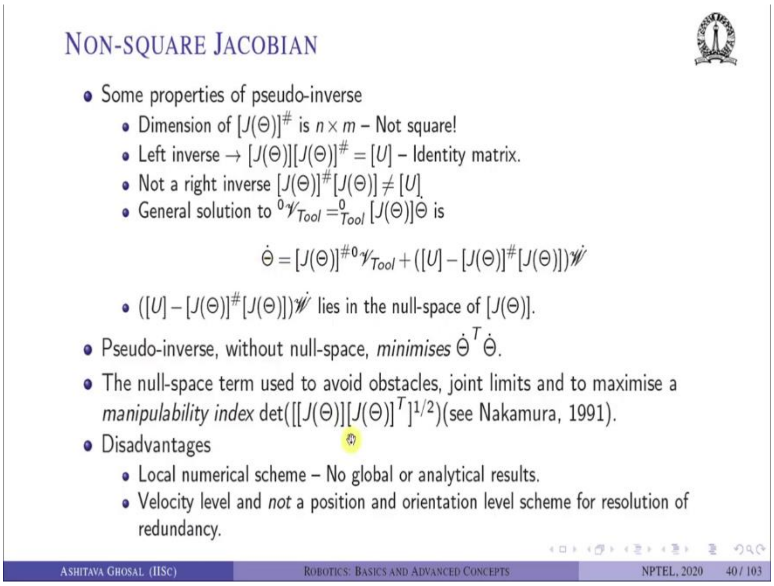

So, let us look at some of the properties of the pseudo inverse. The dimension of this $[J(\Theta)]^{\#}$ is $n \times m$, it is also not square. The left inverse $[J(\Theta)][J(\Theta)]^{\#}$ is identity you can prove this; however, $[J(\Theta)]^{\#}[J(\Theta)]$ is not identity that is it is not a right inverse, ok. So, remember normal matrix $n \times n$ matrix square matrix $A^{-1}A$ is identity and AA^{-1} is also identity, both the left and right inverse exist and gives you identity matrix. In this case it is not true, ok.

The general solution to this linear equation which is the linear and angular velocity of the Tool is $[J(\Theta)]\dot{\Theta}$, ok. When $[J(\Theta)]$ is non-square can be written as $\dot{\Theta} = [J(\Theta)]^{\#}\ {}^0\mathcal{V}_{Tool}$ plus another term, ok. This term is $([I] - [J(\Theta)]^{\#}[J(\Theta)])$ into something $\dot{\mathcal{W}}$, called omega dot. So, this quantity here lies in the null space of $[J(\Theta)]$, ok. So, this part is this, simple pseudo inverse part and this is the null space term.

So, the pseudo inverse without the null space minimizes $\dot{\Theta}^T\dot{\Theta}$. So, we can show that if you give ${}^0\mathcal{V}_{Tool} = [J(\Theta)]\dot{\Theta}$. So, there are many $\dot{\theta}$s which satisfies this linear and angular velocity which is given.

If you minimize $\dot{\Theta}^T\dot{\Theta}$, if you find the solution which minimizes $\dot{\Theta}^T\dot{\Theta}$ then that quantity is this $[J(\Theta)]^\#$, ok. So, $\dot{\Theta} = [J(\Theta)]^\#\ {}^0\mathcal{V}_{Tool}$ not this part only this first part is the minimum $\dot{\Theta}^T\dot{\Theta}$ from the infinitely many $\dot{\Theta}$s which are possible.

The null space term have been used to avoid obstacles joint limits and to maximize a manipulability index which is $\det\left(([J(\Theta)][J(\Theta)]^T)^{\frac{1}{2}}\right)$. Because this null space part exists people have thought of why not use it and they have used it to avoid obstacles joint limits, so if there are ranges of joint limits and to maximize something else. So, there is a book by Nakamura, and we can see this.

The disadvantage is that this is a numerical scheme, ok. This $[J(\Theta)]^\#$ which you obtained numerically and at any instant of time, ok. So, $[J(\Theta)]^\#$ can be obtained at any instant of time it is not a global or an analytical result. And this $\dot{\Theta}$ which you are obtaining it is like resolution of redundancy.

Remember we had looked at finding a solution to the redundant robot when we introduce another constraint, in this case the constraint is we are minimizing $\dot{\Theta}^T\dot{\Theta}$ which in turns gives you this $[J(\Theta)]^\#$. So, this resolution of redundancy is at the velocity level, we are discussing everything at the level of velocity, not at the position and orientation level, ok. So, it is a resolution scheme at the velocity level not at the position and orientation level.

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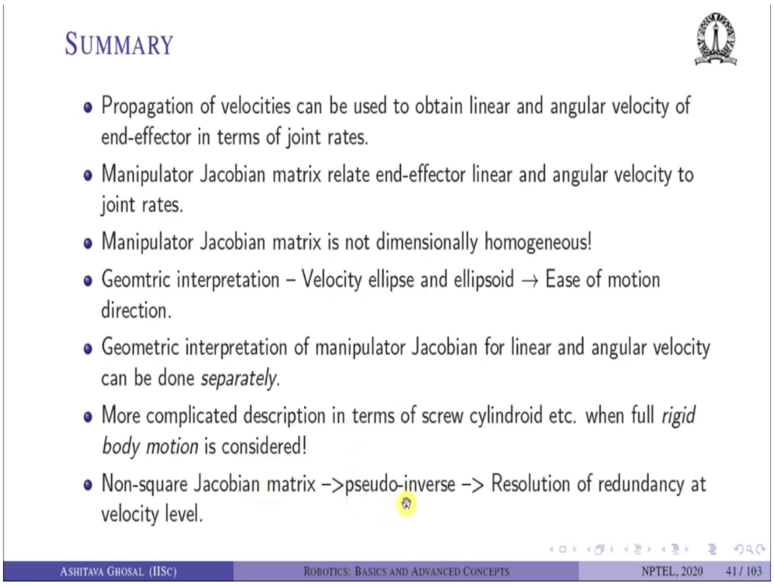

In summary the propagation of linear and angular velocities used to obtain linear and angular velocity of the end effector in terms of joint rates, ok. From these linear and angular velocity of the end effector we can obtain the Jacobian matrix which relate the linear and angular velocity to the joint rates, ok. It must be noted that the manipulator Jacobian matrix is not dimensionally homogeneous, some part contains meters per second or meters and some part is related to radians per second or unit less.

I gave you an interpretation of this Jacobian matrix which is that the Jacobian matrix is related to this velocity ellipse and ellipsoid. There are certain directions which are easy to move and certain directions which are harder to move, ok and they are related to the eigenvalues of this $g_{ij} = [J(\Theta)]^T[J(\Theta)]$, ok.

The geometric interpretation of manipulator Jacobian for linear and angular velocity can be done separately, ok. So, we can look at the linear velocity, angular velocity, and again we have two kinds of ellipses.

If you want to look at it together then it is much more complicated and I have not discussed this but we get things called cylindroids for two degrees of freedom motion, ok. So, when full rigid body motion is considered, full means both position and orientation at the same time together the Jacobian matrix leads to something called screw cylindroid, ok.

If you have a non-square Jacobian matrix, then we have to use the pseudo inverse to obtain theta dot, given the linear and angular velocity. And this can be also thought of as resolution of redundancy at the velocity level because pseudo inverse minimizes the square of the joint rates, ok.

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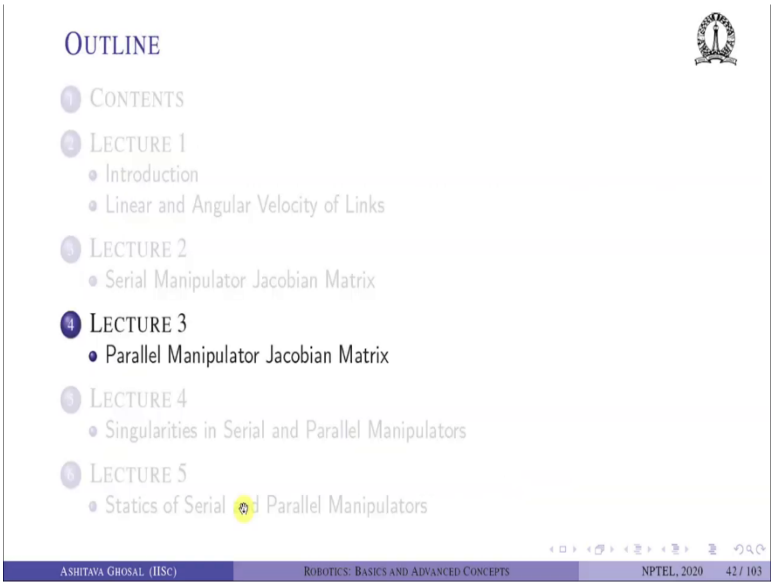

So, with this we will stop here. In the next lecture, we will look at Parallel Manipulator Jacobian Matrix.