Robotics: Basics and Selected Advanced Concepts

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Lecture - 21
Parallel Manipulator Jacobian Matrix

Welcome to this NPTEL lectures on Robotics: Basic and Advanced Concepts. In the last lecture, we had looked at the serial manipulator Jacobian matrix, ok. So, we started with propagation of linear and angular velocity vectors, then we reorganize this angular velocity vectors into something times the joint rates and then we equated that to the linear and angular velocity of the end effector ok.

So, we could start from 0 which is the fixed coordinate system propagate upwards to link 1, link 2 and so on ok. In this lecture we look at Parallel Manipulator Jacobian Matrix. So, in the parallel manipulator Jacobian matrix we do not have a single way of going from the fixed to the chosen end effector. There are multiple parts, there are multiple loops. So, we need to look at how to obtain the parallel manipulator Jacobian matrix ok.

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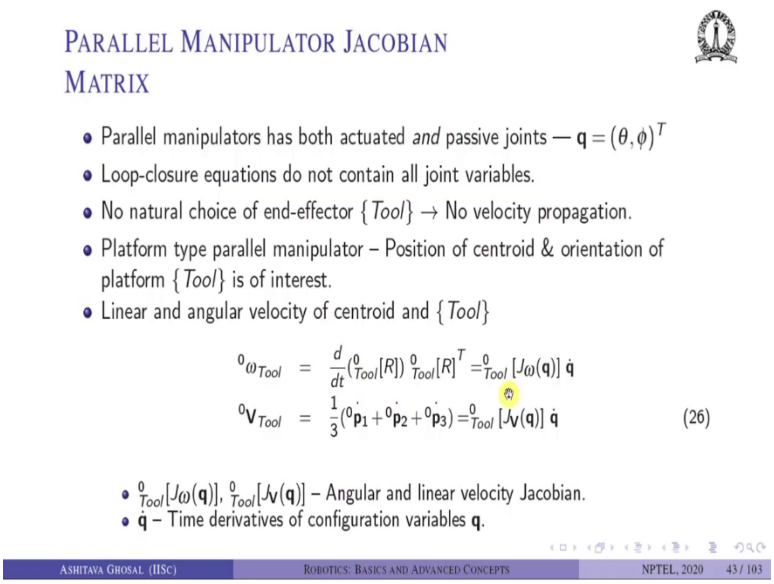

So, in a parallel manipulator it has both actuated and passive joints. So, the configuration vector or configuration space vector $\boldsymbol{q}$ contains both θ and ϕ. And we have been using θ to denote the actuated joints and ϕ to denote the passive joints. The loop closure equations do not contain all the joint variable. So, for example, we had looked at this 3-RPS parallel robot and when we broke up at the spherical joint, then it did not contain the joint variables of the spherical joint.

Other important thing is there is no natural choice of end effector. So, hence we cannot do velocity propagation. So, we do not know which one is the end effector and where to start from and where to end ok. In a platform type parallel manipulator like the Stewart platform or the 3-RPS manipulator the position of centroid and orientation of the platform is of interest most of the time ok.

So, how do we find the linear and angular velocity of the centroid of the top platform and the orientation of the top platform? Ok. So, given the orientation of the top platform and if we choose that the centroid is the point of interest the angular velocity of the top platform or the coordinate system can be obtained by ${}_{Tool}^{0}[\dot{R}]{}_{Tool}^{0}[R]^T$.

So, remember I had shown you that in general the space fixed angular velocity vector can be obtained as ${}_{Tool}^{0}[\dot{R}]{}_{Tool}^{0}[R]^T$. So, that is what is shown here. And we can rearrange this expression once we have obtained the space fixed angular velocity vector into a matrix times $\dot{\boldsymbol{q}}$.

So, now note that there will be both $\dot{\theta}$ and $\dot{\phi}$ because the position vector or the rotation matrix of the top platform or the Tool coordinate system will contain both θ and ϕ. Likewise, the linear velocity of the centroid of the top platform can be obtained as in this case the sum of the linear velocity of the 3 connection points ok.

So, in the 3-RPS case we had 3 connection points, 3 spherical joints and we can obtain the velocity of the centroid as a mean of all these 3 velocity vectors ok with respect to {0} coordinate system. For Stewart platform with many other connection points again we can obtain similar angular velocity and linear velocity vectors.

So, this matrix ${}_{Tool}^{0}[J_\omega(\boldsymbol{q})]$ which is a function of $\boldsymbol{q}$ and ${}_{Tool}^{0}[J_v(\boldsymbol{q})]$ are the angular and linear velocity Jacobian matrices ok and $\dot{\boldsymbol{q}}$ is the time derivative of the configuration variable $\boldsymbol{q}$.

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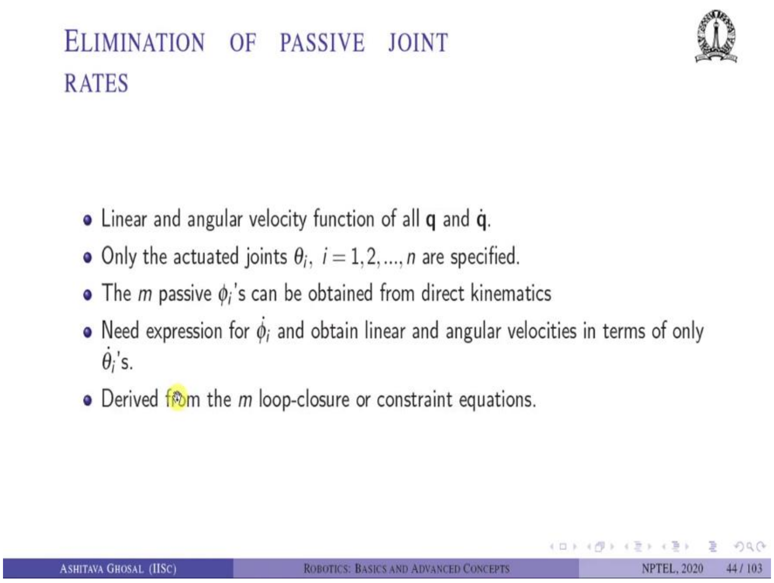

So, the linear and angular velocity vectors are function of all $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$. However, in a parallel manipulator only the actuated joints θ_i are assume there are n of them. The m passive joints can be obtained from direct kinematics ok. We also need expressions for the $\dot{\phi}$'s, the rate of change of the passive joint variables ok.

And obtain linear and angular velocity in terms of only $\dot{\theta}$ because $\dot{\theta}$ are the ones where there are motors we know what is the actuated joint rates. So, we should be able to find the linear and angular velocity of the chosen output link in terms of the actuated joint variables ok. So, basically we need to find $\dot{\phi}$ in terms of $\dot{\theta}$. How do we do that? We can obtain this from the loop closure constraint equations ok.

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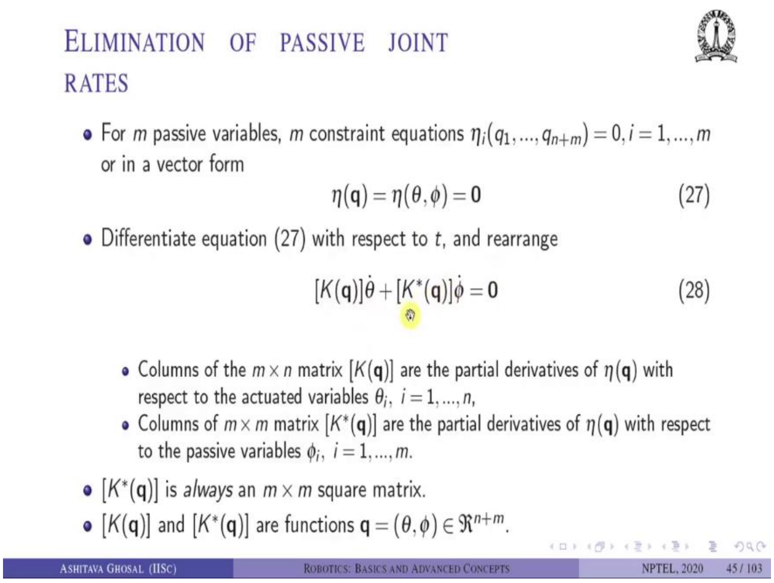

So, let us start. So, the m passive variables we have m constraint equations $\eta_i(q_1, \dots, q_{n+m}) = 0$. So, this is a general form, but basically there are m constraint equations which is of the form $\eta(\boldsymbol{q}) = 0$. So, it is a vector equation. So, we can differentiate this equation with respect to time ok.

So, when we differentiate with respect to time we will have $\dot{\theta}$ and $\dot{\phi}$. And then we can reorganize all terms containing $\dot{\theta}$ in a matrix which I am going to call $[K(\boldsymbol{q})]$, ok. So, $[K(\boldsymbol{q})]$ is a function of all the configuration variables. Likewise, I reorganize the terms containing $\dot{\phi}$ into a matrix which is $[K^*(\boldsymbol{q})]$.

So, we can always do this. Why? Because when we take the derivative of this, constraint equations, we can only get $\dot{\theta}$ and $\dot{\phi}$. We will not get any other non-linear terms in $\dot{\theta}$ and $\dot{\phi}$. So, the columns of this $m \times n$ matrix $[K(\boldsymbol{q})]$ are basically nothing but the partial derivatives of this $\eta(\boldsymbol{q})$ with respect to the actuated joint variables, ok.

So, just by chain rule; so, $\frac{d\eta(q)}{d\theta}\dot{\theta}$ will be this term ok in a matrix form. Likewise, $[K^*(\boldsymbol{q})]$ which is $m \times m$ matrix. Why? Because these are m constraint equations ok and we take the partial derivatives of $\eta(\boldsymbol{q})$ with respect to the passive variables ϕ which are m of them, $\dot{\theta}$ are n of them.

So, this $[K(\boldsymbol{q})]$ is an $m \times n$ matrix whereas, $[K^*(\boldsymbol{q})]$ is always a $m \times m$ matrix, it is a square matrix ok. So, that is what I am mentioning again. $[K^*(\boldsymbol{q})]$ is always a $m \times m$ square matrix ok and $[K(\boldsymbol{q})]$ and the $[K^*(\boldsymbol{q})]$ are functions of both θ and ϕ, ok.

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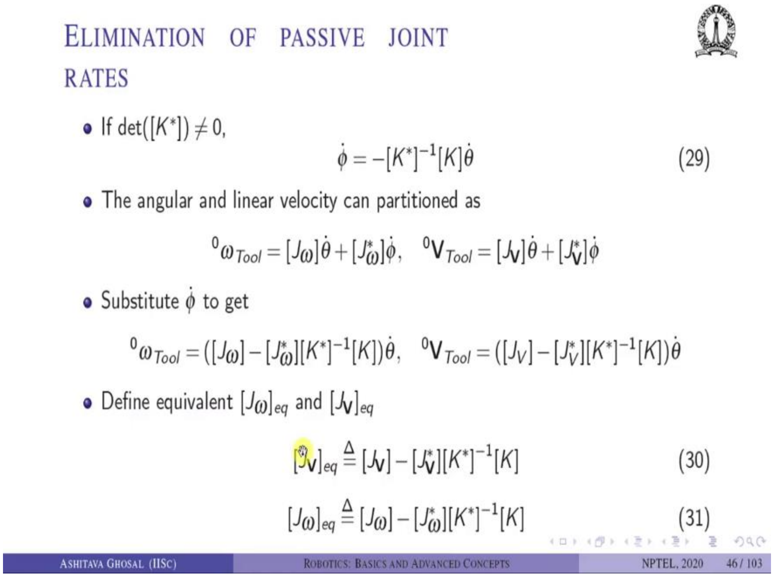

Let us continue. So, if determinant of $[K^*]$ is not equal to 0, then I can solve for $\dot{\phi}$ which is $-[K^*]^{-1}[K]\dot{\theta}$. So, you can see that this is a linear equation. If determinant of $[K^*]$ is not equal to 0, can we take the determinant of $[K^*]$? Yes, it is a square matrix. So, $\dot{\phi} = -[K^*]^{-1}[K]\dot{\theta}$, which is what is written here.

Now, we can go back to the angular velocity expression which is ${}^0\omega_{Tool}$ and partitioned it into two parts. One is $[J_\omega]\dot{\theta}$ and then $[J^*_\omega]\dot{\phi}$. We can always do this because the linear and angular velocity vectors are always linearly related to $\dot{\theta}$ and $\dot{\phi}$.

Similarly, the linear velocity of the Tool ok or chosen point on the platform ok or on the chosen output link can be written in terms of $[J_v]\dot{\theta} + [J^*_v]\dot{\phi}$. Once you have this we can substitute $\dot{\phi}$ in these two equations ok. So, if I substitute $\dot{\phi}$ in the first equation we will get the angular velocity of the Tool is $([J_\omega] - [J^*_\omega][K^*]^{-1}[K])\dot{\theta})$.

So, you can see this $\dot{\phi}$ will become $-[K^*]^{-1}[K]\dot{\theta}$ and then we can take all of this together then we will have $[J_\omega]$ minus this. Likewise, the linear velocity of the Tool will be $([J_v] - [J^*_v][K^*]^{-1}[K])\dot{\theta}$. So, we can now define an equivalent Jacobian matrix for the angular velocity for the parallel robot.

So, $[J_\omega]_{eq}$ and likewise $[J_v]_{eq}$. So, $[J_v]_{eq} = [J_v] - [J^*_v][K^*]^{-1}[K]$. $[J_\omega]_{eq} = [J_\omega] - [J^*_\omega][K^*]^{-1}[K]$. So, it is really straight forward. So, all I have done is if determinant of

$[K^*]$ is not equal to 0, I have solved for $\dot{\phi}$ and substituted back into the expressions for the linear and angular velocity vector and then reorganize them.

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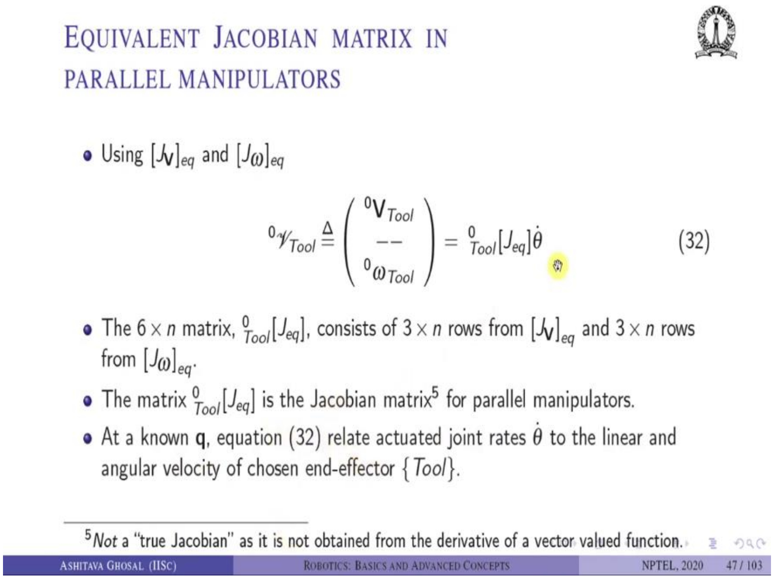

Now, with $[J_v]_{eq}$ and $[J_\omega]_{eq}$ we can write the equivalent Jacobian in this form. So, we have on the left hand side the linear velocity of the Tool and the angular velocity of the Tool they are separated by this dash. So, remember this as meters per second then this as radians per second. These are different units and we can write together ${}_{Tool}^{0}[J_{eq}]\theta$.

So, what is ${}_{Tool}^{0}[J_{eq}]$? The $6 \times n$ matrix here will contain the first $3 \times n$ rows are from $[J_v]_{eq}$ and the next $3 \times n$ rows are from $[J_\omega]_{eq}$. So, this is the matrix which is the parallel manipulator Jacobian matrix ok. So, like in serial manipulators it is not true Jacobian matrix. In the sense that it is not derived from the derivative of a vector valued function ok. It contains the top half contains units of meter, the bottom half contains unitless ok.

So, at a known $\boldsymbol{q}$, we can solve or we can relate the $\dot{\theta}$'s with the linear and angular velocity of the Tool ok. So, if I know what is $\boldsymbol{q}$, this ${}_{Tool}^{0}[J_{eq}]$ is a function of $\boldsymbol{q}$ of the configuration. At every configuration this ${}_{Tool}^{0}[J_{eq}]$ should be different. If I give you $\dot{\theta}$ you can multiply this ${}_{Tool}^{0}[J_{eq}]\dot{\theta}$ and obtain the linear and angular velocity of the Tool ok.

Very similar to the serial manipulator Jacobian matrix except now that we have taken some effort to eliminate the passive joint rates and obtain an equivalent Jacobian in terms of only the actuated joint rates ok.

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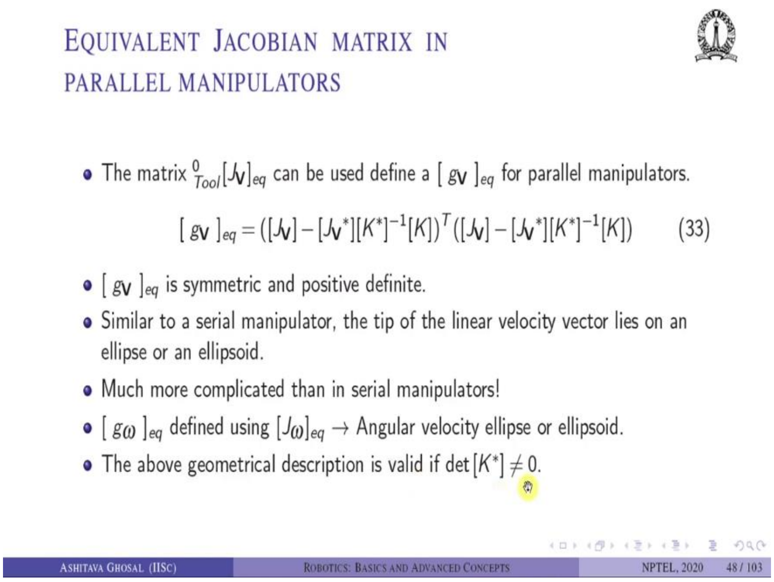

So, like serial robots ok or serial manipulators this $[J_v]_{eq}$ can be also be used to define a $[g_v]_{eq}$ for parallel manipulators. So, remember we have in during the geometric interpretation of the Jacobian matrix we have taken $[J]^T[J]$ and then we had found the eigenvalues of the equivalent $[J]^T[J]$.

So, in this case we have to find the $[g_v]_{eq}$ which is $([J_v] - [J_v^*][K^*]^{-1}[K])^T([J_v] - [J_v^*][K^*]^{-1}[K])$. So, this $[g_v]_{eq}$ is symmetric and positive definite ok. It is symmetric because it is like $A^T A$. And similar to serial manipulator the tip of the linear velocity vector lies on an ellipse or an ellipsoid. So, if it is moving in a plane, it lies on the ellipse. If platform is moving in 3D space the linear velocity vector traces an ellipsoid.

However, as you can see this is so much more complicated. It is not simply $[J_v]^T[J_v]$. We have all these $[J_v^*]$, $[K^*]^{-1}$, $[K]$. So, it also contains the loop closure constraint equations and the effect of the loop closure constraint equations because we have taken derivatives of those and obtained $[K]$ and $[K^*]$ matrices ok.

We can also find $[g_\omega]_{eq}$ again using $[J_\omega]_{eq}$ and again the angular velocity can be described as an ellipse. The tip of the angular velocity method this is an ellipse or an ellipsoid. So, all this is happening only if and only if determinant of $[K^*]$ not equal to 0. So, if $[K^*]$ were equal to 0, then we could not solve $\dot{\phi}$ because it has $[K^*]^{-1}$.

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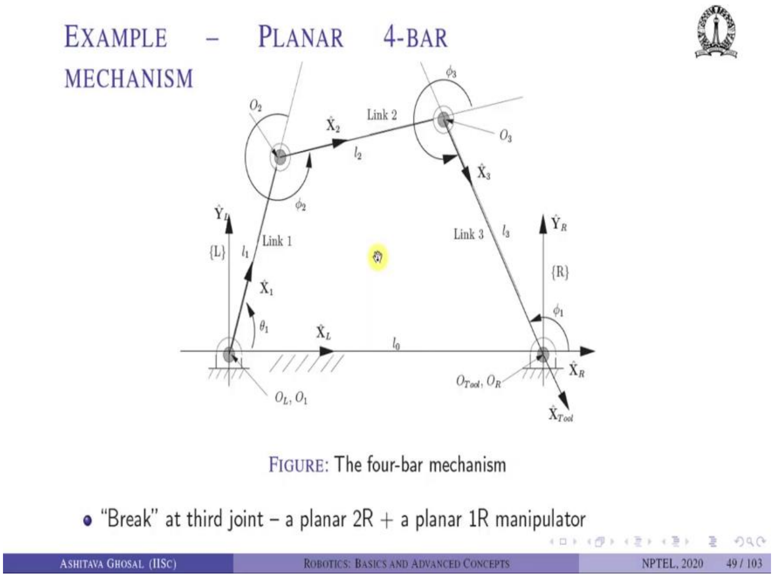

So, let us take an example. So, again we will take the simplest possible example, which is a planar 4-bar mechanism. So, we break this 4-bar mechanism at this 3rd joint. So, we will get one planar 2R and a single 1R manipulator. So, in the planar 2R, we have θ_1 and ϕ_2 these 2 angles and l_1 and l_2 whereas, the 1R we will have ϕ_1 and l_3, these are the main important things.

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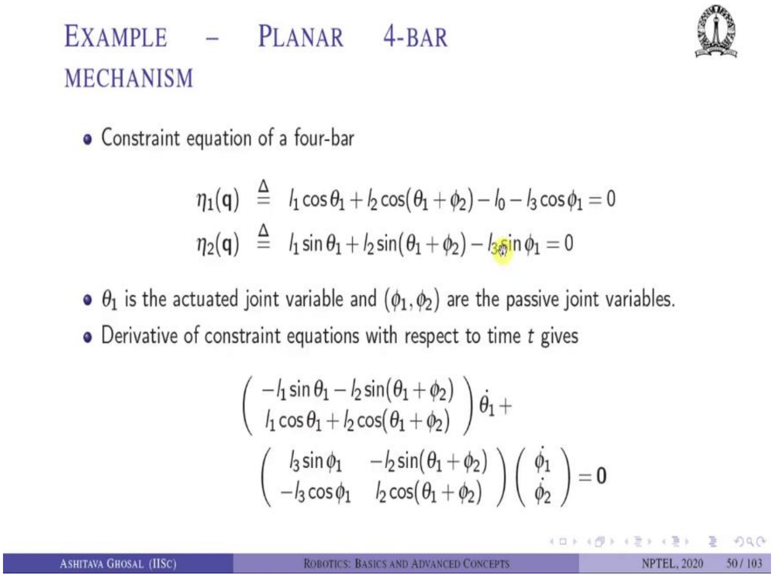

So, the constraint equation of the 4-bar for breaking at the third joint we are seen this earlier. It is $l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \phi_2) - l_0 - l_3 \cos\phi_1 = 0$. So, this is the x component. Similarly, the y component is $l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \phi_2) - l_3 \sin\phi_1 = 0$.

So, in these two constraint equations θ_1 is the actuated joint variable and ϕ_1 and ϕ_2 are the passive joint variables. So, we can take the derivative of these two equations with respect to time and then take all terms containing $\dot{\theta}_1$ and all terms containing $\dot{\phi}_1$ and $\dot{\phi}_2$ and write it in this form.

So, the first term is a column vector which is $\begin{pmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \phi_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \phi_2) \end{pmatrix}$. When you take the derivative you will get these terms times $\dot{\theta}_1$; and then the next term. So, this is the $[K]$ matrix. So, this is $[K]\dot{\theta}_1$.

Then we can also find the derivative its $[K^*]$ matrix and rearrange the derivatives and then we can write $\begin{pmatrix} l_3 \sin\phi_1 & -l_2 \sin(\theta_1 + \phi_2) \\ -l_3 \cos\phi_1 & l_2 \cos(\theta_1 + \phi_2) \end{pmatrix}$ and multiplying $\dot{\phi}_1$ and $\dot{\phi}_2$. So, this is the $[K^*]$ matrix. So, this is $[K]\dot{\theta} + [K^*]\dot{\phi} = 0$.

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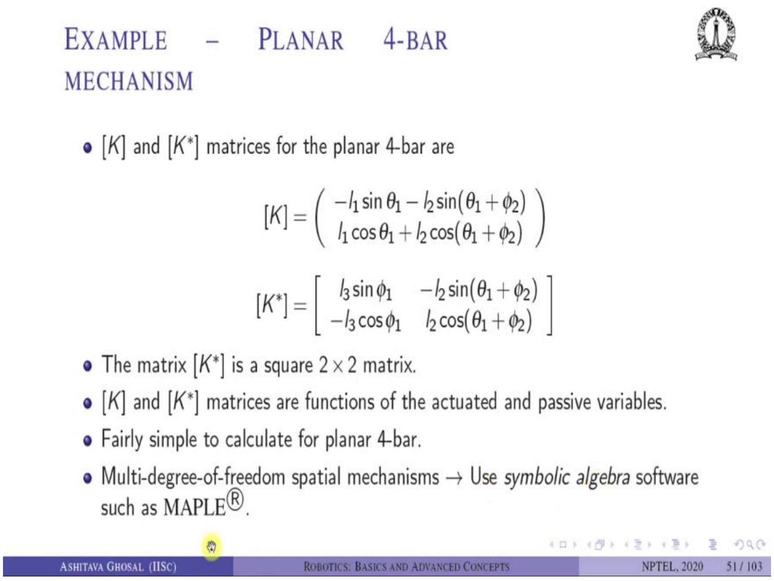

So, the $[K]$ and $[K^*]$ matrix for the planar 4-bar are. So, $[K]$ is a column vector actually, $[K^*]$ is a 2×2 matrix. So, that as expected the $[K]$ and $[K^*]$ matrices are functions of both the actuated joint variables which is θ_1 and the passive joint variables with this ϕ_1 and ϕ_2.

So, ϕ_1 does not appear in $[K]$, but it does appear in $[K^*]$. So, as you can see it is very very easy to calculate for a planar 4-bar. For multi degree of freedom spacial mechanisms this is not so easy to do ok; because the loop closer equations will be much more complex and then we have to take the derivatives, and then we have to organize each one of these into something into $\dot{\theta}$ and something into $\dot{\phi}$.

So, it requires a lot of effort, but we can use a symbolics algebra software such as MAPLE to do this ok. And in fact, I will show you examples later where we have use this symbolic software package called MAPLE ok and then derive the $[K]$ and $[K^*]$ matrices ok.

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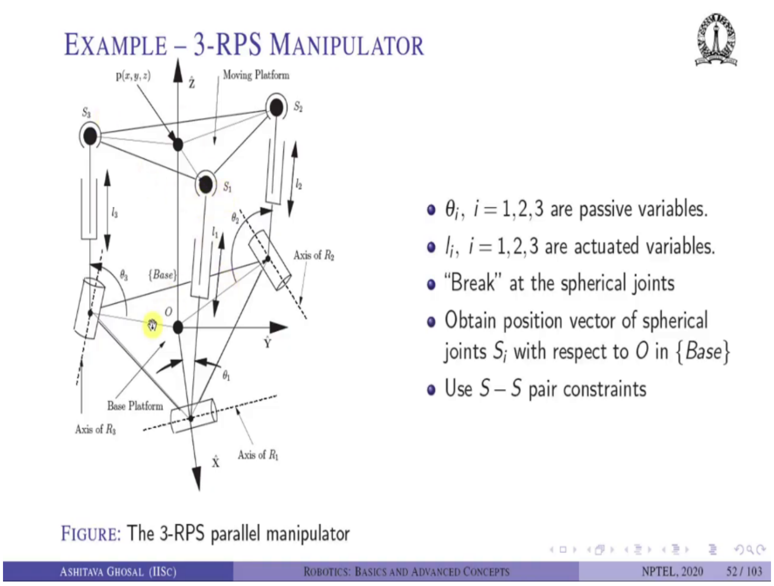

So, let us take another example which is a spatial 3 degree of freedom manipulator. So, this is that well known 3-RPS manipulator which we have looked at earlier. So, there is a fixed base. There is a moving platform. Then each point the top platform is connected to the point in the fixed base by RPS ok.

So, R rotation is θ, actuated joint variable is l and S is the spherical joint which since we are going to break it. At the spherical joints the angles at the spherical joints do not appear. So, there are 3 actuated joints l_1, l_2, l_3 and there are 3 passive joints θ_1, θ_2, θ_3. So, as I said we break at the spherical joint.

Then we obtain the position vector of S_1 which is spherical joint one S_2 and S_3 with respect to the fixed coordinate system which is at the centroid of the bottom platform ok. And the point which is of interest the output link is the moving platform and the point of interest or the linear velocity which we will calculate is for the centroid of the top platform, which is $p(x, y, z)$. How do I finds a loop closer equations?

As I have shown you earlier we find these vectors with respect to $\{0\}$ and then $\{0\}$ to $\{S_1\}$, $\{0\}$ to $\{S_2\}$, $\{0\}$ to $\{S_3\}$ ok. And then we show that the distance between S_1 and S_2 is constant, distance between S_1 and S_3 is constant and between S_2 and S_3 is constant. So, we use the S-S pair constraint equations.

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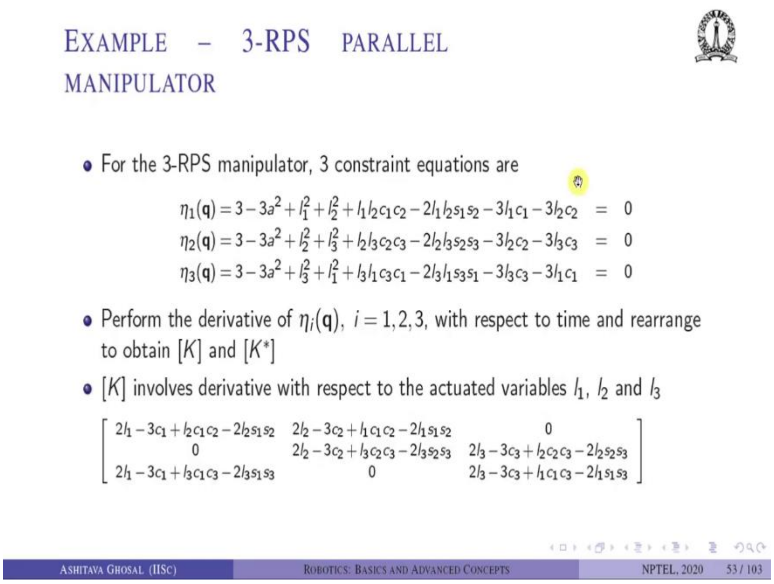

So, for the 3-RPS manipulator the 3 constraint equations are first equation is $3 - 3a^2 + l_1^2 + l_2^2 + l_1 l_2 c_1 c_2$ and so on ok. So, here l_1 and l_2 are the actuations at the 2 prismatic joints, first and second prismatic joint; a is one of the sides of the top platform, b is taken to be 1 very similar to what we have done earlier and c_1 means $\cos\theta_1$ ok, s_1 means, $\sin\theta_1$.

The second equation is the distance between 2 and 3 ok. So, it contains θ_2 and θ_3 and l_2 and l_3 point spherical joint S_2 and S_3. The third equation is the distance between spherical joint S_3 and spherical joint S_1 is constant ok and hence we get another equation.

So, they all look similar except the angles are different slightly. So, we can again take the derivatives of this 3 constraint equations with respect to time and rearrange to find $[K]$ and $[K^*]$ matrix ok. So, $[K]$ will involve the derivatives with respect to the actuated joints variables l_1, l_2, l_3 ok. So, you can see that if I take the derivative of the first equation with

l_1, I will get one term like this. So, it is $2l_1$, $-3c_1$ from here and then $l_2c_1c_2$, and $-2l_2s_1s_2$ from here.

Second term is derivative with respect to l_2. So, we can again this term. There is no l_3 appearing in this first equation. So, that term will be 0 ok. Likewise, if the second row l_1 does not appear.

So, when you take the derivative of the second equation with respect to l_1 partial derivatives, it will be 0 something and something and the third row will be again there is l_1 appearing. So, partial derivatives there will be a term and you can see there will be $2l_1$ and then $l_3c_3c_1$ and so on ok and you can just by inspection see that this is the $[K]$ matrix.

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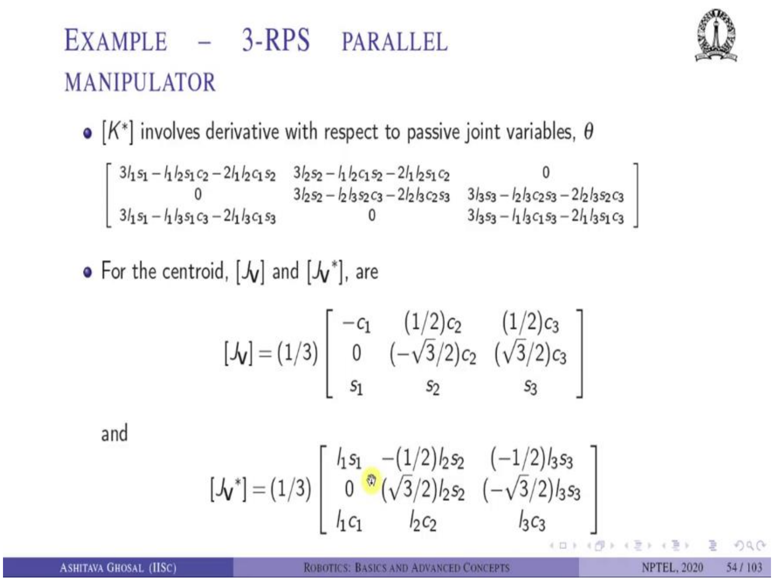

The $[K^*]$ matrix involves partial derivatives with respect to the joint variables. In this case the passive joint variables are θ ok. So, we have to take derivatives of this with respect to θ_1. So, derivative of c_1 will become $-\sin\theta_1$ and so on. So, this will not be there ok.

So, you can see that the partial derivative with the first equation with respect to θ_2, in the second term is with respect to θ_2 ok, θ_2 is also here. There is no θ_3 occurring in this equation ok. So, the term will be 0.

Likewise, there is no θ_1 occurring in the second equation because it is the distance between the second and the third spherical joint. And in the third equation one more term will be 0 ok. So, for the centroid $[J_v]$ and $[J_\omega^*]$ are given by this. $[J_v]$is this and $[J_v^*]$ can also be computed; so, $[J_v^*]$ corresponding to the $[J]$ corresponding to the passive joint variables, ok.

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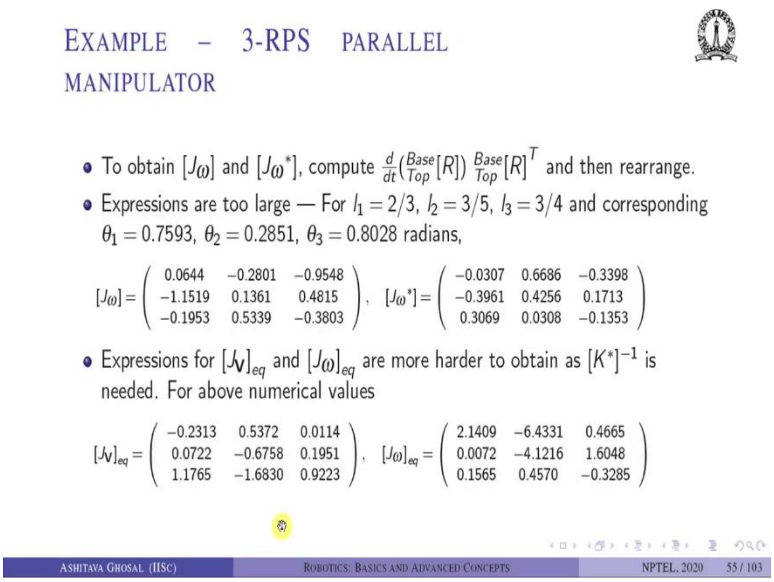

So, we have $[K]$, we have $[K^*]$, we have $[J_v]$ and we have $[J_v^*]$ ok. To obtain $[J_\omega]$ and $[J_\omega^*]$ we have to take the $[\dot{R}][R]^T$ first ok and then rearrange. So, we have to find the angular velocity vector using $[\dot{R}][R]^T$. Then rearrange it into something into $\dot{l}$ and something into $\dot{\theta}$, $\dot{l}$ is the actuated joint variable. So, it will be $[J_\omega]\dot{l}$ and $[J_\omega^*]\dot{\theta}$.

So, these expressions are very large. As you can see the rotation matrix derivatives, then, post multiplying where the transpose it will become too much ok. So, we have done this using MAPLE, but for simplifying the task of MAPLE we have taken $l_1 = 2/3$, $l_2 = 3/5$, and $l_3 = 3/4$.

And the corresponding to this l_1, l_2, l_3 we can find θ_1, θ_2, θ_3 using the direct kinematics of this 3-RPS manipulator ok which we have done earlier ok. Remember we get an 8th degree polynomial and so on. And substituting this l_1, l_2, l_3 and $\theta_1, \theta_2, \theta_3$ in this $[\dot{R}][R]^T$ and simplifying we will get $[J_\omega]$ as a 3×3 matrix ok.

So, these are terms and $[J_\omega^*]$ is another 3×3 matrix. Likewise, for this set of l_1, l_2, l_3 and $\theta_1, \theta_2, \theta_3$ we can also find $[J_v]_{eq}$ and $[J_\omega]_{eq}$.

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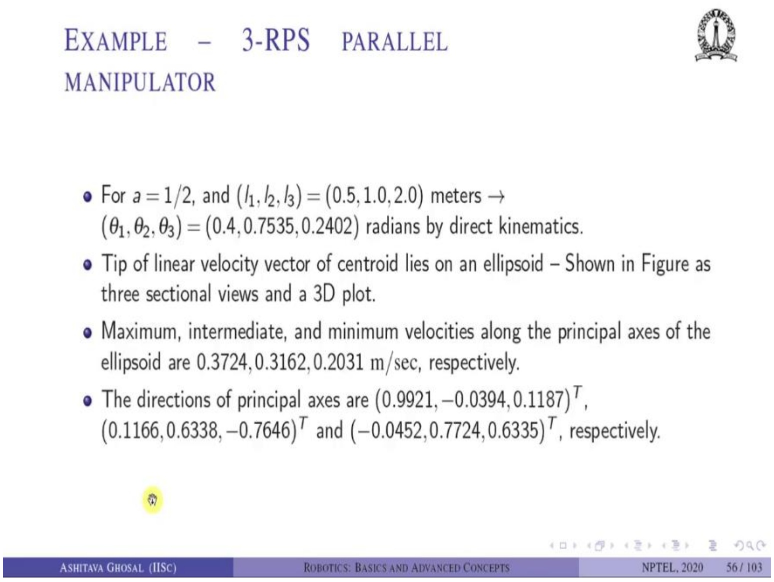

And for a equals half, l_1, l_2, l_3 is equal to 0.5, 1.0, and 2.0 meters. We can solve for $\theta_1, \theta_2, \theta_3$, we will get this by direct kinematics ok. So, I am picking a point and then at that point, I have going to draw the tip of the linear velocity vector ok. So, this is a motion in 3D. So, we expect that the tip of the linear velocity vector traces an ellipsoid ok.

So, I am going to show you the picture of this ellipsoid obtained numerically. And the maximum intermediate and minimum velocities along the principal axes of the ellipsoid are again we can compute because they are the square root of the eigenvalues of the equivalent Jacobian matrix ok. So, 0.3724, 0.3162, 0.2031 meters per second and the direction of the principal axes also we can find out, how is the ellipsoid located in 3D space with respect to the {0} coordinate we will get these vectors ok.

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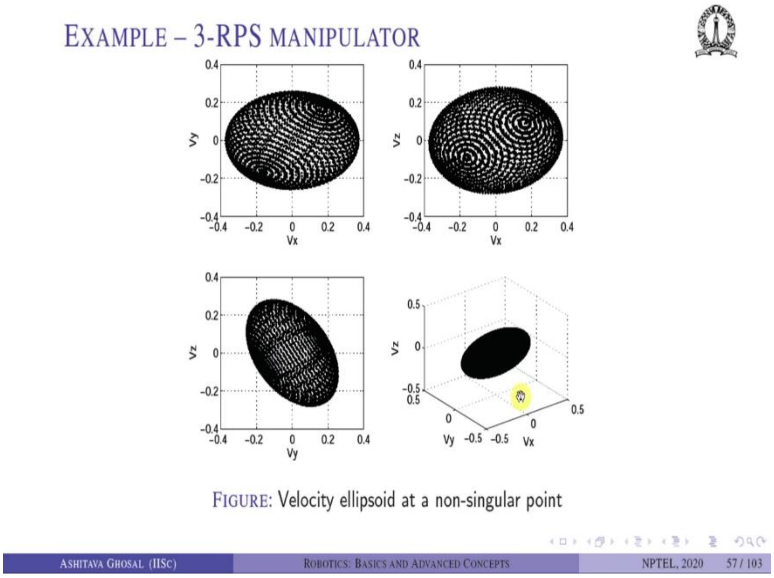

So, this is what is shown in this picture. So, this view is a 3D view 3D plot in MATLAB. So, it shows you that this is V_x, V_y, V_z. So, from (0, 0, 0) which is the center here the tip traces this ellipsoid. We can also look at the sectional views. We can see what is happening in the Z-Y plane or the Z-X plane or the X-Y plane and you can see all of them are ellipses which is what is expected.

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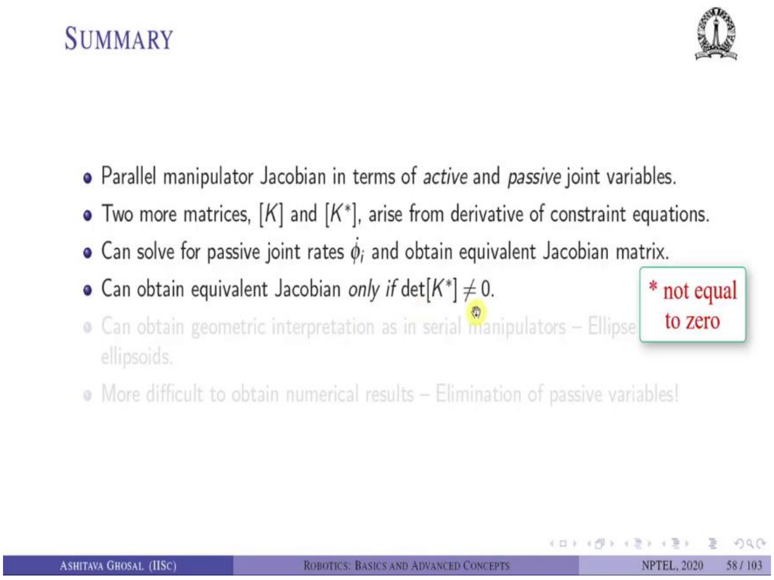

So, if you look at an ellipsoid in these sections we will get ellipses. So, in summary the parallel manipulator Jacobian in terms of active and passive joint variables we can derive. Two more matrices $[K]$ and $[K^*]$ occur ok. So, these come from the derivative of the constraint equations.

We can solve for the passive joints rates $\dot{\phi}_i$ in terms of the actuated joint rates and then obtain an equivalent Jacobian. So, we can substitute $\dot{\phi}$ into the Jacobian matrix ok. It will be some $[J_\omega]\dot{\theta} + [J_\omega^*]\,\dot{\phi}$, we substitute $\dot{\phi}$ and get a $[J_\omega]_{eq}$. We can obtain this equivalent Jacobian only if determinant of $[K^*]$ is equal to 0. Why? Because otherwise we cannot solve for ϕ.

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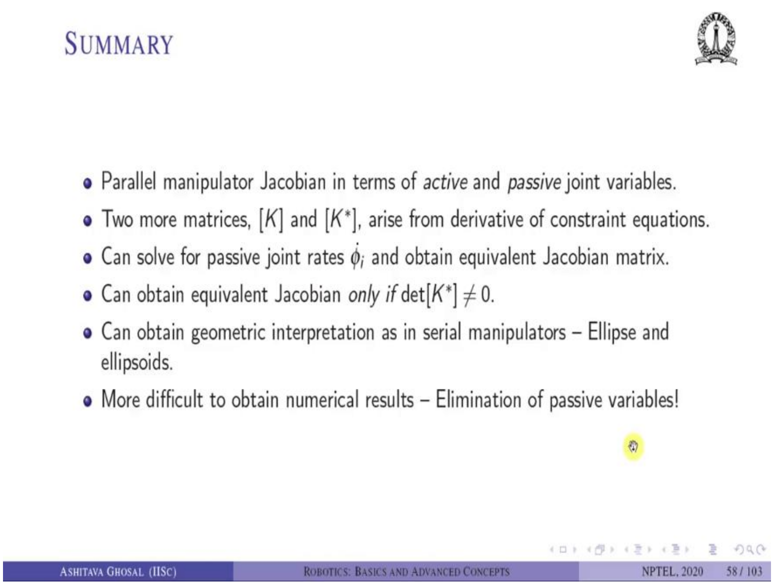

And we can obtain a geometric interpretation as in the case of the serial manipulator. So, we can show that the tip of the centroid or tip of the point chosen on the output link traces an ellipse if it is in moving in a plane and or if it is an ellipsoid. So, the 3-RPS it traced an ellipsoid ok.

So, it is clearly much more difficult than serial manipulators right because there are several problems. We had to eliminate the passive variables; we had to find the point where you know for actuated joint variables what were the passive joint variables. We also have to compute this $[K^*]^{-1}$, $[K]$ and $[J_v^*]$ and $[J_\omega^*]$ it is a many many terms are there ok, but it can be done numerically as I have shown you in these two examples.

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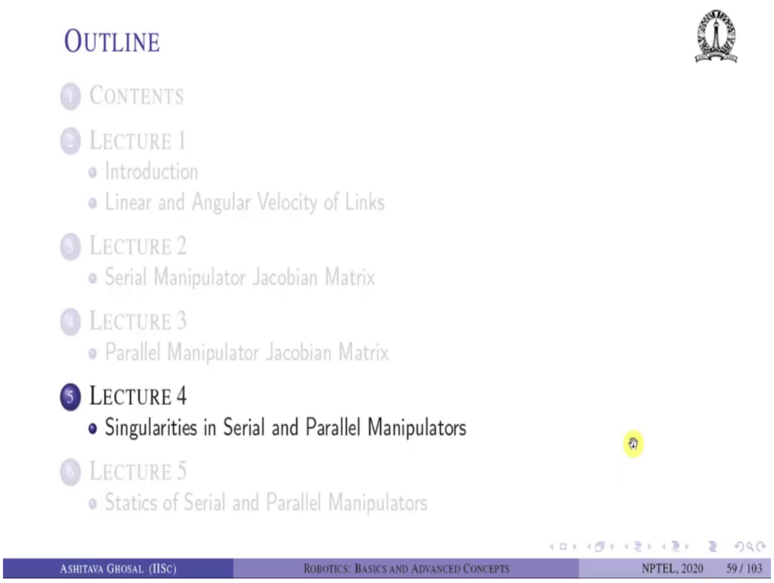

So, with this we will stop this lecture. In the next lecture we will look at the Singularities in Serial and Parallel Manipulators.