Robotics: Basics and Selected Advanced Concepts
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Lecture - 22
Singularities in Serial and Parallel Manipulators

Welcome to this NPTEL lectures on Robotics: Basic and Advanced Concepts. In the last lecture, we looked at parallel manipulator Jacobian matrix ok. In this lecture, we look at the Singularities in Serial and Parallel Manipulators.

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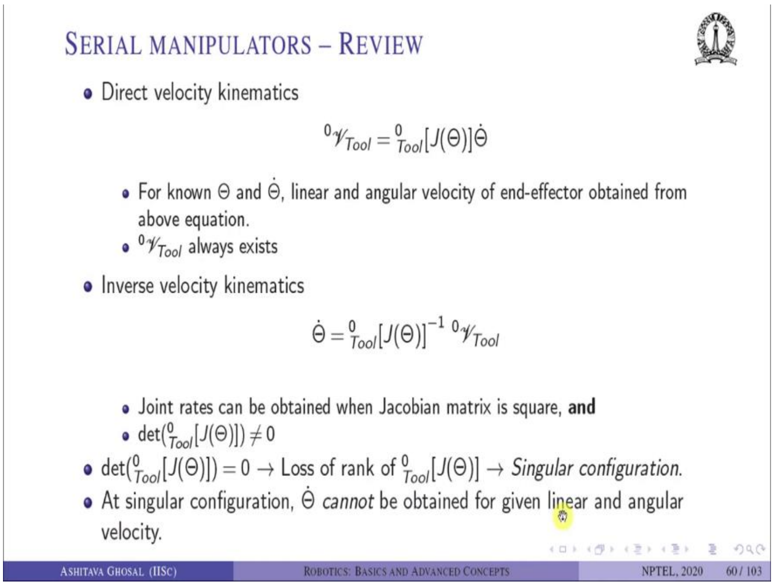

Just a review of the serial manipulator velocity kinematics. So, the direct velocity kinematics is given by on the left-hand side of this equation, we have linear and angular velocity of the Tool coordinate system which is related to the joint rates $\dot{\Theta}$ by this Jacobian matrix; so, ${}_{Tool}^{0}[J(\Theta)]\dot{\Theta}$ will give the linear and angular velocity of the end effector or the $\{\text{Tool}\}$ coordinate system.

So, for known Θ and $\dot{\Theta}$, linear and angular velocity of the end-effector can be obtained from the above equation and this ${}^{0}\mathcal{V}_{Tool}$ always exists ok. So, remember this is not a proper vector, this is not a proper matrix because part of the matrix contains meters per second or meters ok and the bottom half the angular velocity is in radians per second.

So, the inverse velocity kinematics exists if this Jacobian matrix is square and the inverse of the Jacobian matrix exists. So, given linear and angular velocity of the Tool coordinate system, if the inverse of Jacobian matrix exist, you can multiply ${}^{0}_{Tool}[J(\Theta)]^{-1}\ {}^{0}\mathcal{V}_{Tool}$ and we will get $\dot{\Theta}$. So, the joint rates in summary can be obtained when the Jacobian matrix is square and the determinant of this Jacobian matrix is not 0.

When the determinant of the Jacobian matrix is 0, what happens is there is a loss of rank of this Jacobian matrix ok, this from linear algebra, straightforward linear algebra. If you have a matrix a, $n \times n$ matrix and the determinant of that matrix is 0, then the rank of that matrix is less than n. In robotics, these are called singular configuration. At a singular configuration, $\dot{\Theta}$ cannot be obtained for a given linear and angular velocity.

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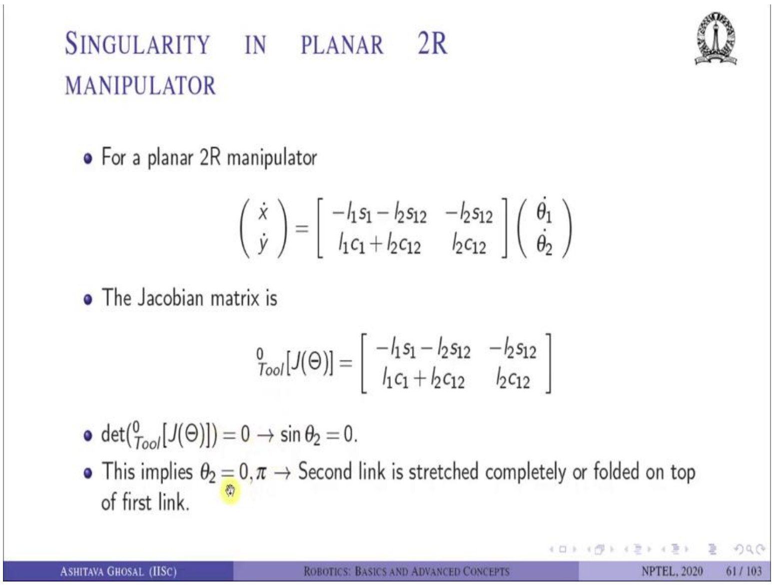

So, example for a planar 2R manipulator $(\dot{x}, \dot{y})^T$ can be written in terms of $\dot{\theta}_1$ and $\dot{\theta}_2$ dot. $\dot{\theta}_1$ and $\dot{\theta}_2$ are the joint rates in terms of sine and cosine of the angles and l_1, l_2 and so on ok, this we have seen earlier also. So, the Jacobian matrix in this case is the quantity in the square bracket.

So, the first element is $(-l_1s_1 - l_2s_{12})$, the second element is $(-l_2s_{12})$, the second row first element is $(l_1c_1 + l_2c_{12})$ and two, two element is (l_2c_{12}), c_{12} stands for $\cos(\theta_1 + \theta_2)$, s_{12} stands for $\sin(\theta_1 + \theta_2)$ and s_1 and s_2 and so on are the sin and cosine of the angles.

So, the determinant of this can be easily computed you know this into this minus this into this and it turns out that the determinant is 0 when $\sin\theta_2 = 0$. The determinant will simplify simple $\sin\theta_2$ and this is 0, when $\theta_2 = 0$ or π.

So, basically the determinant of the Jacobian is 0 when θ_2 is 0 and π and what is 0 and π? If we go back and try to visualize what the planer 2R robot was doing, the second joint is 0 so, that means, it is completely stretched out second link or it is completely folded on top of the first link ok.

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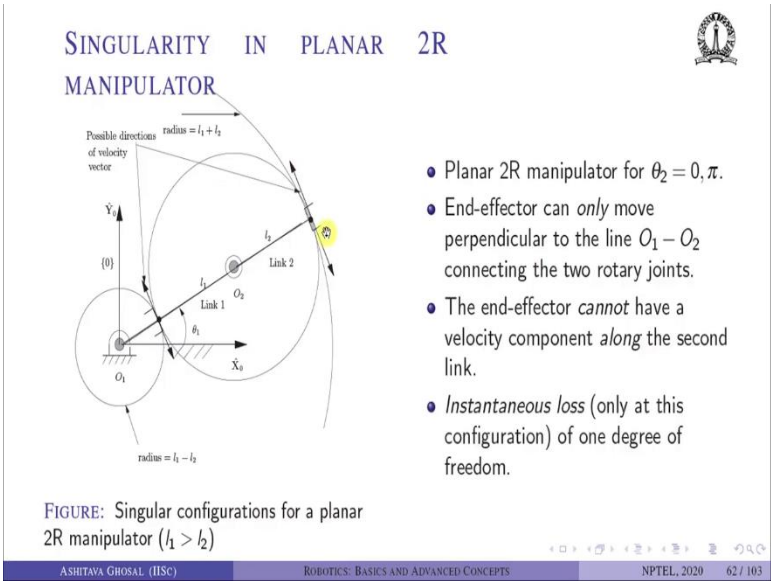

So, let us draw these two configurations. So, we have a 2R robot which is l_1 and l_2. So, if it is completely stretched out, the point is on the boundary ok. Remember the for the planar robot, the maximum reach is at radius $(l_1 + l_2)$ and the minimum it can come to near to the origin is $(l_1 - l_2)$.

So, for θ_2 as 0 and π, the end effector is completely stressed out or completely folded in and as you can see that the velocity vector of this end-effector can be only tangent to this outer circle or tangent to this inner circle.

So, it can only move perpendicular to the line connecting O_1, O_2 and extended both directions ok. So, the end effector velocity component cannot be along the second link ok, it must be perpendicular to the second link.

So, as a result, as you can clearly see there is an instantaneous loss of one-degree of freedom, 2R robot has two-degrees of freedom, but the end of the robot can only move tangent to the circle perpendicular to the links ok, line connecting the two origins. So, this is happening instantaneously only when $\theta_2 = 0, \pi$ ok. When it is π, you can see again it is tangent to this circle and it is perpendicular to the link.

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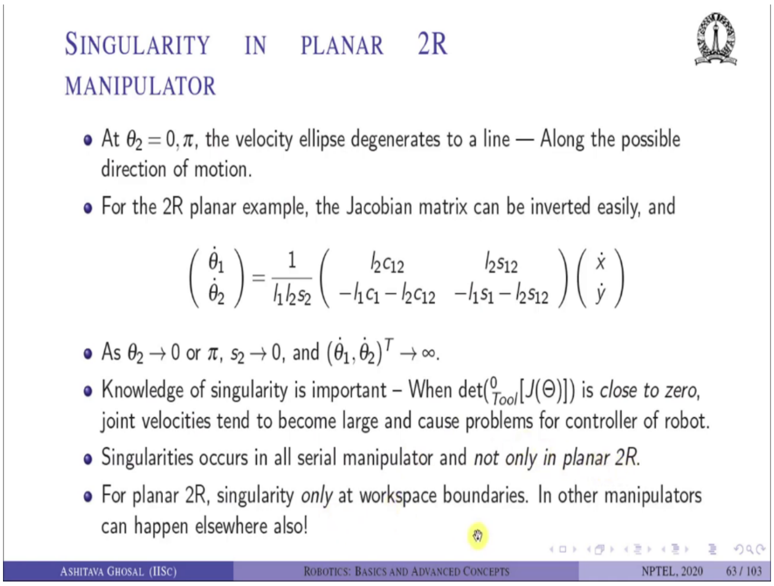

It is so at $\theta_2 = 0$ or π, the velocity ellipse degenerates to a line. Remember in at any other point, I can draw the tip of the velocity vector as $\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 = 1$. So, it traces an ellipse in the plane. However, when it is completely stretched out or completely folded, it can be only along a line. So, the velocity ellipse degenerates to a line and this is only along the direction of motion, possible direction of motion.

For 2R planar example, the Jacobian matrix can also be inverted easily, it is a simple 2×2 matrix and we can write

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = \frac{1}{l_1 l_2 s_2} \begin{pmatrix} l_2 c_{12} & l_2 s_{12} \\ -l_1 c_1 - l_2 c_{12} & -l_1 s_1 - l_2 s_{12} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

So, I can find $\dot{\theta}_1$ is something times $\dot{x}$ divided by $(l_1 l_2 s_2)$. So, that is the important part. $\dot{\theta}_2$ is also divided by $(l_1 l_2 s_2)$.

So, when θ_2 tends to 0 which is determinant of the Jacobian going to 0 or π, s_2 will go to 0 and $\dot{\theta}_1$, $\dot{\theta}_2$ tends to infinity. This is also a very important result. So, when the tip of the manipulator is very close to the boundary ok, not at the boundary, but close to the boundary, the $\dot{\theta}_1$, $\dot{\theta}_2$ will become very large ok, it tends to infinity as it comes close to the boundary.

So, this knowledge of singularity is very important ok. So, even when this determinant of the Jacobian is close to zero, the joint velocities tend to become very large for a desired $(\dot{x}, \dot{y})$ and this can cause problems with the controller of the robot. The controller or the motors may not be able to supply the joint rates required.

Singularities occur in all serial manipulators and not only planar 2R. This is a statement, and you can verify it and it is well-known ok. For planar 2R, singularity occurs only at the workspace boundaries. As you can see, it is only θ_2 is 0 or π only at the complete out outside stretched or completely folded. In other manipulators, for example, in spatial six-degree of freedom manipulators, it can happen inside the workspace also, at other points also ok.

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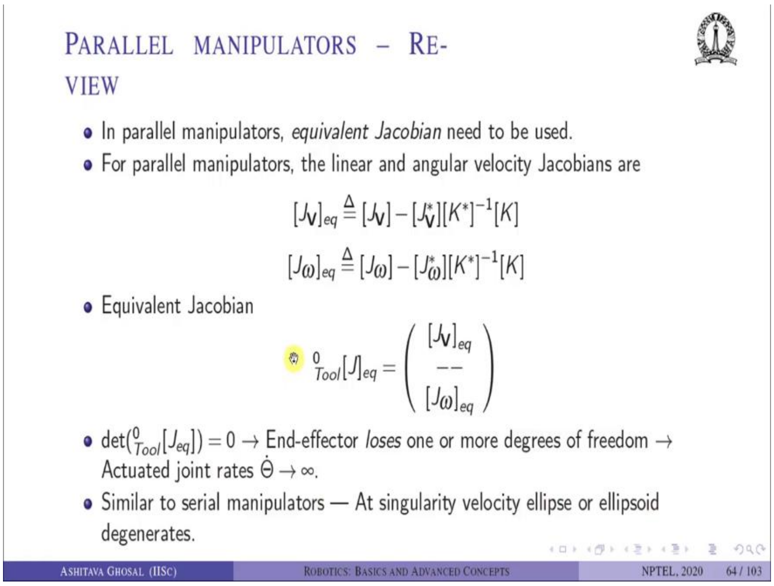

So, let us review parallel manipulator once more. In parallel manipulator, the equivalent Jacobian can be found and we can use the equivalent Jacobian ok. So, for parallel manipulators, the linear and angular velocity Jacobians are $[J_v]_{eq} = [J_v] - [J_v^*][K^*]^{-1}[K]$ and $[J_\omega]_{eq} = [J_\omega] - [J_\omega^*][K^*]^{-1}[K]$.

So, the equivalent Jacobian can be formed by concatenating $[J_v]_{eq}$ and $[J_\omega]_{eq}$ one on top of the other and again, we can see that the determinant of $[J]_{eq}$ equal to 0, the end effector will again lose one or more degrees of freedom ok. We can also show or see the $\dot{\Theta}$ will tend to infinity because $\dot{\Theta}$ will be equal to some matrix divided by the determinant of the equivalent Jacobian ok.

And similar to serial robots, at the singularity velocity ellipse or ellipsoid degenerates ok. So, in the case of planar 2R, it degenerated from an ellipse to a straight line ok. In the case in 3D, we can show that the ellipsoid can degenerate to an ellipse or the ellipse can degenerate to a straight line.

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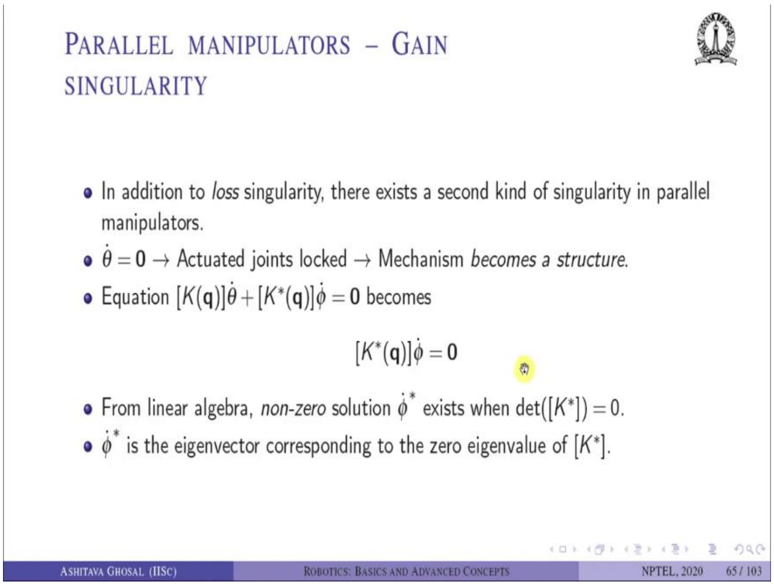

So, in addition to loss singularity where we lose one or more degrees of freedom in parallel manipulator, there exists another kind of singularity and this is what we will discuss next, this is called gain singularity ok. So, let us continue. So, let us consider $\dot{\theta}$ as 0 ok, the case when $\dot{\theta}$ is 0.

What is $\dot{\theta}$? These are the actuated joints. So, $\dot{\theta}$ means actuated joints are locked. So, if you lock all the actuated joints, the mechanism should become a structure right because those are the degrees of freedom ok, then you lock them and then, it should become a structure.

Now, equation look closure equation, when we took the partial derivatives or derivative with respect to time and rearranged, it could be written as $[K]\dot{\theta} + [K^*]\dot{\phi} = 0$; now, for

$\dot{\theta} = 0$, when the actuated joints are locked, we are left with $[K^*]\dot{\phi} = 0$ ok. So, the first term goes to 0, the second term remains.

So, from linear algebra we know that if you have a matrix equation $Ax = 0$, it can have a non-zero x when the determinant of the matrix is 0. So, in this case, we can get a non-zero $\dot{\phi}$ denoted by $\dot{\phi}^*$ when the determinant of $[K^*]$ is 0 ok. So, and what is this solution of $\dot{\phi}^*$? It is nothing but the eigenvector corresponding to the zero eigenvalue of $[K^*]$ ok. So, this is well-known in linear algebra that $Ax = 0$ has a non-trivial x when determinant of A is equal to 0.

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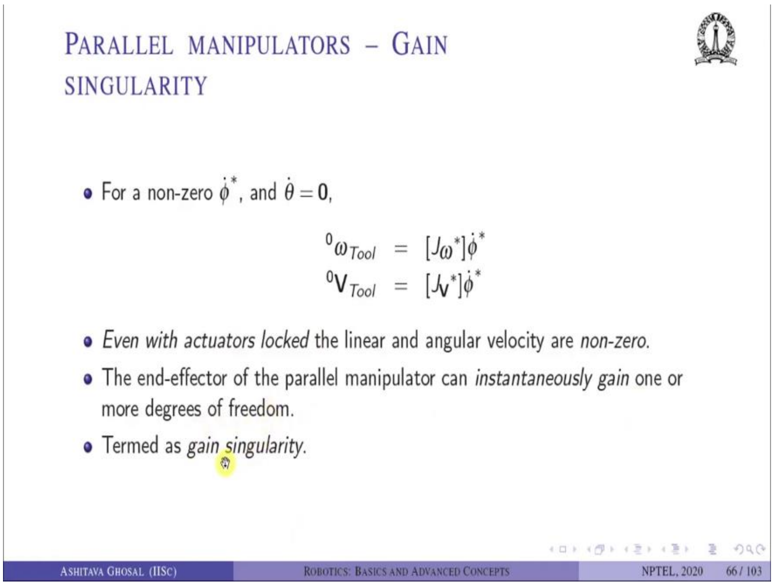

So, for this non-zero $\dot{\phi}^*$ and $\dot{\theta} = 0$, we can substitute back in the expression with the angular velocity of the Tool and the linear velocity of the Tool and we are now left with terms containing phi dot star so, it will be $[J_\omega^*]\dot{\phi}^*$ and linear velocity is $[J_v^*]\dot{\phi}^*$. So, what do we have? Even with the actuators locked, the linear and angular velocity are not 0 ok upto the {Tool} coordinate system ok.

So, this is an important observation that the end-effector of a parallel manipulator can instantaneously gain one or more degrees of freedom and this is called a gain singularity. Why is it instantaneous? Because this only happens at the place when this determinant of $[K^*]$ is equal to 0 and determinant of $[K^*]$ is a function of $\boldsymbol{q}$ so, it can happen only at some $\boldsymbol{q}$.

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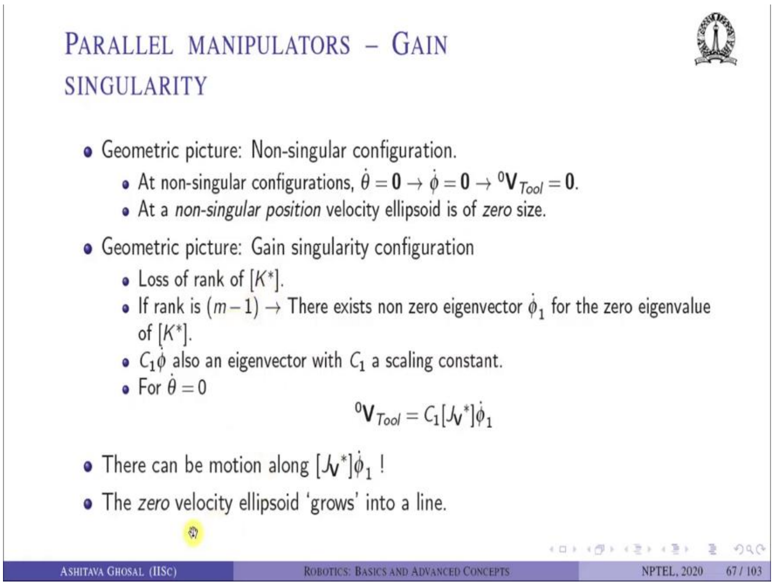

So, a geometric picture of a parallel manipulator with gain singularity. Basically, at a non-singular configuration, if $\dot{\theta} = 0$, $\dot{\phi} = 0$; if determinant of $[K^*]$ is not equal to 0, both $\dot{\theta}$ and $\dot{\phi}$ will be equal to 0 and hence, the linear velocity of the Tool will be 0 ok.

So, at a non-singular configuration, the position velocity ellipsoid is of zero size ok, the velocity vector is 0 magnitude or you can think of it as a ellipsoid of zero size ok. And a gain singularity configuration, there is a loss of rank of $[K^*]$. So, if the rank of $[K^*]$ is $m-1$, normally $[K]$ is $m \times m$ matrix, if the rank is $m-1$, there exists a non-zero eigenvector $\dot{\phi}_1$ for the zero eigenvalue of $[K^*]$.

So, $\dot{\phi}_1$ is an eigenvector, then $C_1\dot{\phi}_1$ is also an eigenvector with C_1 a scaling constant. So, for $\dot{\theta} = 0$, the linear velocity is now $C_1[J_v^*]\dot{\phi}_1$. So, there can be a motion along this vector $[J_v^*]\dot{\phi}_1$.

So, this is an eigenvector and $[J_v^*]$ multiplied by eigenvector will give a certain direction. So, what do we have? When determinant of $[K^*] = 0$, there is a loss of rank to let us say $m-1$, then the zero velocity ellipsoid grows into a line, it can move around this direction $[J_v^*]\dot{\phi}_1$.

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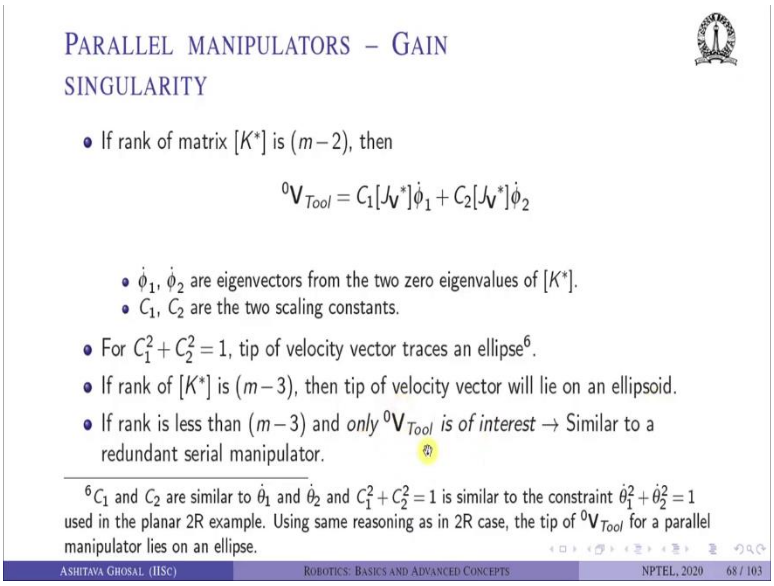

If the rank of matrix $[K^*]$ is $m-2$ which means it has fallen by 2, then the linear velocity of the Tool can be written as some $C_1[J_v^*]\dot{\phi}_1^* + C_2[J_v^*]\dot{\phi}_2^*$. So, there are two eigenvectors corresponding to the two zero eigenvalues of $[K^*]$ and C_1 and C_2 are two scaling constants.

So, for $C_1^2 + C_2^2 = 1$, the tip of the velocity vector traces an ellipse ok. So, C_1 and C_2 are similar to $\dot{\theta}_1$ and $\dot{\theta}_2$ and $C_1^2 + C_2^2 = 1$ is same as the constraint $\dot{\theta}_1^2 + \dot{\theta}_2^2 = 1$ which we used in the planar 2R example ok.

And exactly the same reasoning as a planar 2R when we discussed the geometric picture of the Jacobian matrix, we can show that the tip of the velocity vector, linear velocity vector for the parallel robot also lies on an ellipse ok.

So, if the rank of $[K^*]$ is $m-3$, then the tip of the velocity vector will lie on an ellipsoid ok. How about if it is less than $m-3$, then and only the linear velocity is of interest, then it is similar to a redundant serial manipulator.

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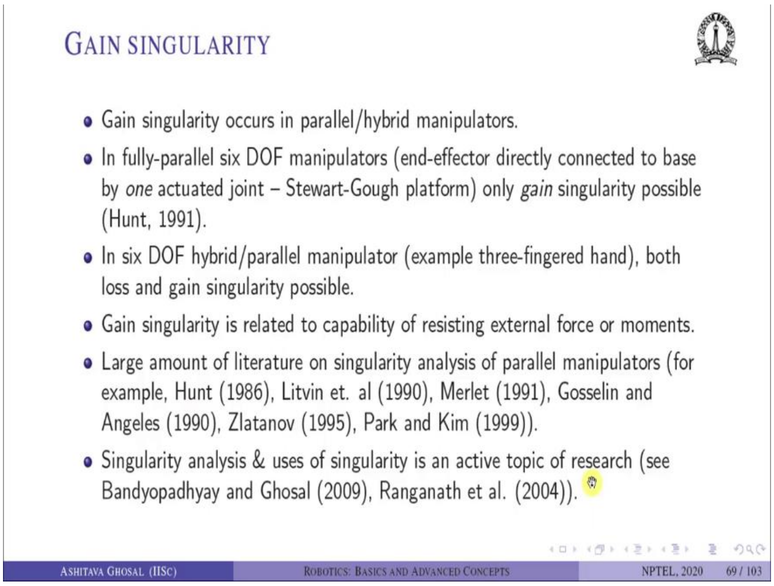

So, this gain singularity occurs in parallel hybrid manipulators ok. In a fully parallel six degrees of freedom manipulator, what is a fully parallel six degree of freedom manipulator? The end effector is directly connected to the base by one actuated joint ok as in the Stewart Gough platform ok.

In such fully parallel six degree of freedom manipulators only gain singularity possible; this was shown by Hunt ok. We cannot have loss of degrees of freedom because it is a prismatic joint and at the end only joint limits happen ok, there is nothing else happening in the range of motion of the joints.

In six degree of freedom hybrid parallel manipulators where it is not fully in parallel so, for example of three degree of freedom hand ok. So, in each finger there were three joints ok, it was not directly connected to the gripped object, both loss and gain of singularity are possible ok. So, the gain singularity is related to the capability of resisting external forces or moments this we will see in the next lecture ok. So, at a gain singularity, the manipulator cannot resist even infinitesimally small external forces.

There are large amount of literature on singularity analysis of parallel manipulators ok, it is a very very important topic. We need to know what is happening at the gain singularity and again from the point of view of control ok, it has nothing to do with joint limits ok, it is happening because at some configuration determinant of $[K^*] = 0$.

If you think about it, the determinant of $[K^*]$ is related to the loop closure constraint equations ok and there is something which is happening to the loop closure constraint equation which results in determinant of $[K^*] = 0$ ok and if you think a little bit, the rows of $[K^*]$ are not independent anymore which you think a little bit more deeply, we can show that the constraint equations are not independent.

So, we would like to know when the constraint equations are not independent when you get this additional degree of freedom. What happens to the control of the parallel robot at a gain singularity ok, you cannot control it has been shown by researchers. So, singularity analysis is an active topic of research ok, it has we have also done a lot of work at IISc.

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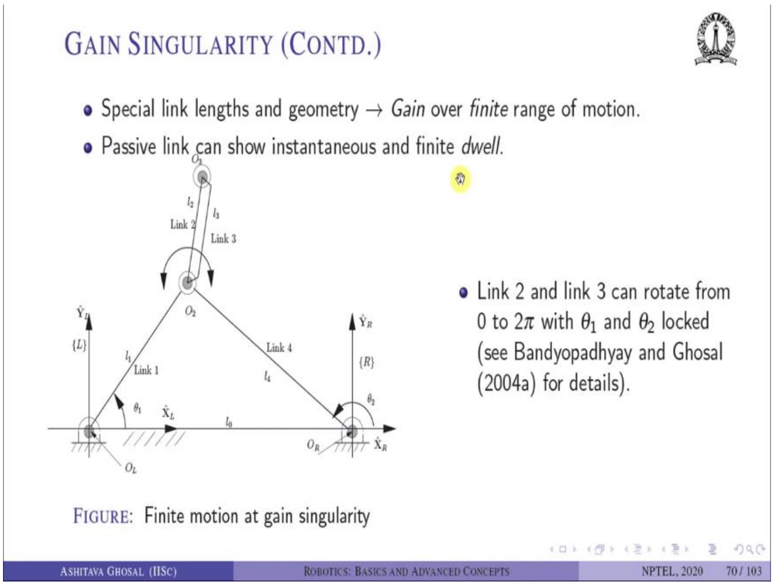

So, the special link geometry, link lengths and geometry, we can also have gain over the finite range of motion ok. So, this is an example of a 5-bar mechanism. So, this is θ_1, θ_2, a 5-bar mechanism has two degrees of freedom ok you can show. So, there has 1, 2, 3, 4 and 5 joints ok. So, there is link l_1, l_2, l_3, l_4 and l_0. So, the for a special link length which is $l_2 = l_3$ and this geometry, this geometry is such that this l_1, l_4 and l_0 from form a triangle and $l_2 = l_3$.

You can see that these two links can rotate completely ok even when θ_1 and θ_2 are locked. So, these are typically the actuated joint variables in a 5-bar mechanism ok. So, if I lock θ_1 and θ_2, this link or this joint angle can rotate freely and it can rotate over 0 to 2π. So, this happens only because of some special link lengths and geometry. So, the passive link can show instantaneous and finite dwell. Dwell means it is not moving ok so, that is also possible.

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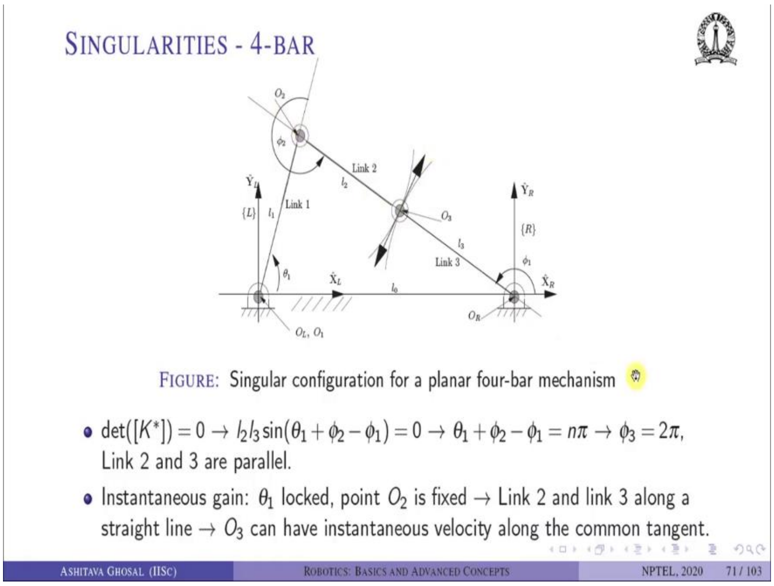

Let us look at the similarities in a 4-bar mechanism. So, we have this 4-bar mechanism Link 1, Link 2, Link 3 ok. So, the determinant of $[K^*]$ for the four-bar mechanism equal to 0 leads to this equation which is $l_2 l_3 \sin(\theta_1 + \phi_2 - \phi_1) = 0$. So, $\theta_1 + \phi_2 - \phi_1 = n\pi$ or $\phi_3 = 2\pi$ because these for angles form the inside of a triangle which means that Link 2 and Link 3 are parallel ok.

So, this is what determinant of $[K^*]$ equal to 0 mean as far as the geometry of the configuration of the 4-bar mechanism is concerned. So, with θ_1 locked. If θ_1 is locked, then this second joint is also locked right so, O_2 is also fixed, but Link 2 and Link 3 are along a straight line with a joint in between. So, if you think about it, this point O_3 can instantaneously move perpendicular to this line ok.

So, this point if you look at it from Link 2 lies on a circle, this point if you look at it from Link 3 lies on a circle so, the common tangent to this circle O_3 point can move instantaneously along in this direction ok. So, this is meaning of this instantaneous gain of degree of freedom. So, although θ_1 is locked, this joint is locked, this joint is also locked, this point can move instantaneously up and down in this perpendicular to the two links.

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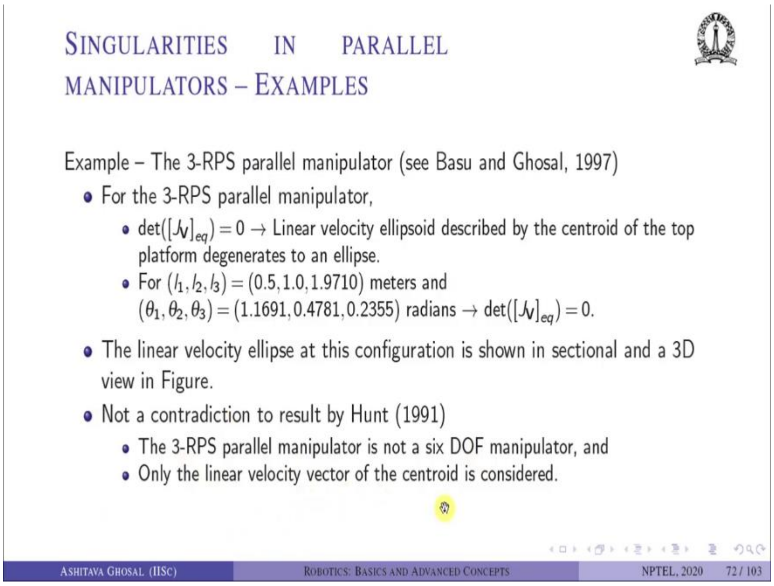

For three-degree of freedom, 3-RPS manipulator or a manipulator moving spatially ok. The determinant of $[J_v]_{eq}$ equal to 0 implies that the linear velocity ellipsoid described by the centroid of the top platform degenerates into an ellipse ok. This was worked by one of my students a long time back, these numbers are from that work.

So, for l_1, l_2, l_3 given by 0.5, 1.0, 1.9710 meters, you can solve the direct kinematics for this 3-RPS parallel manipulator and find that θ_1, θ_2, θ_3 are given by these values 1.169, 0.478, 0.2355 radians and for this combination of actuated and passive joint variables determinant of $[J_v]_{eq} = 0$.

So, the linear velocity at this configuration is shown in the sectional and 3D view next ok. So, normally, we expect an ellipsoid and the sectional views should be ellipses ok. We will see later that it is no longer on the ellipsoid and ellipses ok.

So, this is not a contradiction according to Hunt ok, this is a fully in parallel manipulators. According to Hunt, there should not be any loss of degrees of freedom. However, this is not a contradiction because this 3-RPS manipulator this is not a six-degree of freedom manipulator and only we are interested in the linear velocity of the centroid ok we are not considering both the linear and the angular velocity and it is not a six-degree of freedom manipulator.

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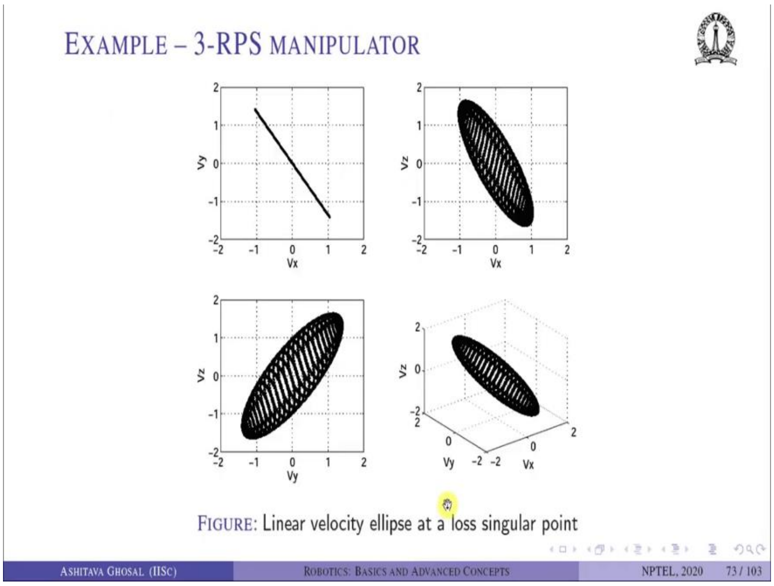

So, as I said, we can plot the tip of the linear velocity vector and it turns out that this is a 3D plot V_x, V_y, V_z it lies in an ellipse ok. The fact that it is at the singular configuration can be seen in this $V_x - V_y$ plot, it is no longer an ellipse, it is lying on a straight line ok. The other two views are still ellipse, it is an ellipse. If this were an ellipsoid, in this view also we would get an ellipse ok. So, the linear velocity ellipse at a loss singular point degenerates to a line ok.

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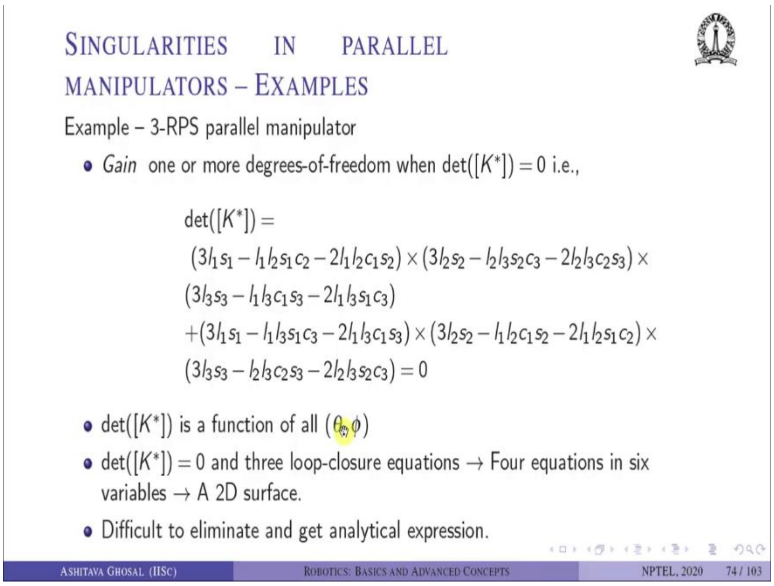

Let us continue and look at this 3-RPS parallel manipulator and look at the possibility of gain singularity ok. So, as mentioned this 3-RPS parallel manipulator will gain one or more degrees of freedom when determinant of $[K^*] = 0$. So, I have shown you what the $[K^*]$

matrix was, and I can compute the determinant of the $[K^*]$ and it will be a long expression ok. So, it is one product plus another product.

So, and it contains l_1, l_2, l_3 these are the actuated joint variables and θ_1, θ_2, θ_3 using sines and cosines ok in both the steps. So, the determinant of $[K^*]$ equal to 0 is a function of all the three θ's and the three ϕ's. So, in this case, ϕ's are the passive joint variables and also the θ's thetas, and the actuated joint thetas are l_1, l_2, l_3.

So, determinant of $[K^*]$ is an expression involving all these $\boldsymbol{q}$, all the actuated and passive joint variables ok. We also have three loop closure constraint equations remember. The distance between the two spherical joints is given and pairwise so, we have three equations there and this together will be four equations and six unknowns. What are the six unknowns? l_1, l_2, l_3, θ_1, θ_2, θ_3.

So, conceptually, it is four equation and six variables which basically means it is a 2D surface ok, but it is very difficult to eliminate and get analytical expression of this surface where the determinant of $[K^*]$ is equal to 0 and where the 3-RPS manipulator is gaining one-degree of freedom ok.

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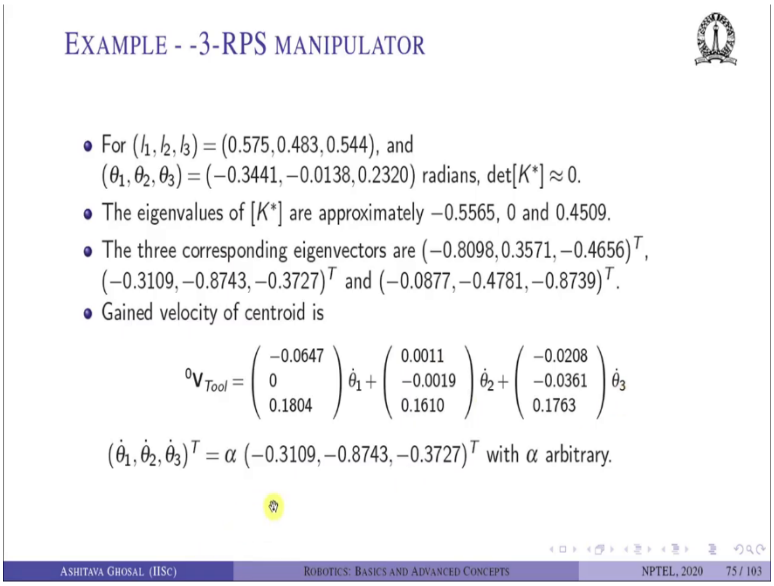

Let us do numerically. So, for l_1, l_2, l_3 given by 0.575, 0.483 and 0.544, we can solve for θ_1, θ_2, θ_3 and we get these values -0.34, -0.0138, 0.2320 radians ok and you can show that this determinant of $[K^*]$ is approximately equal to 0, very close to 0 because this is a numerical approach ok. So, when we solve for that direct kinematics, it will never be exactly equal to 0, but ok this is like 10^{-5}.

The three eigenvalues of $[K^*]$ are approximately -0.5565, very close to 0 and 0.4509 and the corresponding eigenvectors for -0.5565, 0 and this are given by this so, which is the direction in which the 3-RPS manipulator has gained a degree of freedom, it corresponds to the eigenvector for the 0 eigenvalue ok.

So, when the gained velocity of the centroid is given by this vector into $\dot{\theta}_1$, this vector into $\dot{\theta}_2$ and this vector into θ_3 ok. So, this 0.3109, -0.8743 and minus -0.3727 is the eigenvector corresponding to the 0 eigenvalue. So, α times that is also an eigenvector and that is what has been used here and when you substitute that in the linear velocity vector, you will get this as a function of $\dot{\theta}_1$, $\dot{\theta}_2$ and $\dot{\theta}_3$. So, it has gained the degree of freedom.

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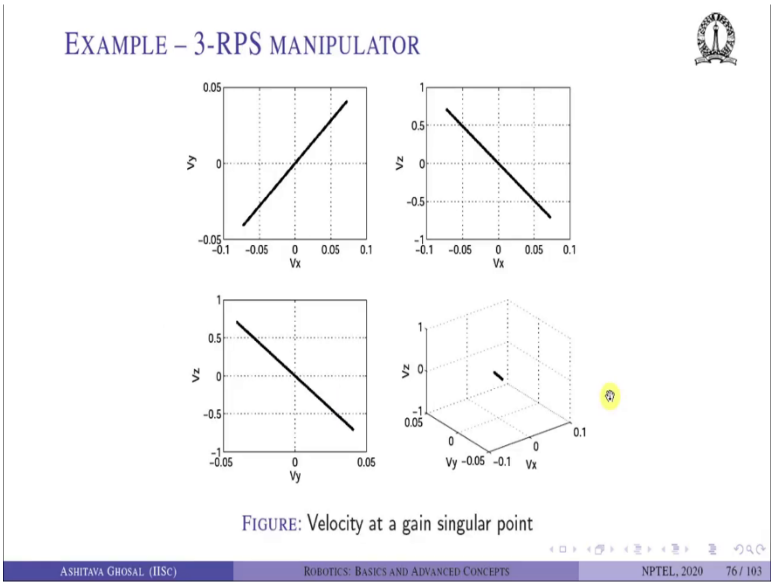

We can also now plot what is this velocity vector which it has gained in 3D plot x, y, z plot? You can see that it moves along, it can move along this direction. In the $V_x - V_y$ section view, the plane velocity vector is along this direction. In $V_z - V_x$, it is along this direction and $V_z - V_y$ it is along this direction. So, the end-effector can move in this direction.

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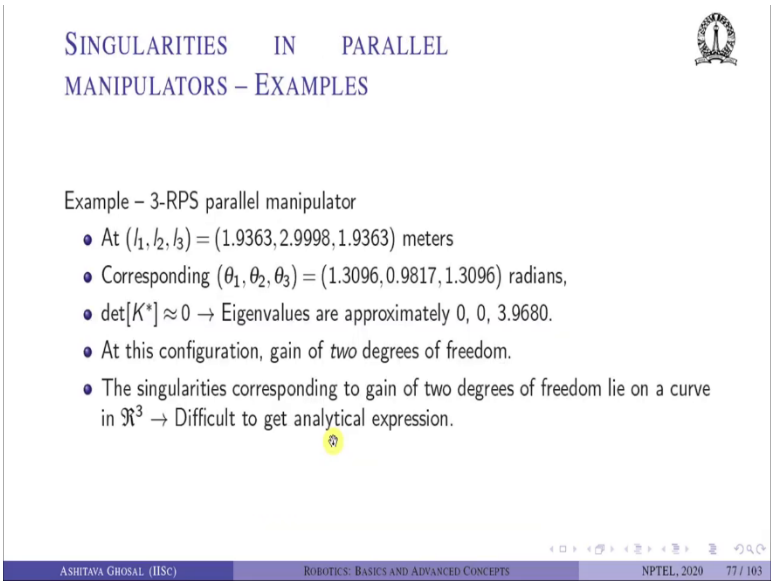

Let us continue this 3-RPS is a simple manipulator, we can do more work with it. So, at l_1, l_2, l_3 given by these values 1.9363, 2.9998, 1.9363 meters. The corresponding θ_1, θ_2, θ_3 are given by this ok and it turns out determinant of $[K^*]$ is also very close to 0.

So, how would we find these numbers l_1, l_2, l_3? We just do a search ok same story in the previous case when we when we gain one-degree of freedom. In this case, the eigenvalues are 0, 0, 3.9680. So, there are two 0 eigenvalues. So, which means what it has gained two degrees of freedom.

So, the singularities correspond to the gain of two degrees of freedom lie on a curve. The singularity correspond to gain of one degree of freedom lies on a 2D surface. So, we have one more equation so, hence it will lie on the curve. Again, it is very difficult to get analytical expression for that curve, but we can again plot it.

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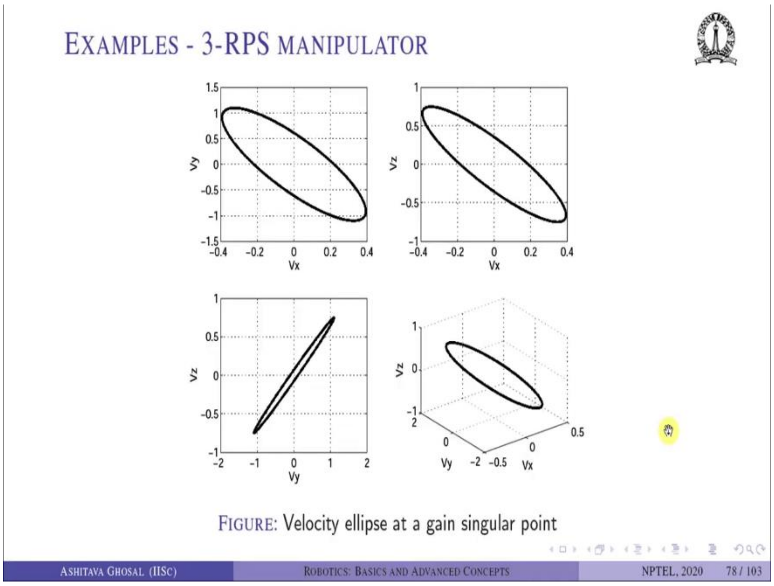

So, we can plot the gain to degree of freedom, and you can see in both in the 3D plot as well as in the sectional plots, we get ellipses ok. So, the tip of the linear velocity vector lies on an ellipse in 3D space and also in the three sectional views.

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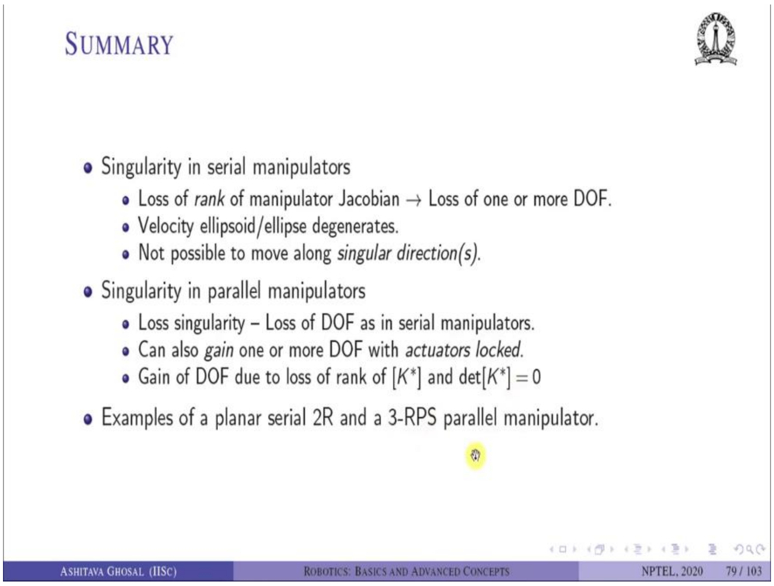

So, in summary the singularity in serial manipulators leads to a loss of rank of the manipulator Jacobian, it also implies loss of one or more degrees of freedom. Geometrically, the velocity ellipsoid or ellipse degenerates. So, ellipsoid will become an ellipse and ellipse might become a line. It is not possible to move along the singular direction. So, what is the singular direction? It is perpendicular to the direction of motion.

The singularity in parallel manipulator, you can also lose one or more degrees of freedom very similar to the loss of degree of freedom as in a serial robot. But it can also gain one or more degrees of freedom with actuators locked ok. So, the gain of degree of freedom is due to the loss of rank of $[K^*]$ and determinant of $[K^*]$ equal to 0. So, I showed you examples of planar 2R and a 3-RPS parallel manipulator ok.

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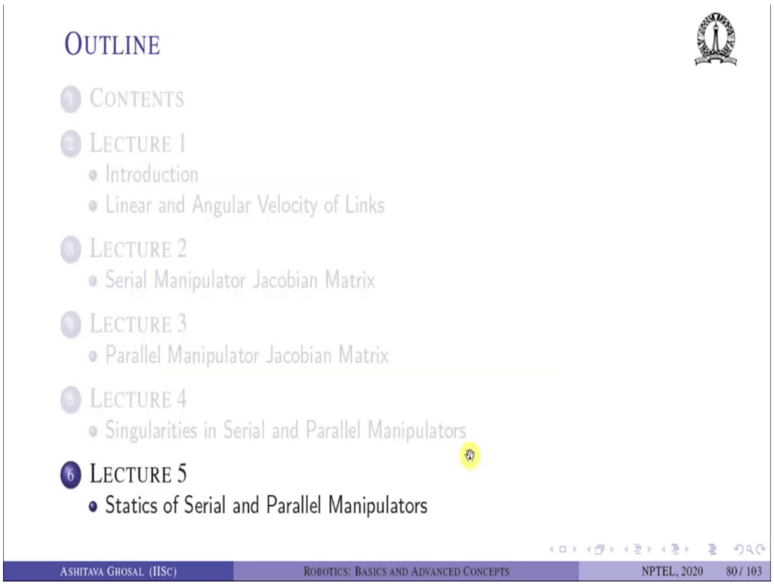

So, with this we come to an end of this discussion on singularities in serial and parallel manipulators. In the next lecture, we will look at the statics of serial and parallel manipulators.