Robotics: Basics and Selected Advanced Concepts
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Lecture - 23
Statics of Serial and Parallel Manipulators

Welcome to this NPTEL lectures on Robotics: Basics and Advanced Concepts. In the last few lectures we have done the velocity analysis of serial and parallel robots. In this last lecture, we look at the Statics of Serial and Parallel Manipulators. We will see that there is an intimate connection between the velocity kinematics and the statics.

In particular, we will see that the Jacobian matrix which appears when we do the velocity kinematics also appears in the statics of serial and parallel manipulators. So, let us continue. This is the last lecture on statics of serial and parallel manipulators.

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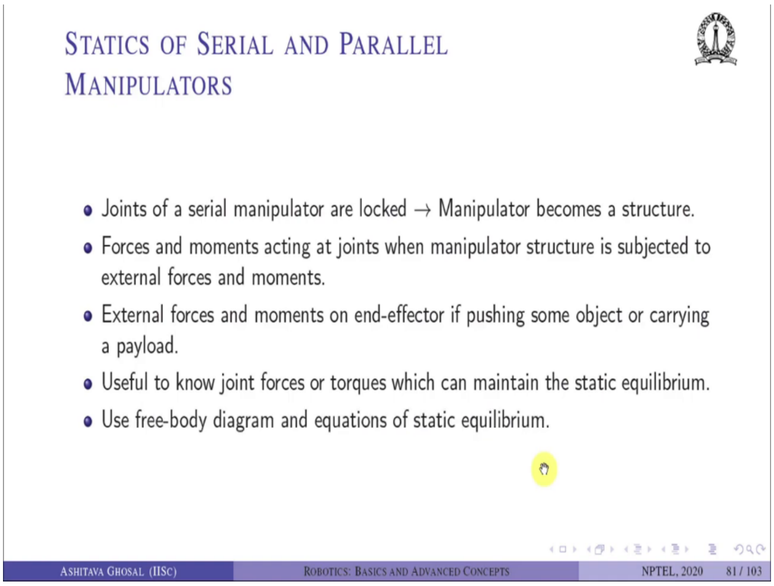

So, what do we mean by statics we basically lock all the joints in a serial robot and if we lock all the joints in a serial manipulator, the manipulator becomes a structure. So, if you have three degrees of freedom serial robot and so, there are three motors and if you lock all these three motors then the manipulator will not move.

So, the forces and moments acting at the joints when the manipulator structure is subjected to external forces and moment is the topic of statics of serial manipulators. The same thing will happen when we look at the statics of parallel manipulators.

So, basically we are interested in finding out what is happening at the joints when the manipulator structure is subjected to external forces and moment. Remember, the joints are locked. The actuated joints are locked in a parallel manipulator, ok. Another problem which we are interested in the statics of serial and parallel manipulator is the following.

So, let us assume that the manipulator is pushing some object or carrying a payload ok. So, when it is pushing an object it is moving very slowly, it is more or less static, ok. We would like to know what are the forces and moments which are acting at the joints due to this pushing force which this end-effector is applying, ok.

So, we want to know what are the forces and torques which need to be applied at the joints; such that there is static equilibrium when it is pushing an object or when it is holding on to a payload.

And, how do we obtain these forces and moments at the joints when there is an external force or a moment acting on the manipulator structure? Basically we have to use the notion of free-body diagram and equations of static equilibrium. So, this is a very well known well understood and in undergraduate many places.

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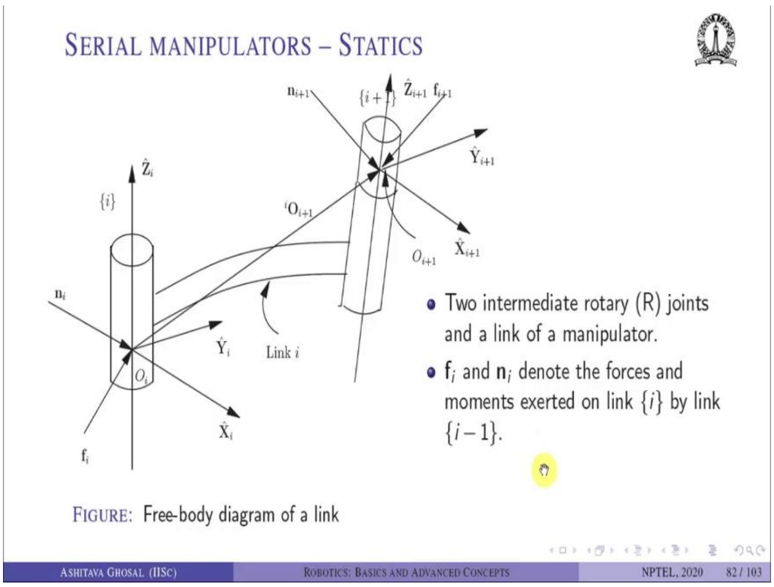

So, basically what we want to do is we want to draw the free-body diagram. So, basically the free-body diagram of a link will consist of the link and we show all the external forces and moments which are acting on this link. So, this is a link i. So, Z-axis is along the joint axis, the link $i+1$ is after this joint axis same as before there is no change in our convention.

So, the only thing is the X, Y and Z-axis and the origin are shown for link i. Similarly, the X, Y and Z axis $\{i+1\}$ coordinate system and the origin $\boldsymbol{O}_{i+1}$ is shown. There is a vector which locates the $(i+1)^{\text{th}}$ origin with respect to the i^{th} origin. So, this is this vector ${}^{i}\boldsymbol{O}_{i+1}$.

Now, let us store all the forces and moments, which are acting on this link i. So, we will denote all the moments which are acting on this link i by $\boldsymbol{n}_i$ the symbol $\boldsymbol{n}_i$ bold phase $\boldsymbol{n}_i$ and all the forces which are acting on link i by $\boldsymbol{f}_i$ ok. So, what is this $\boldsymbol{n}_i$ and $\boldsymbol{f}_i$ when you draw the free-body diagram?

These are basically the forces and moments acting on link i from the link $i-1$, ok, this is important. So, it is the forces and moments exerted on link i by link $i-1$ ok. So, what is $\boldsymbol{f}_{i+1}$ and $\boldsymbol{n}_{i+1}$ it is the forces and moments acting on link i from link $i+1$, ok.

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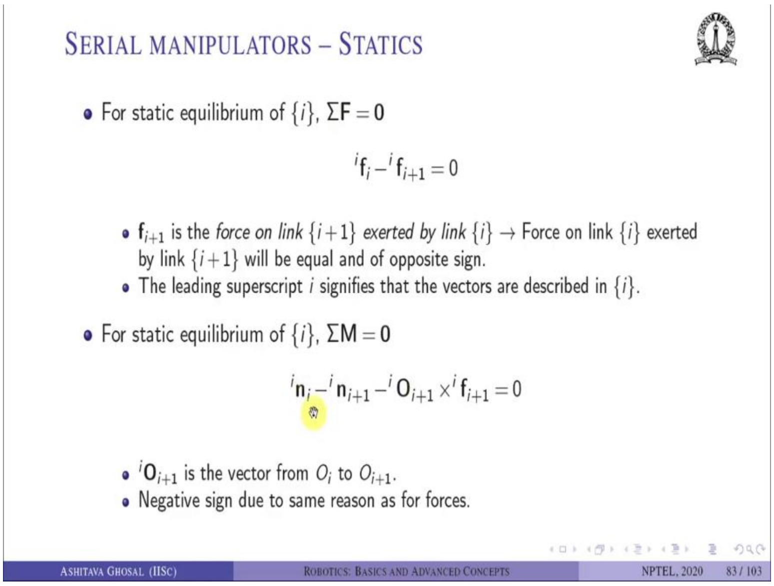

For static equilibrium the sum of all the forces must be equal to 0. So, ${}^{i}\boldsymbol{f}_i - {}^{i}\boldsymbol{f}_{i+1} = 0$. Ok where is this minus sign coming? Because $\boldsymbol{f}_{i+1}$ is the force which is acting on the link

$i+1$. So, the equal and opposite force is acting on link i, ok. So, is that clear? And, we are going to do vector summation. So, they must be written in the same coordinate system.

So, we are describing this $\boldsymbol{f}_i$ in the i^{th} coordinate system and a ${}^{i}\boldsymbol{f}_{i+1}$ in the i^{th} coordinate system. So, some of these two forces must be equal to 0 ok. So, as I said $\boldsymbol{f}_{i+1}$ is the force and link $i+1$ exerted by link i. Hence the force on link i exerted by link $i+1$ will be equal and opposite sign and the leading superscript i signifies that the vectors are described in $\{i\}$.

For static equilibrium not only the sum of the forces must be equal to 0, the sum of the moments must also be equal to 0. So, what is the moment on a link i from $i-1$? That is $\boldsymbol{n}_i$. $\boldsymbol{n}_{i+1}$ is the moment acting on link $i+1$ from link i. So, an equal and opposite moment will act on link i which is this term and then we have the $\boldsymbol{r} \times \boldsymbol{f}$ term.

So, basically a moment generated by this force $\boldsymbol{f}_{i+1}$ at a distance $\boldsymbol{O}_{i+1}$, ok. So, this is like standard undergraduate statics. So, the moment due to a force is $\boldsymbol{r}$ cross that force; cross means cross product, ok. So, here also we have I have ${}^{i}O_{i+1} \times {}^{i}\boldsymbol{f}_{i+1}$.

This minus sign again comes from the fact that $\boldsymbol{f}_{i+1}$ is the force acting on the link $i+1$, so, equal and opposite force is acting on link i, ok. So, again the negative sign is due to the same reason as for forces. So, we have these two equations some of the forces acting on link i equal to 0.

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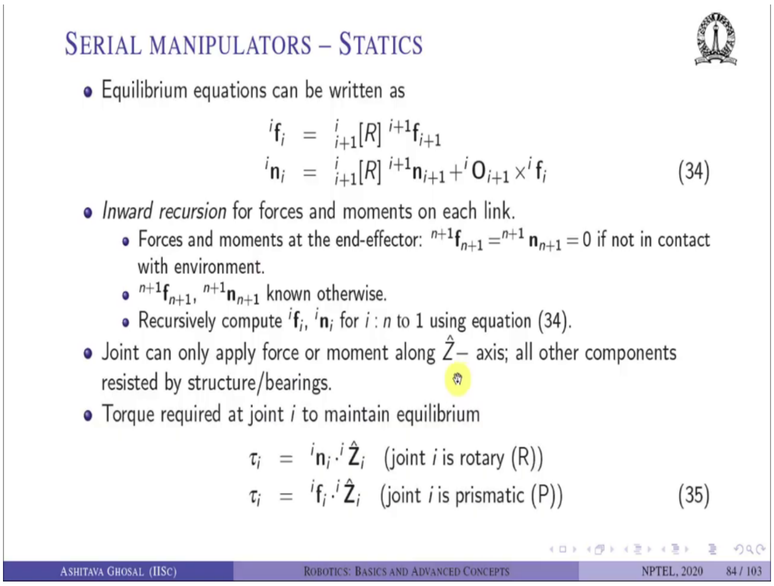

And, some of the moments acting on link i is equal to 0. We can rewrite these two equations in this form. It is a nice iterative form. So, ${}^{i}\boldsymbol{f}_i$ was equal to ${}^{i}\boldsymbol{f}_{i+1}$ previous ${}^{i}\boldsymbol{f}_i$ is ${}^{i}\boldsymbol{f}_{i+1}$, but we can rewrite this by pre multiplying by a rotation matrix ${}_{i+1}^{i}[R]^{i+1}\boldsymbol{f}_{i+1}$.

So, what is this rotation matrix? This is the link $i+1$ in with respect to the i^{th} coordinate system. Similarly, ${}^{i}\boldsymbol{n}_i$ can be written as ${}_{i+1}^{i}[R]\ {}^{i+1}\boldsymbol{n}_{i+1} + {}^{i}\boldsymbol{O}_{i+1} \times {}^{i}\boldsymbol{f}_i$. There is a reason why we want to write it in this way because we will go backwards, ok. So, we will do what is called as an inward recursion for forces and moments on each link.

So, typically the forces and moments at the end-effector which is the $n+1$ link; remember the end-effector is the after the joint n, ok. So, the force and moment which are acting on the end-effector either in its own coordinate system or in some other coordinate system is known, ok.

So, most of the time if it is not in contact with the environment if it is free space, then it is 0 ok, otherwise if it is pushing against something or lifting some payload or doing some work then this force which is acting on the end-effector is known. So, knowing the force and the moment acting on the end-effector we can substitute on the right hand side and we can find out what is happening to the force and moment in that previous link.

And, likewise if I know on the $n-1$ is link I can go back and calculate on the $n-2$th link and so on ok. So, this is the idea of invert recursion. Remember, in the case of velocity we went out what we started from the fixed base and went to the end-effector. In the case of forces we are going backwards and it is intuitively correct; in the case of velocities the base velocity is fixed or 0.

In the case of forces external force is acting on the end-effector and that is known. So, we can recursively compute ${}^{i}\boldsymbol{f}_i$, ${}^{i}\boldsymbol{n}_i$ for $i = n$ to 1 using this equation. So, the joint can only apply force and moment along the Z-axis. We are dealing with one degree of freedom rotary or prismatic joints. So, a rotary joint can apply a moment about the joint axis, a sliding joint can apply a force along the joint axis.

All other components of the force $\boldsymbol{f}_i$ and $\boldsymbol{n}_i$ which is acting at the joint are resisted by the structure, ok. So, there are bearings which will take care of radial loads and other kinds of loads. So, the torque required to maintain equilibrium at joint i is nothing but the component of the moment along the Z-axis if it is a rotary joint or the component of the force along the Z-axis if it is a prismatic joint ok.

So, you can think of that there is an external force acting. How much force or how much torque should I apply at a joint i such that the whole manipulator remains as a structure. So, we know that the motors can only apply along the joint axis or rotate about the joint axis. So, hence we can find the Z-component of this $\boldsymbol{n}_i$ or the $\boldsymbol{f}_i$ vector, ok; $\boldsymbol{f}_i$ if it is a prismatic joint, $\boldsymbol{n}_i$ if it was a rotary joint.

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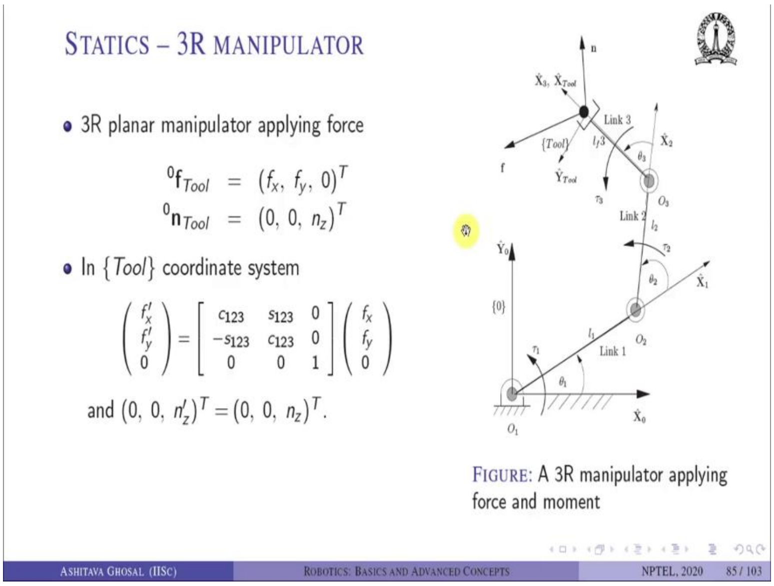

So, example: So, let us consider a planar 3R manipulator applying a force environment. So, the forces $(f_x, f_y, 0)^T$ and it is with respect to the 0th coordinate system. So, somehow we have figured out that this planar 3R manipulator is applying a force with components $(f_x, f_y, 0)^T$; 0 because it is in a planar example and it is with respect to the 0th coordinate system.

It is also applying a moment about the environment. So, you can think of it as carrying a wrench and it is trying to tighten a nut for example, ok. And, again the moment is only along the z axis the x and y components are 0. So, first thing to do is we want to convert this force and moment into the Tool coordinate system ok because remember we need ${}^{i+1}\boldsymbol{f}_{i+1}$.

So, we need the force in its own coordinate system, the moment in its own coordinate system, ok. So, how do we do that? Let us call this force is $(f_x', f_y', 0)^T$, we have to pre multiply by a rotation matrix. So, instead of writing it in the 0th coordinate system we can write with respect to ${}^{Tool}_{0}[R]$ times this will give me that force in its own coordinate system.

And, rotation matrix if you think a little bit is this it is nothing but $\begin{bmatrix} c_{123} & s_{123} & 0 \\ -s_{123} & c_{123} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and the z-axis is remaining same. The moment in it is own coordinate system is a single vector along the z-axis and it will be the same, ok.

It does not matter whether it is you know because only rotation is about z axis. So, if the moment is already about z axis nothing happens ok. So, ${}^{i+1}\boldsymbol{n}_i$ for the tool coordinate system is $(0,0,n_z\,)^T$, ${}^{i+1}\boldsymbol{f}_i$ for the force in the Tool coordinate system is given by $(f_x', f_y', 0)^T$.

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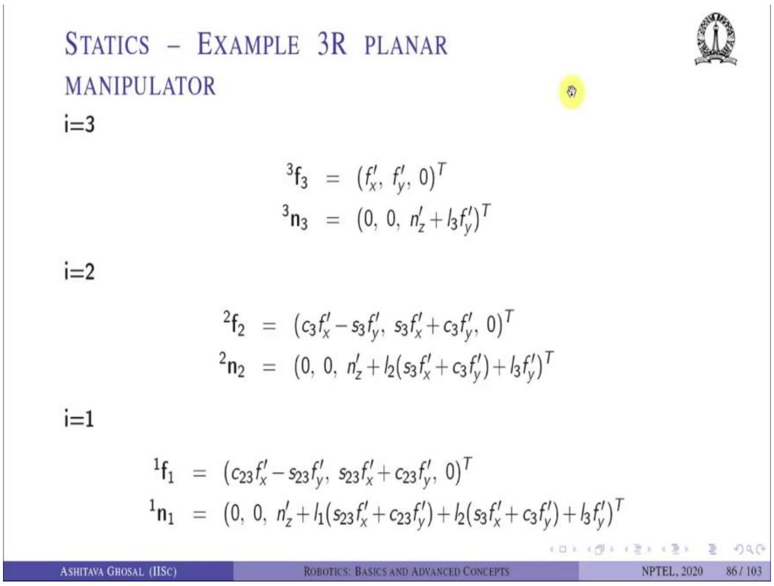

So, for $i = 3$, we know ${}^{i+1}\boldsymbol{f}_i$ which is $(f_x', f_y', 0)^T$. So, we can compute ${}^3\boldsymbol{f}_3$ ok. Then for $i = 3$, we know what is ${}^3\boldsymbol{n}_3$, how do I find out ${}^3\boldsymbol{n}_3$? That is by based on this inward recursion formula. So, basically we can show that it is a moment plus some $l_3 f_y'$. So, there is a moment which is acting here, but then it will have some $l_3 f_y'$.

So, $f_y l_3$ will give another moment which is along the z axis. The $f_x l_3$ is along the same direction and will not contribute to the moment. So, ${}^3\boldsymbol{n}_3$ will be given by this. Then we substitute $i = 2$ in the recursion formula and we can obtain ${}^2\boldsymbol{f}_2$ and it turns out to be $\left(c_3 f_x' - s_3 f_y', s_3 f_x' + c_3 f_y', 0\right)^T$.

And, moment is $\boldsymbol{n}_z{}'$ which is there then $l_3 f_y{}'$ which is there, but then we have an additional component coming because of the moment arm at 2. So, $l_2\left(s_3 f_x' + c_3 f_y'\right)$. Finally, we can substitute $i = 1$, in the recursion formula and we can get ${}^1\boldsymbol{f}_1$ and ${}^1\boldsymbol{n}_1$, ok.

So, ${}^1\boldsymbol{f}_1$ is some $\cos(\theta_2 + \theta_3)\, f_x' - \sin(\theta_2 + \theta_3)\, f_y{}'$ and so on, ok. And, similarly the moment is $\boldsymbol{n}_z{}'$ then $l_3 f_y{}'$ then this term and then one more additional term due to l_1 ok.

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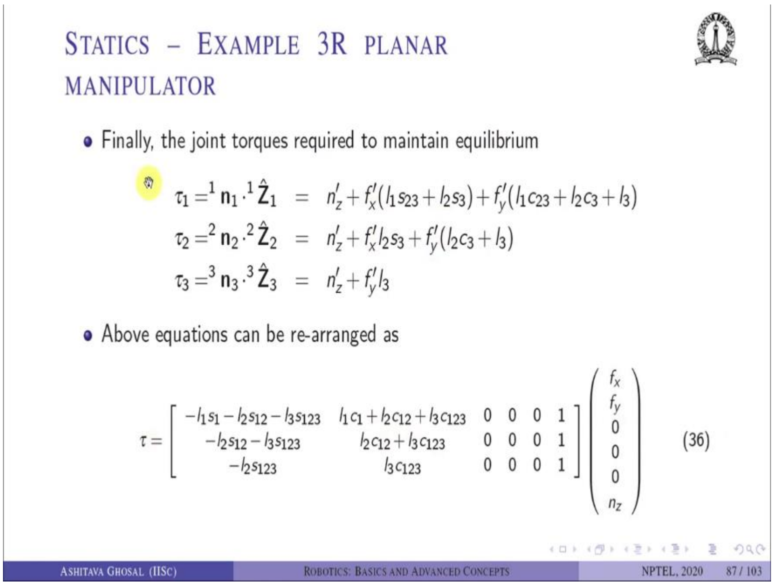

So, and what are now the joint torques which are required to maintain equilibrium? We just take the Z component of ${}^1\boldsymbol{n}_1$ for τ_1, Z component of ${}^2\boldsymbol{n}_2$ for τ_2 and again Z component of ${}^3\boldsymbol{n}_3$ for τ_3. So, in this case we are looking at the moments because all the joints are rotary joints and they can only apply a moment.

So, if you take the Z component we will get some $\boldsymbol{n}_z' + \boldsymbol{f}_x'(l_1 s_{23} + l_2 s_3) + f_y{}'(l_1 c_{23} l_2 c_3 + l_3)$ and so on; τ_3 will be $\boldsymbol{n}_z' + f_y' l_3$. So, these above equations these three equations can be rearranged to be written like this. So, the left hand side is this vector τ which is τ_1, τ_2, τ_3 the right hand side is $\left(f_x, f_y, 0,0,0, n_z\right)^T$.

So, we have a force which is acting at that end in the 0th coordinate system we have reconverted it back to f_x, f_y, n_z not primes anymore, ok. And, the left hand side is τ_1, τ_2, τ_3 and this we have a matrix with elements like this. So, the first element is $(-l_1 s_1 - l_2 s_{12} - l_3 s_{123})$ then $(l_1 c_1 + l_2 c_{12} + l_3 c_{123})$ and three 0s 1 ok.

The last row for example, is minus $(-l_3 s_{123}, l_3 c_{123}, 0,0,0,1)$. So, if we look a little bit carefully at this matrix you can see that you have seen this matrix before, ok. So, if we go back to velocity kinematics, you may recall what is this matrix for the planar 3R manipulator.

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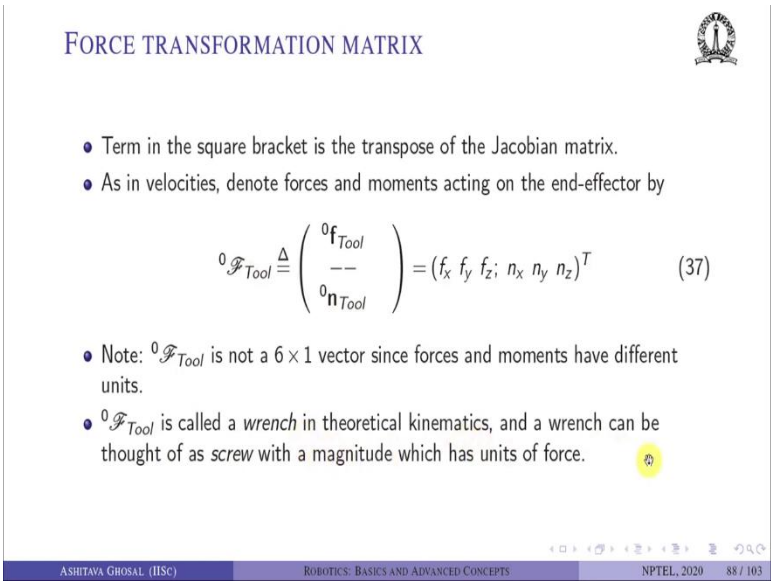

So, it turns out that the terms in this square bracket is nothing but the transpose of the Jacobian matrix remember we had derived Jacobian matrix using velocity propagation and then again rearranging terms for the planar 3R example. So, the term here what we get is transpose of that Jacobian matrix which we have already seen, ok. So, this is the interesting part.

We are doing statics, but the Jacobian matrix which we are derived for velocity kinematics appears not exactly the Jacobian matrix, but the transpose of the Jacobian matrix appears, ok. So, as in velocities, we can denote forces and moment acting on the end-effector by 6×1 entity not really a vector.

So, first top three is the force acting at the end-effector and the bottom three are the three components of the moment which are acting at the end-effector. So, basically $(f_x, f_y, f_z; n_x, n_y, n_z)^T$ so, this is not really a vector because the force and moment have different units, ok. So, what is force? Newton. What is moment? Newton meters ok. So, they have different units.

This quantity is ${}^0\mathcal{F}_{Tool}$ is called a wrench in theoretical kinematics, ok. So, basically a wrench can be thought of as a screw with a magnitude which has units of force, ok. Let us not worry about it, but in theoretical kinematics a combination like this using some mindset approaches like using homogeneous coordinates can be thought of as a 6×1 entity proper entity without this messing up of units.

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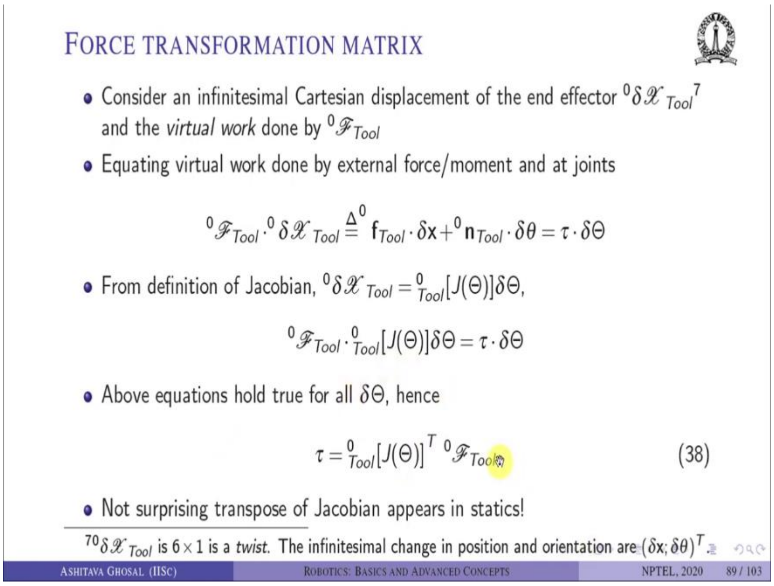

So, now consider an infinitesimal Cartesian displacement of the end-effector, ok. So, what do we mean by Cartesian displacement. So, there is a displacement of $(\delta_x, \delta_y, \delta_z)$ and also let us think of a small change in the orientation given by δ_θ, ok. So, a little bit of hand waving, but we know that we can derive the three Euler angles and then we can think of the change in orientation roughly in terms of δ_θ, ok.

So, the and the virtual work done by this force and moment which is acting on this end-effector. So, the virtual work done by the external forces moments can be equated to the virtual work done by at the joints. So, what is the virtual work done by the end-effector at the end-effector? It is the force and moment dot product with this infinitesimal displacement ${}^{0}\delta\,\mathcal{X}_{Tool}$, ok. So, it is ${}^{0}\mathcal{F}_{Tool} \cdot {}^{0}\delta\,\mathcal{X}_{Tool}$ which is nothing but ${}^{0}\boldsymbol{f}_{Tool} \cdot \delta\boldsymbol{x} + {}^{0}\boldsymbol{n}_{Tool} \cdot \delta\theta$.

And, what is the virtual work done at the joints? It is $\tau \cdot \delta\Theta$. So, from the definition of Jacobian, what is this Jacobian? This is ${}^{0}\delta\,\mathcal{X}_{Tool}$ is related to ${}^{0}_{Tool}[J(\Theta)]\delta\Theta$. So, it is very similar to velocity is equal to ${}^{0}_{Tool}[J(\Theta)]\dot{\Theta}$. So, now, we can say that this ${}^{0}\mathcal{F}_{Tool} \cdot {}^{0}\delta\,\mathcal{X}_{Tool} = {}^{0}\mathcal{F}_{Tool} \cdot {}^{0}_{Tool}[J(\Theta)]\delta\Theta = \tau \cdot \delta\Theta$.

So, the above equation holds for all $\delta\Theta$, this is the standard way we use this principle of virtual work. And hence we can show that this $\tau = {}^{0}_{Tool}[J(\Theta)]^{T}\,{}^{0}\mathcal{F}_{Tool}$ because this dot product $a \cdot b$ is nothing but ${}^{0}_{Tool}[J(\Theta)]^{T}\,{}^{0}\mathcal{F}_{Tool}$, ok. So, if you think a little bit if you think a little bit we can see that this torque at the joints is related to the external forces and moments by this ${}^{0}_{Tool}[J(\Theta)]^{T}$, ok.

And, hence it is not very surprising that the Jacobian appears in statics ok. So, this is a very very useful expression it tells you that if you apply a force on the end-effector in a serial robot you need to apply some torque to keep it in equilibrium and this torque is given by ${}^{0}_{Tool}[J(\Theta)]^{T}\,{}^{0}\mathcal{F}_{Tool}$.

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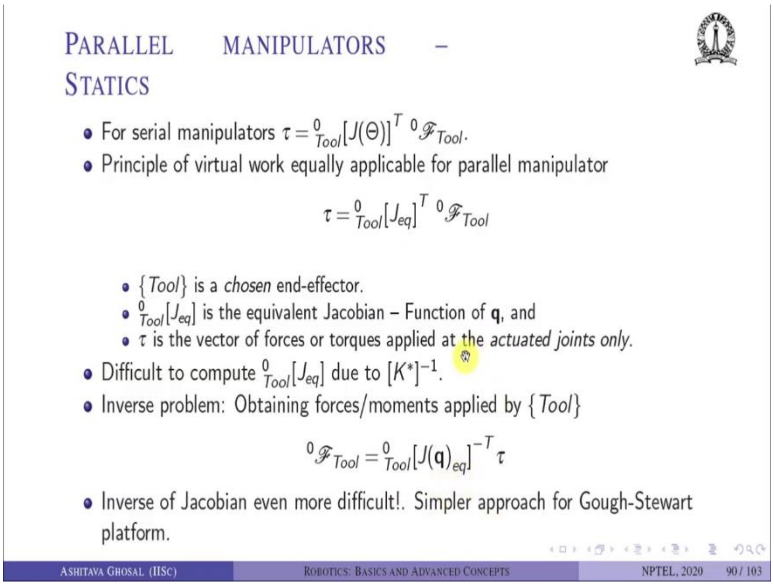

Our parallel manipulator statics: so, for serial manipulators $\tau = {}_{Tool}^{0}[J(\Theta)]^{T}\ {}^{0}\mathcal{F}_{Tool}$. The principle of virtual work is equally applicable for parallel manipulator, ok. So, all we need to do is we need to say instead of ${}_{Tool}^{0}[J]$ we have ${}_{Tool}^{0}[J_{eq}]$ and what was ${}_{Tool}^{0}[J_{eq}]$? ${}_{Tool}^{0}[J_{eq}]$ had to do deal with $[J_v]$, $[J_v^*]$, $[K^*]$, $[K]$ and so on, and similarly, $[J_\omega^*]$, $[K^*]$, and all these various things which we are derived in the velocity analysis.

So, what is the subtle difference? Basically, we have to say that the {Tool} is the chosen end-effector, ok. In a serial manipulator the end-effector is very natural; in a parallel manipulator we have to derive this Jacobian for the chosen end-effector and this $[J_{eq}]$ is the is the equivalent Jacobian, it is a function of all the passive and active joints. It is a function of the $\boldsymbol{q}$ the configuration variables.

And, τ is the vector of forces or torque applied at the actuated joints only, ok. So, in a parallel manipulator we have actuated joints and passive joints. You are not going to apply any torque on the passive joints. So, τ is only the forces and moments which are applied on the actuated joints.

It is quite difficult to compute ${}_{Tool}^{0}[J_{eq}]$ basically because there is a $[K^*]^{-1}$, if you go back and see the notes you can see the ${}_{Tool}^{0}[J_{eq}]$ in a parallel robot contains is $[K]$ and $[K^*]^{-1}$. The inverse problem is to obtain the forces and moments applied by the Tool, ok.

So, suppose I give you the joint torques that we are applying so much moment or so much torque at the joints, what is the end-effector applying to the environment? So, we could symbolically write it as inverse of this matrix which is what is written here ${}_{Tool}^{0}\left[J_{eq}(\mathrm{q})\right]^{-T}$. So, that is the force which the end-effector is applying on to the environment.

So, this is clearly very very hard to obtain, right because ${}_{Tool}^{0}[J_{eq}(\mathbf{q})]$ already had some $[K]$, $[K^*]^{-1}$. Now, you have to find the inverse of that whole quantity, ok. So, this is not possible, who? For most parallel robots; however, it is required to be solved for several robots and it can solve very simply for the Gough-Stewart platform, ok.

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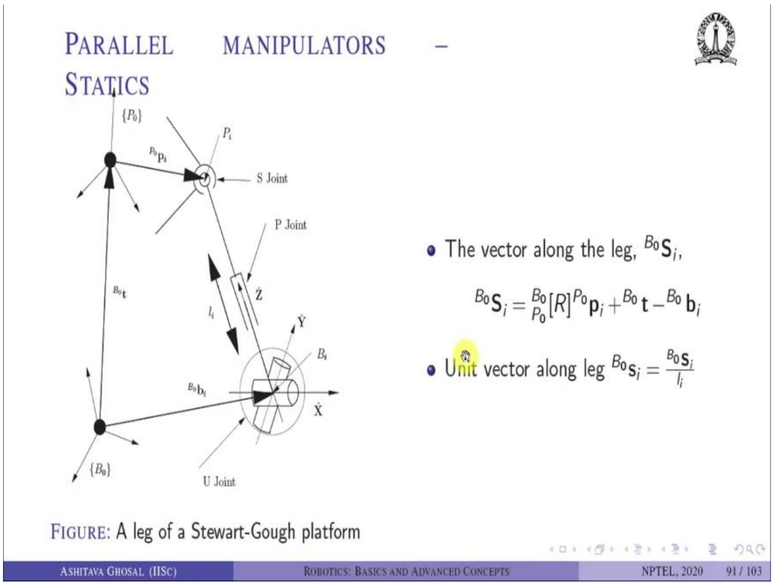

FIGURE: A leg of a Stewart-Gough platform

So, it can be shown to be very easily solvable for as fully in parallel manipulator meaning that between the output and the input that is only one actuated joined. So, let us go back and look at this in a little bit more detail. So, the leg of a Stewart platform consists of one P joint, one S joint at the top platform moving platform and a U joint on the base fixed base.

So, there is a vector $^{B_0}\boldsymbol{b}_i$, there is a vector $^{B_0}\boldsymbol{t}$ to the origin of the moving platform, there is a local vector which locates the spherical joint in the moving coordinate system and then there is this two rotations at this hook joint, ok. So, we can rewrite this vector from B_i to S_i ok. So, basically the vector along the leg is nothing but $^{P_0}\boldsymbol{p}_i$ this plus this minus this vector, ok.

So, $^{B_0}\boldsymbol{t}$ plus $^{P_0}\boldsymbol{p}_i$ rotated back into the base coordinate system in the $\{B_0\}$ coordinate system minus this. So, that will give this vector. The unit vector along this leg is nothing but this vector divided by l_i because the magnitude of this vector is the translation at the prismatic joint.

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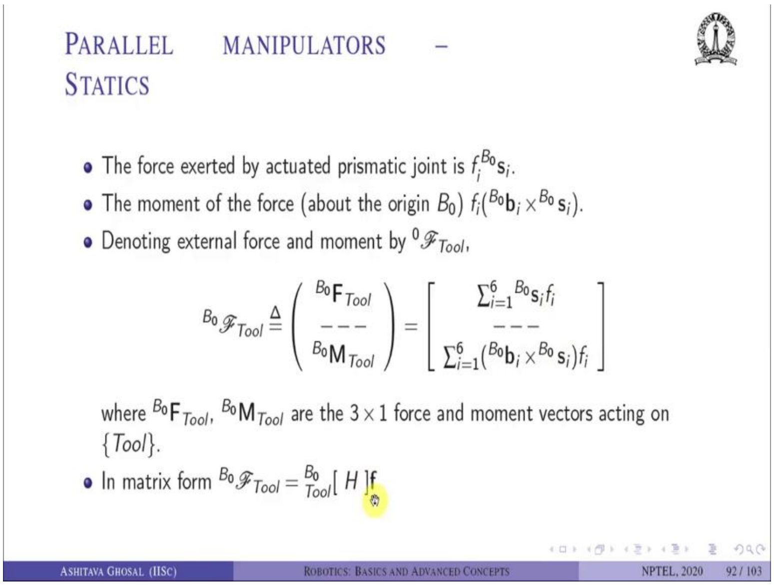

So, the force exerted by the actuated prismatic joint is nothing, but $f_i{}^{B_0}\boldsymbol{s}_i$, so, the from the unit vector. The moment of the force about the origin $\{B_0\}$ ok; so, the unit vectors there is a force acting this. So, the moment will be this vector cross that force which is what is written here $f_i({}^{B_0}\boldsymbol{b}_i \times {}^{B_0}\boldsymbol{s}_i)$.

So, if you denote the external force and moment by ${}^0\mathcal{F}_{Tool}$ which is nothing, but the force acting on the end-effector and the moment acting on the end-effector. This must be balanced by the force along the leg and the moment which is acting with respect to the fixed coordinate system. So, this must be equal to this, ok.

So, in matrix form we can rewrite this portion as some matrix ${}^{B_0}_{Tool}[H]\boldsymbol{f}$, ok. So, the end effective force and the end-effector moment is equal to ${}^{B_0}\boldsymbol{s}_i f_i$ and $({}^{B_0}\boldsymbol{b}_i \times {}^{B_0}\boldsymbol{s}_i) f_i$. So, we can rewrite this in a matrix form and times $\boldsymbol{f}$. So, what is this matrix? It is nothing but the unit vectors and the moments of the unit vectors. What is this $\boldsymbol{f}$? That is f_1, f_2, f_3 altogether till f_6, ok.

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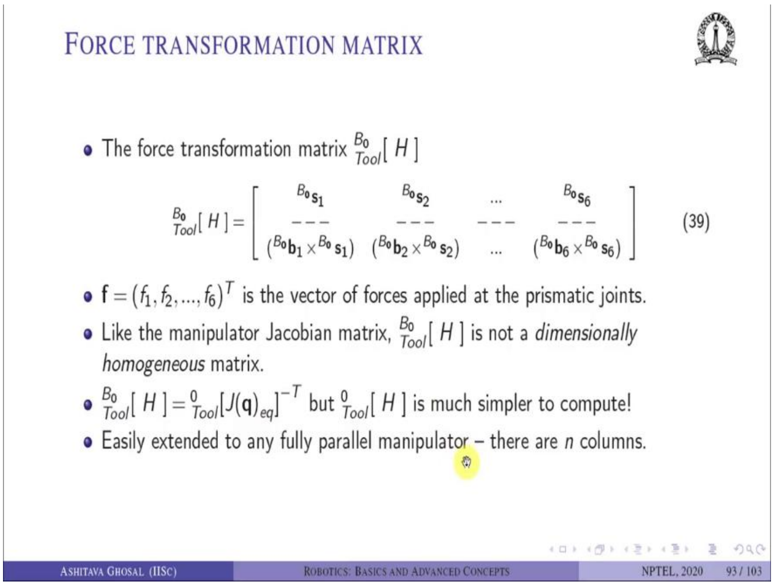

So, we have the force transformation matrix as I said the top portion is this unit vector along the leg and the bottom portion is the moment of that unit vector from the fixed origin. Second column is the second unit vector along the second leg and the moment of the second unit vectors and so on till the sixth unit vector along the sixth leg and the moment of that sixth unit vector.

So, in a Stewart Gough platform there are six legs and so, the H matrix consists of these six column vectors and it is multiplied by $\boldsymbol{f}$ which is nothing but the f_1 through f_6. The prismatic joints can only apply force along that like along the direction of the prismatic joint. So, like the Jacobian matrix, this matrix is also not a dimensionally homogeneous matrix. Why?

Because this is a unit vector, but this has some length cross that unit vector. So, this portion has units of length, this portion does not have any units, ok. But, as you can see this is so much easier to compute or so much easier to evaluate corresponding to ${}_{Tool}^{0}[J(\mathbf{q})]^{-T}$; both must be same, ok.

In previous case also we obtained the external force in terms of the leg forces using ${}_{Tool}^{0}[J(\mathbf{q})]^{-T}$ and in this case it is in terms of a matrix which we call as the force transformation matrix. But, this force transformation matrix is much simpler to compute. And, you can easily extend to any fully in parallel manipulator and then in that case, there will be n columns, ok.

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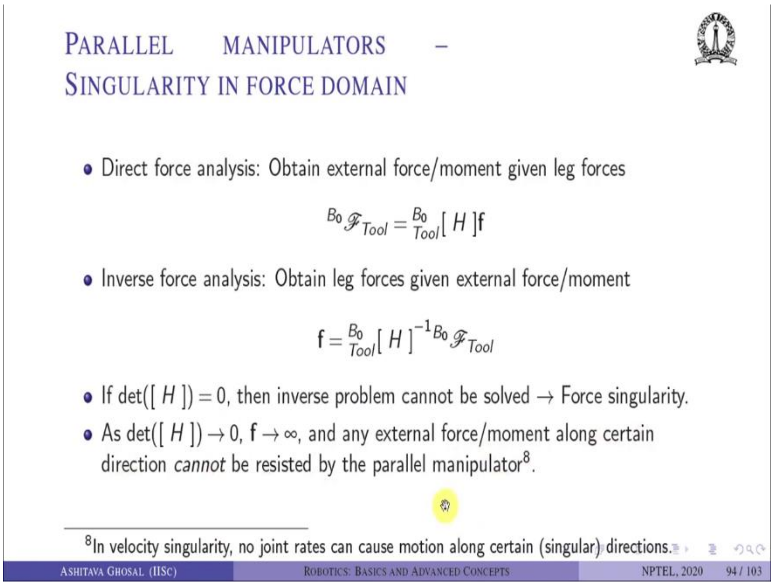

So, now, let us look at the singularity in the force domain, ok. So, what is the direct force analysis? Obtain the external forces and moment given the leg forces. So, what is the leg forces? This vector $\boldsymbol{f}$ and if we know this force transformation matrix, we can find the external forces and moments.

How about inverse? Again, the leg forces given the external forces and moments. So, if I tell you that the top platform of a Stewart platform, moving platform is experiencing some force and moment, an external force and moment is acting on the top platform. What is the forces required in the prismatic joints to keep it in equilibrium? The answer is $\boldsymbol{f} = {}^{B_0}_{Tool}[H]^{-1}\,{}^{B_0}\mathcal{F}_{Tool}$.

So, there is an inverse which is coming in; if the determinant of $[H]$ is equal to 0 then the inverse problem cannot be solved that is obvious, right. So, determinant of $[H]$ equal to 0, I cannot find a $\boldsymbol{f}$ for the external force and moments which are acting on the moving platform of this Stewart platform. So, this is called as force singularity, ok. So, as determinant of $[H]$ tends to 0, this $\boldsymbol{f}$ will tend to infinity and any external force moment along certain directions cannot be resisted by the parallel manipulator, ok.

So, analogous thing in velocity singularity, no joint rates can cause motion along certain singular direction, but $\dot{\theta}$'s goes to infinity if you want to move in the singular direction. Here any external force/moment along certain singular directions cannot be resisted by the parallel manipulator. So, thing is one is cannot be resisted and in velocity it cannot be applied, ok.

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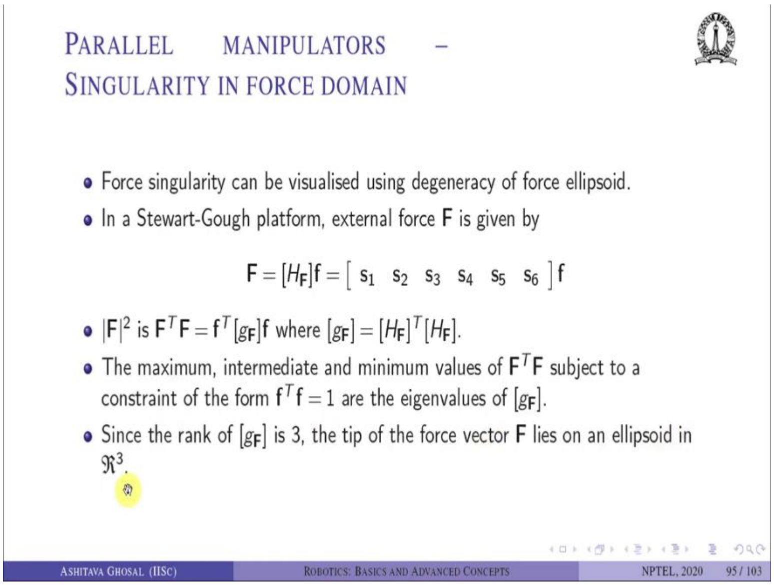

So, the force singularity can be visualized as using degeneracy of the force ellipsoid, ok. So, what is the force ellipsoid. So, in a Stewart Gough platform the external force is given by $\boldsymbol{F} = [H_{\boldsymbol{F}}]\boldsymbol{f}$. So, we are taking some part of the $[H]$ matrix which is the top half $\boldsymbol{s}_1$, $\boldsymbol{s}_2$ all the way till a 6 times $\boldsymbol{f}$.

So, the square of this force or the magnitude of the force can be written as $\boldsymbol{f}^T[g_{\boldsymbol{F}}]\boldsymbol{f}$, where $[g_{\boldsymbol{F}}] = [H_{\boldsymbol{F}}]^T[H_{\boldsymbol{F}}]$. So, just like we did for the velocity analysis, velocity ellipse and velocity ellipsoid.

So, the maximum intermediate and minimum values of this force, magnitude of this force subject to a constraint of the form $\boldsymbol{f}^T\boldsymbol{f} = 1$, very similar to what we did for velocity $\dot{\Theta}^T\dot{\Theta} = 1$ and eigenvalues of $[g_{\boldsymbol{F}}]$, ok. So, since the rank of $[g_{\boldsymbol{F}}]$ is 3, the tip of the force vector lies on the ellipsoid in $\Re^3$.

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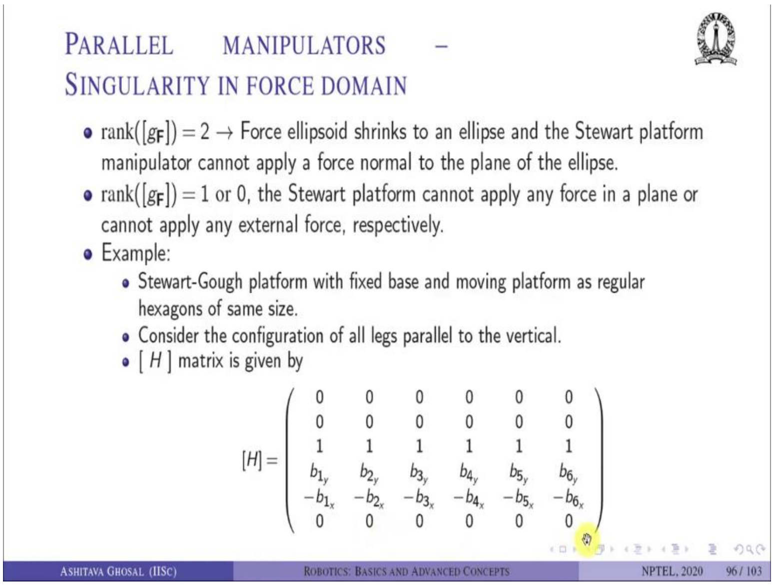

If the rank of $[g_F]$ is 2, the force ellipsoid shrinks to an ellipse and this Stewart platform manipulator cannot apply a force normal to the plane of the ellipse. If the rank of $[g_F]$ is 1 the Stewart platform manipulator cannot apply a force in a plane or cannot apply any external force respectively; if it becomes 0, then it cannot apply any external force, ok.

Example: Stewart platform with fixed base and moving platform as regular hexagon. So, the top is a hexagon regular hexagon 6 sided equal and the bottom is also a regular hexagon with all the 6 sides been equal, ok. And, we consider all the legs as parallel. So, basically considered that hexagons are equal in size.

So, for such a configuration the $[H]$ matrix will be all the leg vectors are along the Z-axis (0 0 1) (0 0 1) and so on and the moment of this Z-axis can be written in terms of y and x. So, b_{1y}, $-b\_1_x$, b_{2y}, $-b_{2x}$ and the z components will be 0. So, there is a vertical line and there is a vector which is in the x-y plane. So, the moment is $\boldsymbol{r}$ cross that line and this is what you will get.

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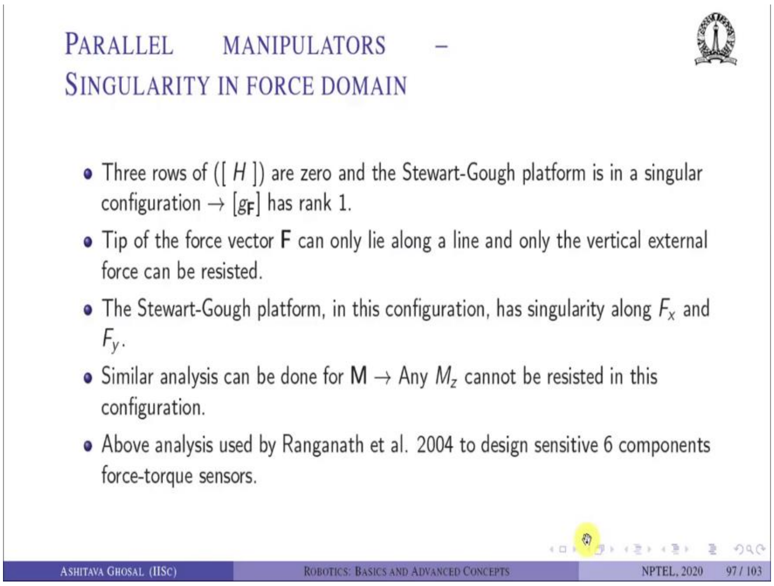

So, now, you can see that the three rows of $[H]$ are 0, ok. So, 1, 2 and 3 rows of $[H]$ are 0. So, the $[H]$ which is the top portion, two rows are 0 and the bottom portion only one row is 0, ok. So, $[g_{\boldsymbol{F}}]$ has a rank 1; obviously, right because there is only one column vector. So, it is 0 0 1 everywhere, so it has rank 1.

So, the tip of the force vector can only lie along a line, ok. It has dropped by two ranks and only a vertical external force can be resisted is not that obvious? That we have a hexagonal top and a hexagon on bottom all the legs are exactly vertical, ok. So, you can only apply a force from the top and that will be resisted. Any other force in the x or y direction cannot be resisted, the Stewart platform will just follow over, ok.

So, the Stewart platform in this configuration which is the you know very simple configuration which you can visualize as singularity along F_x and F_y, ok. The singularity is also along M_z, why? Because you can see the last row is also 0 0 0. So, if I apply a moment about the Z-axis again it cannot be resisted, ok.

So, this kind of analysis was used in a work by one of our IISc students and he showed how we can design a sensitive Stewart platform. Why? Because if you can make the Stewart platform very close to a singular configuration a very small force will lead to a very large value of force in the legs.

Remember it is divided by the determinant, ok. So, he showed that you can design very sensitive 6 component force tourqe sensors and these sensors are basically Stewart platform based sensors which are very close to a singular configuration not exactly a singular configuration, ok.

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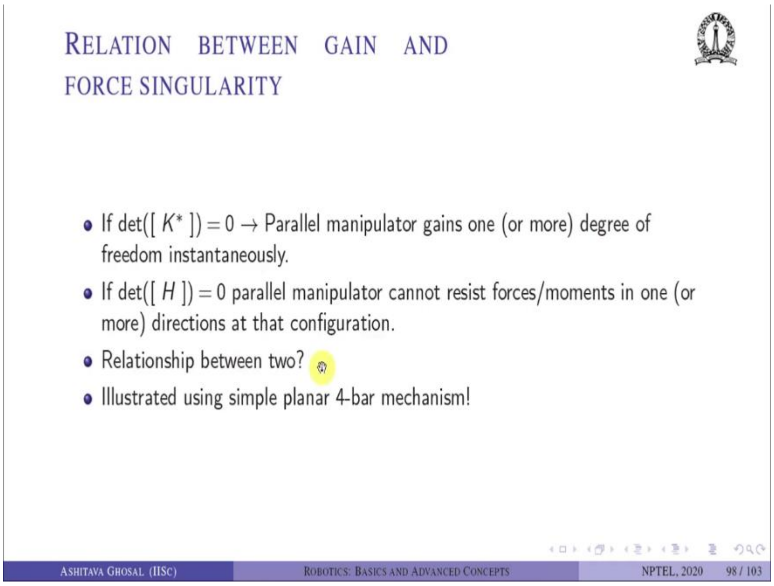

Finally, let us look at the relationship between gain and force singularity. Remember, in a parallel robot we had something called as a gain singularity and to recollect if determinant of $[K^*]$ equal to 0, the parallel manipulator gains one or more degrees of freedom instantaneously, ok. I have shown you for the 4-bar as well as for the 3 RPS robot.

If determinant of $[H]$ equal to 0, the parallel manipulator cannot resist forces or moment in one or more directions at that configuration this is what I have shown in the last few slides. So, the question is, is there a relationship between the two? What happens when you cannot resist forces and what happens when you gain one or more degrees of freedom, ok. So, what I will do is, I will illustrate what is happening in these two cases by using a simple 4-bar mechanism because that can be worked out.

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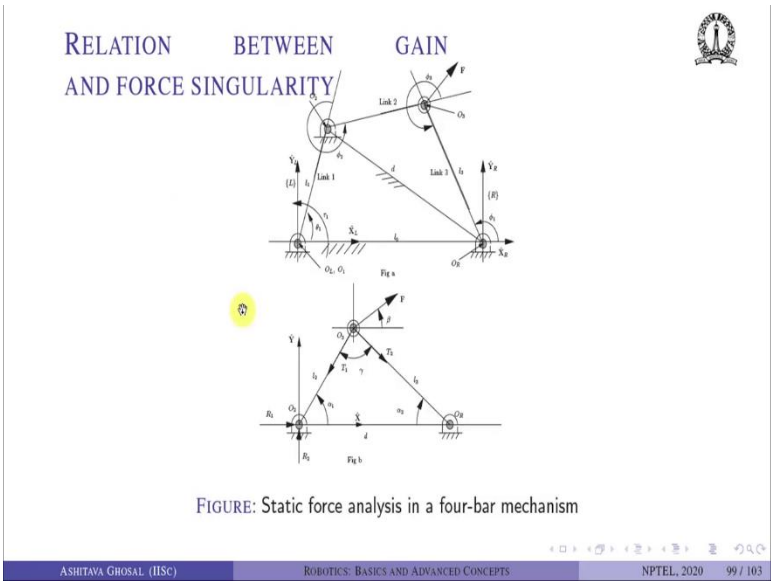

So, let us look at a 4-bar mechanism. So, we have this 4-bar mechanism and there is a force which is acting at third joint. So, this is that vector $\boldsymbol{F}$ which is acting. So, when we are looking at singularity the actuator joint is locked. So, θ_1 is locked. So, which means basically this point O_2 is also fixed. So, what do we have? We have like a truss member, we have one link, one joint, another link and this is the fixed link. So, this is drawn here in more detail.

So, we have a fixed base and we have a truss, we have three links and there is a force acting $\boldsymbol{F}$ maybe at an angle β to the horizontal axis and then we can define these angles α_1 and α_2. So, α_1 is this angle and α_2 is this angle. So, this is not ϕ_1 or ϕ_2, but it is sort of related to ϕ_1 and ϕ_2.

So, if you have a force which is acting at an angle β, this will lead to two forces in these two links. Let us call them T_1 and T_2 and the angle with these two links makes at this vertex is γ. Because of this force or because of this forces along the axial members between these this link or along this link, we will have reactions R_1 and R_2 at this joint ok.

There will be an action at this joint also, but we can just concentrate on what is happening to R_1 and R_2 the reaction. So, we will try to find out what is R_1 and R_2 as a function of $\boldsymbol{F}$ this angle α_1, γ and β ok.

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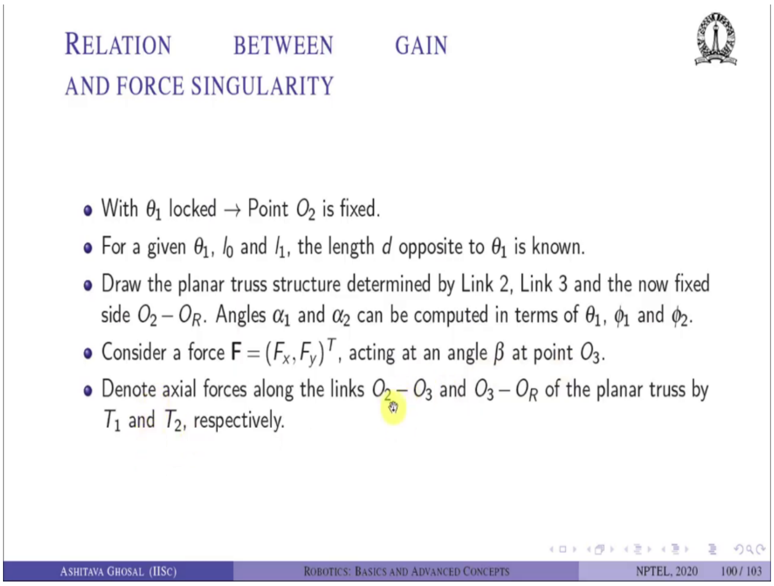

So, let us repeat with theta 1 locked, point O_2 is fixed. For a given θ_1, l_0 and l_1 the length d opposite to θ_1 is known. Draw the planar truss structure determined by link 2, link 3 and the now fixed side $O_2 - O_R$. Angles α_1 and α_2 can be computed in terms of theta 1, ϕ_1 and ϕ_2, not important, but if necessary we can find out depending on the lengths.

Now, consider a force $(F_x, F_y)^T$ acting at an angle at point O_3 whatever I have shown you in the previous drawing and we want to find the axial forces T_1 and T_2 acting along the links $O_2 - O_3$ and $O_3 - O_R$ of the planar truss.

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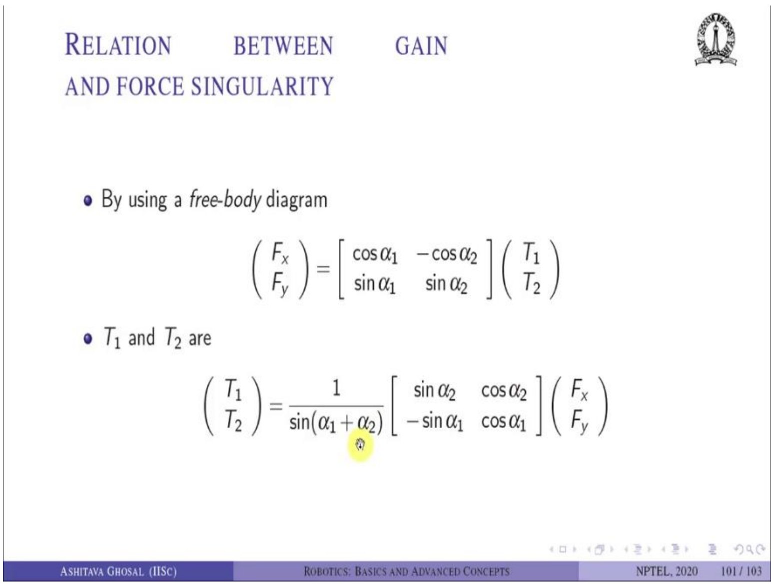

How do we find this we draw the free-body diagram and you can show that

$$\begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} \cos\alpha_1 & -\cos\alpha_2 \\ \sin\alpha_1 & \sin\alpha_2 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$$

This is a standard problem which we do in statics in undergraduate. So, there is a three link we have a force. Only difference here is the force is not acting vertically, it is acting at some random angle β.

So, once we have F_x, F_y in some function of T_1, T_2 we can also find out T_1, T_2 as a function of F_x and F_y, ok. So, basically what you will end up is something which is some matrix $\frac{1}{\sin(\alpha_1+\alpha_2)}\begin{pmatrix} \sin\alpha_2 & \cos\alpha_2 \\ -\sin\alpha_1 & \cos\alpha_1 \end{pmatrix}$.

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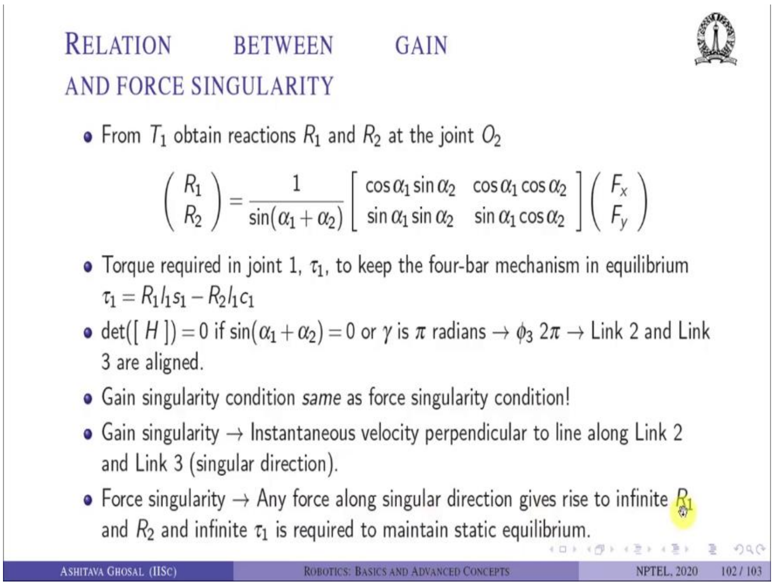

So, from T_1 we can now obtain the reactions R_1 and R_2 at joint O_2. So, let us go back and see this picture once more. So, I have a force which is acting at an angle β this components F_x and F_y they can be related to T_1, T_2 by simple force equilibrium at this node. And, then from T_1, I can find R_1 and R_2 and it turns out that you will get these angles α_1, α_2 and so on, ok.

So, R_1 and R_2 is obtained as $1/\sin(\alpha_1+\alpha_2)$ times some matrix which now contains both $\cos\alpha_1 \sin\alpha_2$ and so on times $\left(F_x, F_y\right)^T$. So, the torque required at joint 1 to keep the four-bar mechanism in equilibrium is $\tau_1 = R_1 l_1 s_1 - R_2 l_1 c_1$. How did we get this? This is obtained again from equilibrium.

So, if you look at this point so, there is R_1 and R_2 and τ_1 will not appear. So, equilibrium at this point moment equilibrium will give me what is the relationship between R_1, R_2 and this force what is a torque which is acting at the joint which is along the z-axis ok. So, this is the expression that you will get.

If determinant $[H]$ is 0, i.e. $\sin(\alpha_1 + \alpha_2) = 0$. So, we have α_1, α_2 and γ ok. So, if $\sin(\alpha_1 + \alpha_2) = 0$ or $\gamma = \pi$, because the sum of the three angles inside the triangle will be π, ok, so, then link 2 and link 3 will be aligned, ok. So, think of it this way. Let us go back to this figure once more.

So, when determinant $[H]$ is 0, $\sin(\alpha_1 + \alpha_2) = 0$. So, $\alpha_1 + \alpha_2 = 0$. So, basically this link and this link at parallel are lying on top of this base ok, they are aligned. So, this is exactly the same as what was happening when you had gain singularity.

So, in the case of gain singularity if θ_1 is locked, this second link and the third link were aligned ok. So, gain singularity is same as force singularity. So, determinant of $[H]$ equal to 0 is same as determinant of $[K^*]$ equal to 0, ok. So, gain singularity: Instantaneous velocity is perpendicular to link 2 and 3.

Force singularity: Any force along the single direction gives rise to infinite R_1 and R_2 and infinite τ_1. So, if you apply a force along the singular direction which is perpendicular to link 2 and 3, then you need infinite thought to keep it in equilibrium ok. So, the bottom line is the gain singularity which is determinant of $[K^*]$ equal to 0 is same as determinant of $[H]$ equals to 0, which is the force singularity at least for the planar four-bar mechanism, ok. And, it turns out it is true for all parallel mechanisms.

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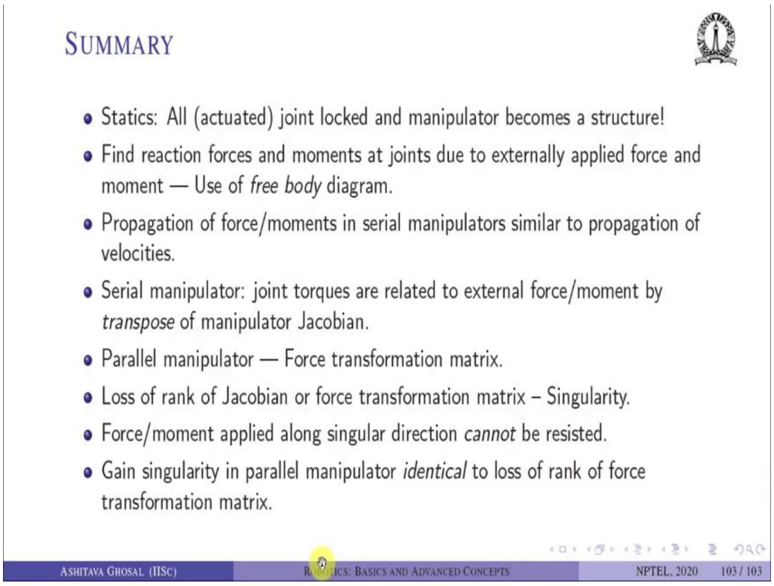

So, in summary, when we look at statics all the actuated joints are locked and the manipulator becomes a structure. So, we can find the reaction forces and moments at the joints due to externally applied force and moment using free-body diagrams, ok. Propagation of forces and moments in serial manipulator is similar to propagation of velocities.

So, remember ${}^{i}\boldsymbol{f}_{i}$ is something which is ${}^{i}_{i+1}[R]\,{}^{i+1}\boldsymbol{f}_{i+1}$, except in a serial manipulator it is going backwards. For velocity analysis it went from the fixed base to the end-effector, in the case of serial manipulator statics the forces and moments go backwards from the end-effector to the fixed base.

So, in serial manipulator joint torques are related to external forces moments by the transpose of the manipulator Jacobian. So, we have this well known expression $\tau = [J]^{T}\boldsymbol{f}$. In velocity kinematics it was $\mathcal{V} = [J]\dot{\Theta}$. For parallel manipulators especially in parallel manipulators we can define something called as the force transformation matrix.

So, the loss of rank of Jacobian or force transformation matrix is this singularity, ok. So, force/moment applied along the singular direction cannot be resisted. And, in gain singularity in parallel manipulator is identical to the loss of rank of force transformation matrix, ok.

So, determinant of $[H]$ equal to 0, which is loss of rank of force transformation matrix is same as determinant of $[K^{*}]$ equal to 0, ok. So, with this we will stop this discussion on velocity analysis and static analysis of serial and parallel manipulators. Thank you.