Robotics: Basics and Selected Advanced Concepts
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Lecture - 27
Introduction, Lagrangian formulation

Welcome to this NPTEL lectures on Robotics. So, in this set of lectures, we will look at the Dynamics of Serial and Parallel Robots.

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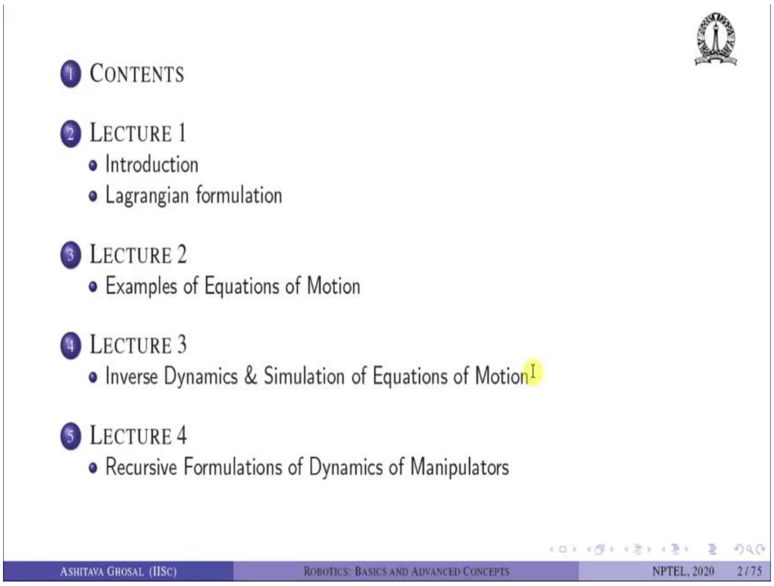

So, there will be 4 lectures in this content. So, first we will introduce the problem of dynamics of serial and parallel robots then, we will look at the way to derive the equations of motion using something called as the Lagrangian formulation. In lecture 2, we will look at examples of equation of motion, for a planar 2 degree of freedom 2R robot and also for a parallel mechanism, closed loop mechanism.

Then, on 3rd lecture we look at the inverse dynamics and simulation of equations of motion and finally, we will look at recursive formulation of dynamics of serial and parallel manipulators, ok.

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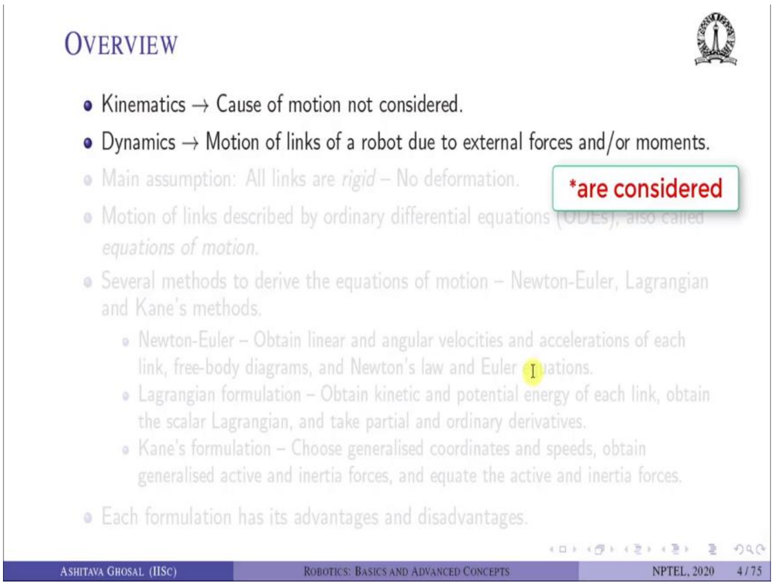

So let us start. So, as an overview in kinematics, we looked at the motion without worrying about what is the cause of the motion. So, the cause of motion was not considered. In dynamics the motion of links of a robot due to external forces and moments are considered ok. So, they are the topic of dynamics.

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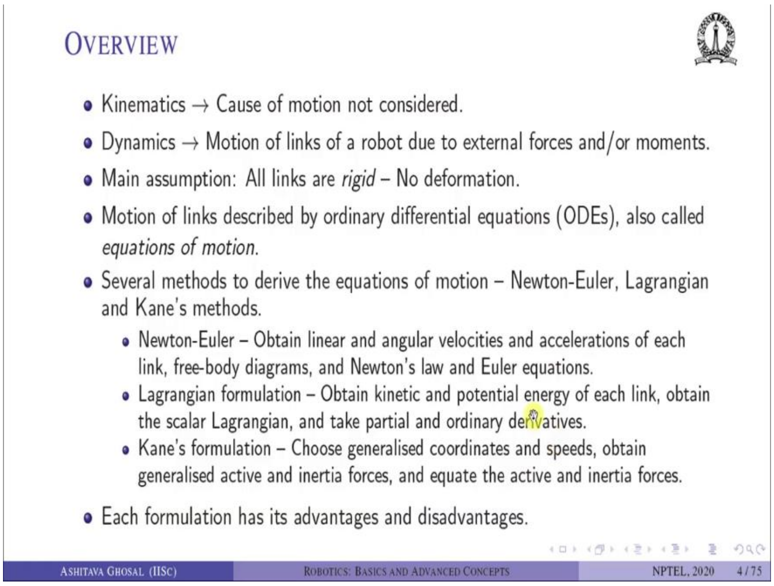

So, main assumption in these lectures will be that all links are rigid. So, basically there is no deformation there is no flexibility in any of the links. The motion of the links are described by ordinary differential equation, also called equations of motion. If you have flexibility if you have deformation then, it is no longer ordinary differential equation ok. The equations of motion will contain partial derivatives ok, and then they are PDE's.

There are several methods to derive the equations of motion for example, there is something called as Newton-Euler, there is something called Lagrangian formulation and then there is Kane's method ok. In the Newton-Euler approach basically, we obtain the linear and angular velocities and accelerations of each of the link ok, we use free body diagrams we also use Newton's law which is $F = ma$ and the Euler equation which relates the torque and the moment of inertia of a rigid body.

So, torque is related to angular acceleration and angular velocity. In the Lagrangian formulation we obtain the kinetic and potential energy of each link ok, we obtain something called as a Lagrangian which is a scalar quantity and then, we have to take certain partial and ordinary derivatives, ok. We have to take ordinary derivatives with respect to time and partial derivatives with what are called generalized coordinates, and their rates.

In the Kane's formulation we choose generalized coordinates and speeds ok, we obtain something called as generalized active and inertia forces, and then we equate the active and inertia forces ok. Each of these formulations has its own advantage and disadvantages.

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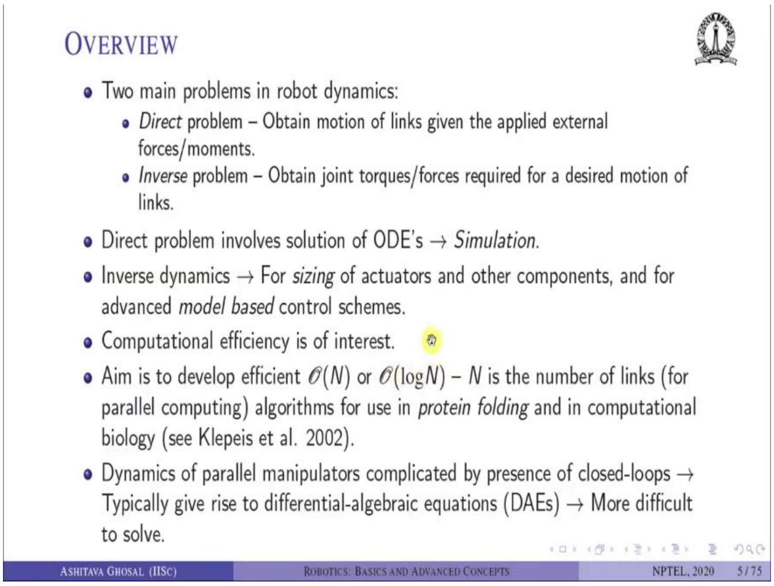

There are two main problems in robot dynamics, first is this direct problem. The direct problem states that we need to obtain the motion of the links given the applied external forces and moments ok. The inverse problem we obtain the joint torques and forces required for a desired motion of the links of the robot.

So, the direct problem involves solution of ODE's this is also sometimes called a simulation. The inverse problem is required for sizing of the actuators and other components, and for advanced model based control schemes ok. So, once you solve the inverse problem we can find the joint torques and forces required to achieve a desired motion ok, and hence that helps you in choosing what is the sizes of the actuators.

This is also very important problem ok, it does not involve solution of ordinary differential equations and it is used extensively in model based control schemes. In both of these direct and inverse problem the computational efficiency is of interest ok. So, we want to do solve this direct and inverse problem in a computationally efficient way. The goal is to develop what is called as $\mathcal{O}(N)$ or $\mathcal{O}(\log N)$ algorithms.

$\mathcal{O}$ here means, that the order of the computational efficiency is linearly related to the number of links ok. So, say if the computation time is N if I increase or double the number of links, the computation time will scale up linearly ok, it will become $2N$ or if even better if we can get a $\log N$ computational efficiency ok.

The reason why this is very important is there are now problems in something called protein folding and in computational biology, where the number of links or number of protein elements and number of chains is very large, it could be 500 it could be 1000 of them ok. And hence, we need to solve this direct and inverse problem in dynamics very very efficiently.

The dynamics of parallel manipulators is complicated by the presence of closed loops ok. So, as we have seen till now, in parallel manipulators there are closed loops there are constraints which involve the passive joint variables ok. So, these constraints are typically directly in terms of the joint variables ok, they are algebraic constraints are constraints involving sin and cos of the angles.

So, these constraints give rise to what are called as differential algebraic equations and these are normally much more difficult to solve, than ordinary differential equations -- ODE's.

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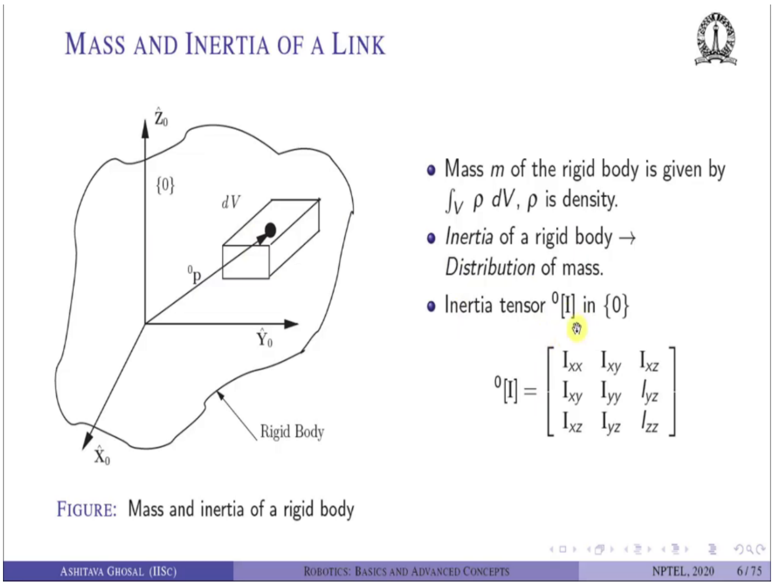

So, the first important thing to understand while deriving the equations of motion of a manipulator is this concept of mass and inertia of a link ok. So, if I have a rigid body in a coordinate system {0} labeled {0}. So, $\hat{X}_0, \hat{Y}_0, \hat{Z}_0$ is the coordinate system attached to the reference coordinate system, and we have a point on this rigid body which is at a distance P ok and there is a volume element dV so, dX, dY, dZ. So, the mass of the rigid body is given $\int_V \rho\, dV$ where ρ is the density.

Now, just the mass is not enough we also need something called the inertia of a rigid body. So, the inertia of a rigid body is nothing but the distribution of the mass, how is this mass distributed. So, we have done in undergraduate. So, you can have a mass which is a plate, but we can also have a mass which is like an annular -- with a hole inside the plate ok. So, the inertia even though the mass of this plate and with the hole and without the hole is same ok, but the inertia will be different.

So, the inertia of a rigid body is given by this inertia tensor ok, which is with respect to a coordinate system in this case 0 denotes that this inertia matrix of this rigid body is with respect to the {0} coordinate system, and it consists of 9 elements ok. So, I_{xx}, I_{xy}, I_{xz}, but $I_{yx} = I_{xy}$, so it is a symmetric tensor. So, out of these 9 elements only 6 are independent.

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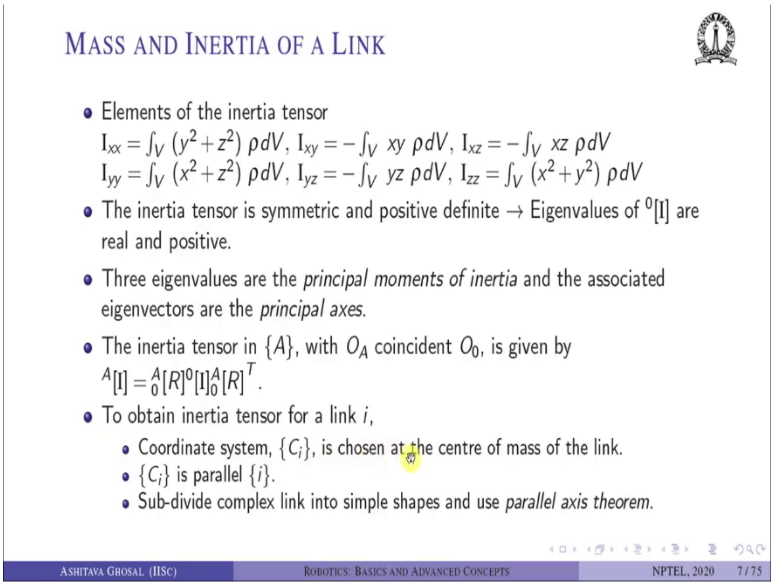

So, the elements of the inertia tensor can be obtained again by volume integral. So, $I_{xx} = \int_V (y^2 + z^2)\,\rho\, dV$, $I_{xy} = -\int_V xy\,\rho\, dV$. So, ok we have seen this earlier. So, $I_{zz} = \int_V (x^2 + y^2)\,\rho\, dV$. So, it tells you how the mass is distributed so, I_{xx} will tell you how the mass is distributed related to y and z and so on.

The inertia tensor is symmetric and positive definite ok, the eigenvalues of this inertia tensor are real and positive, this comes from linear algebra. If you have a 3×3 matrix which is symmetric and positive definite you will have three eigenvalues which are real and positive ok. The three eigenvalues are called the principal moments of inertia and the associated eigenvectors are called the principal axes, ok.

So, we have seen inertia in undergraduate. So, there is something called principal moments of inertia and principal axes. So, they are determined from the eigenvalues and eigenvectors of this inertia matrix. The inertia tensor in another coordinate system {A} with O_A coincident with O_0 can be obtained by rotation matrices ok. So, the point of the coordinate system, the origin of the coordinate system O_A and O_0 must be coincident for this formula to hold through.

So, ${}^{A}[I] = {}^{A}_{0}[R]\ {}^{0}[I]{}^{A}_{0}[R]^{T}$. So, to obtain the inertia for a link i in a robot first thing is we have to choose a coordinate system, so the typically the coordinate system is labeled $\{C_i\}$ it is chosen at the center of mass of the link. The coordinate system $\{C_i\}$ is parallel to the i^{th} link. So, remember the coordinate system for the i^{th} link is on the joint axis i ok, the origin is at the point of intersection of the common perpendicular with the Z axis and so on.

So, we take that coordinate system which is X_i, Y_i, Z_i and O_i and shift it to the center of mass keeping the orientation same ok. So, $\{C_i\}$ is parallel to i, we can also sub-divide a complex link into simple shapes and use parallel axis theorem ok. So, if you have a very strange looking link ok, with maybe holes and maybe some portions which are bent and so on.

We can divide it into simple shapes ok, and then use parallel axis theorem. So, there is something called as a parallel axis theorem which again I am assuming you know, ok. So, with these concepts of inertia we can now get into the Lagrangian formulation to derive the equations of motion.

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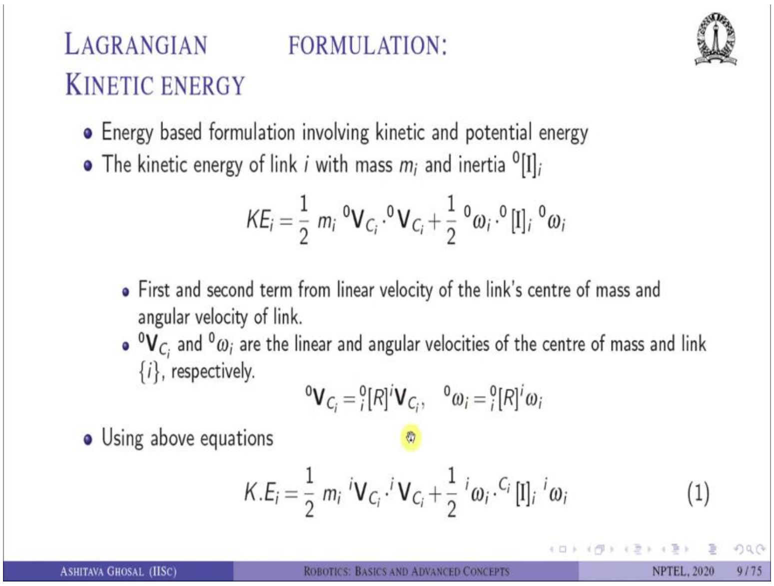

So, the Lagrangian formulation ok is based on kinetic energy and potential energy it is a energy based formulation ok. So, first we will look at the kinetic energy. So, the kinetic energy of a link i with mass m_i and inertia $^{O}[I]_i$ is given by $\left(\frac{1}{2}mV^2 + \frac{1}{2}I\omega^2\right)$, this I am sure you have seen it earlier.

In this case, we are writing it more formally as so, $\frac{1}{2}mV^2$ is nothing but $\frac{1}{2}m_i\ {}^0\boldsymbol{V}_{C_i} \cdot\ {}^0\boldsymbol{V}_{C_i}$so it is the velocity of the center of mass of the i^{th} link square of that and, this is the angular velocity of the i^{th} link in the 0th coordinate system, which we need to use in the $\frac{1}{2}I\omega^2$. So, the first and second terms from the linear velocity of the link center of mass and the angular velocity of the link.

${}^0\boldsymbol{V}_{C_i}$ and ${}^0\omega_i$ at the linear and angular velocity of the center of mass and link i, respectively ok. If you know the linear velocity in its own coordinate system we can convert it back to the 0th coordinate system in using this rotation matrix ok. So, sometimes during the propagation formulas we obtain the linear and angular velocity described in its own coordinate system.

So, that can also be used. So, since kinetic energy is a scalar you can show that $\left(\frac{1}{2}m_i\ {}^i\boldsymbol{V}_{C_i} \cdot\ {}^i\boldsymbol{V}_{C_i} + \frac{1}{2}\ {}^i\omega_i \cdot {}^{C_i}[I]_i\ {}^i\omega_i\right)$ will also give the kinetic energy of the link, it is a scalar. So, it does not really matter from where you are starting, if you are careful.

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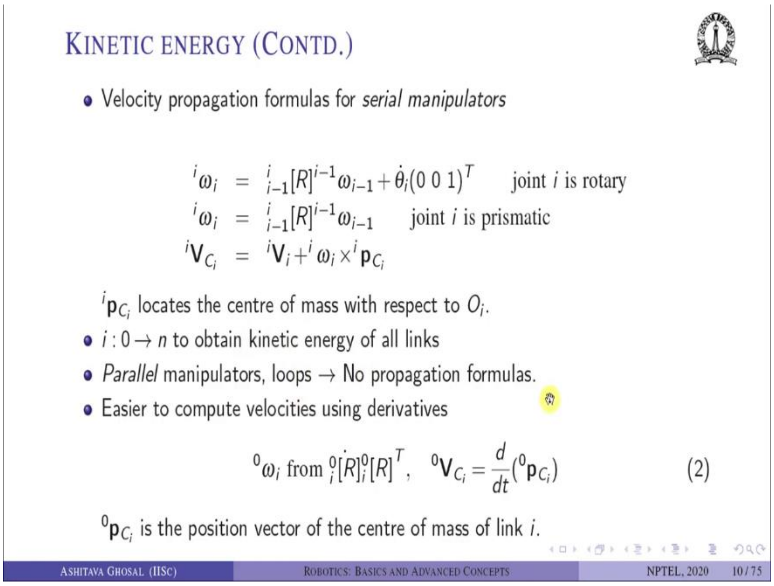

So, we will use this velocity propagation formulas for serial manipulators basically ${}^i\omega_i$ is ${}^{i-1}\omega_{i-1}$ pre multiplied by a rotation matrix added to $\dot{\theta}$ along the z axis. So, this is if the joint is rotary. If the joint is prismatic then ${}^{i}\omega_i$ is same as ${}^{i-1}\omega_{i-1}$ pre multiplied by a rotation matrix.

And the velocity of the center of mass is nothing but the velocity of the origin of the coordinate system into some $\omega \times r$, so location of the center of mass with respect to the origin. So, we do this for i : 0 to n to obtain the kinetic energy of all links, ok. So, $i = 1$ is, so we find ${}^{1}\omega_1$, ${}^{1}\boldsymbol{V}_{C_1}$ and so on for all the links.

In case of parallel manipulators, there are loops, there are no propagation formulas ok, which is not very obvious what are the propagation formulas because, we do not know what is the link or what is the joint and after that link after that joint we can go in two different ways to the same link, but more than two different ways to the same link. In that case, it is easier to compute the angular velocity from ${}^{0}_{i}[\dot{R}]\,{}^{0}_{i}[R]^T$.

So, for the chosen end-effector or chosen link we can find ${}^{0}_{i}[\dot{R}]\,{}^{0}_{i}[R]^T$, and then from that we can this is a skew symmetric matrix and we can extract the space fixed angular velocity vector from that ok. Likewise, from the position vector of this center of mass of the i^{th} link we can take the derivative with respect to time and find the linear velocity of the center of mass of the i^{th} link ok.

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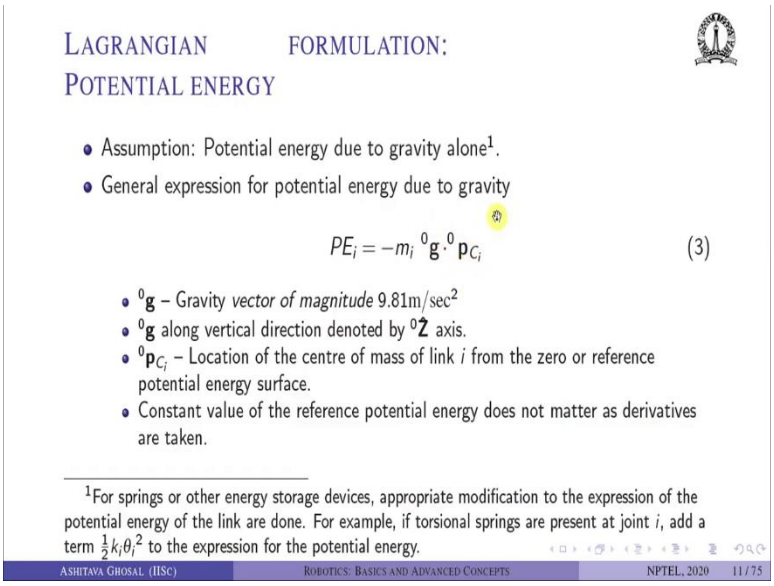

So, ${}^{0}\boldsymbol{p}_{C_i}$ is the position vector of the center of mass of link i. So, the Lagrangian formulation also requires you to compute the potential energy and in this case we will assume that the potential energy is due to gravity alone ok. So, if you have springs or some other energy storage devices, we need to make appropriate modification to the expression of potential energy for a link.

So, for example, if there is a spring torsional spring which is driving a joint ok, then we have to add $\frac{1}{2}k_i\theta_i^2$ where k is the spring constant, but most of the time we will be dealing only with gravity.

So, the general expression for potential energy due to gravity is this dot product of the gravity vector which is most of the time pointing downwards is opposite to the Z-axis most of the time into the position of the CG dot product with vector locating the CG with respect to the 0^{th} coordinate system ok, and there is a minus sign ok.

So, the gravity vector has a magnitude of 9.81 meters per second square this is the standard assumption, ${}^0\boldsymbol{g}$ is normally along the vertical direction along the Z axis, but pointed in the opposite direction. And the position of the center of mass is the location of the center of mass so ${}^0\boldsymbol{p}_{C_i}$ locates the center of mass of link i from the zero or reference potential energy ok.

So, it is is not very important to know what is the reference potential energy or the zero potential energy because, we will take derivatives. So, if you have a constant value of the reference potential energy which you need to add or subtract that does not matter because finally, we will be taking derivatives of the kinetic and potential energy.

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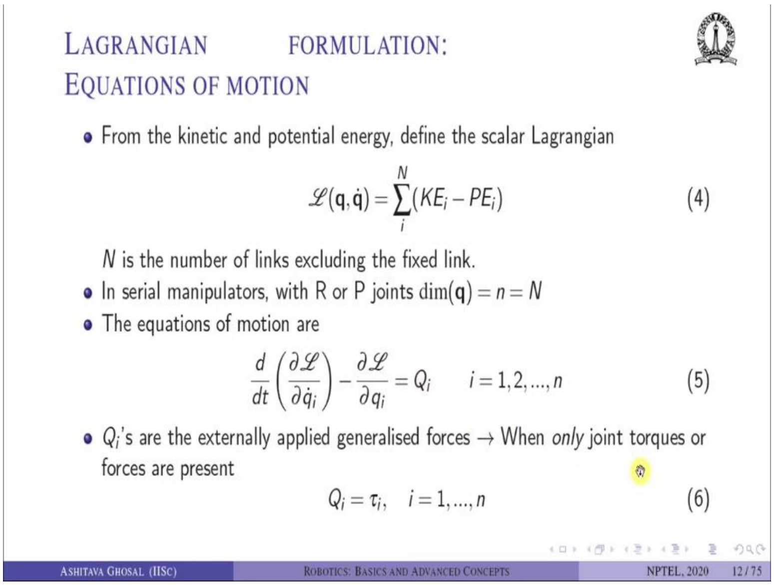

So, from the kinetic and potential energy, we define a scalar Lagrangian which is kinetic energy of link i minus potential energy of link i and then we sum over all the links $i = 1$ to N. So, this is a function of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$. So, $\boldsymbol{q}$ means joint variables and $\dot{\boldsymbol{q}}$ means the rate of change of the joint variables. So, kinetic energy contains $\dot{\boldsymbol{q}}$ and $\boldsymbol{q}$ and potential energy contains $\boldsymbol{q}$ ok.

In a serial robot with R and P joints the dimension of $\boldsymbol{q}$ is n, which is the number of degrees of freedom and that should be equal to the number of links ok, we have seen this earlier. So, once we have this Lagrangian the equations of motion are obtained by following this recipe. So, what do we do?

We take the partial derivative of the Lagrangian with respect to q_i, we take the partial derivative of the Lagrangian with respect to $\dot{q}_i$ and then the time derivative of del Lagrangian with del $\dot{q}_i$ and then, we equate it to Q_i and this is for each one of this i so, i equals 1 through n.

The right hand side capital Q_i are externally applied generalized forces ok. So, if there are forces and moments which are acting on this manipulator they will be accommodated in this Q_i. If there are only joint torques which are acting so, then Q_i is nothing but τ_i, which is the joint torques or forces which are acting at the joints.

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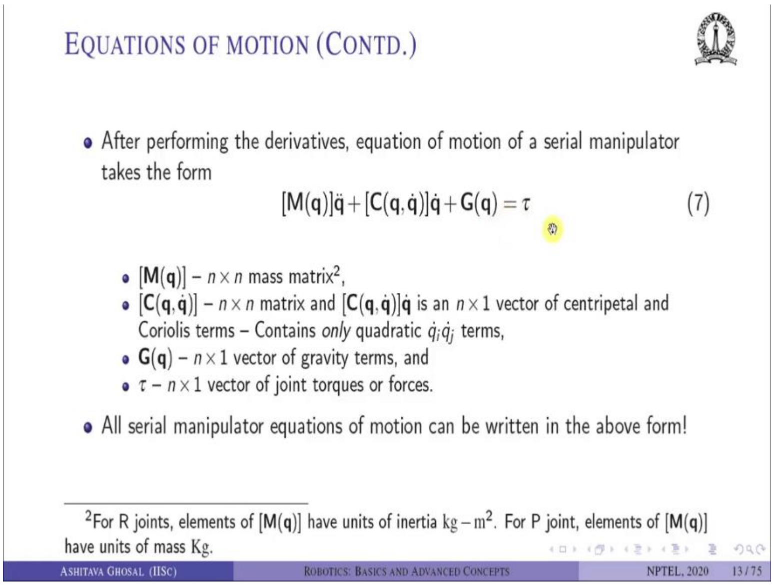

So, after performing all these derivatives, the equation of motion of a serial manipulator takes the form $[\boldsymbol{M}(\boldsymbol{q})]\ddot{\boldsymbol{q}} + [\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})]\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{\tau}$. So, in this equation here all serial manipulators can be written in this form; this $[\boldsymbol{M}(\boldsymbol{q})]$ is called the mass matrix it is an $n \times n$ mass matrix, this matrix $[\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})]$ this is also an $n \times n$ matrix and this $[\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})]\dot{\boldsymbol{q}}$ is a $n \times 1$ vector of centripetal and Coriolis terms ok.

So, it contains terms which are quadratic terms of the form $\dot{q}_i\dot{q}_j$. The term $\boldsymbol{G}(\boldsymbol{q})$ is called the gravity terms, and it is a $n \times 1$ vector and $\boldsymbol{\tau}$ right hand side is a vector of joint torques or forces again this is $n \times 1$.

So, it turns out that all serial manipulator equations of motion can be written in the above form, this is a very strong statement that there is something which is multiplying $\ddot{\boldsymbol{q}}$, there is something which is centripetal and Coriolis term, there is something which is gravity term which is only a function of $\boldsymbol{q}$, and on the right hand side we have the external torques which are acting at the joints.

So, for R joints, the elements of $[\boldsymbol{M}(\boldsymbol{q})]$ have units kg meter square so, this is like inertia ok I. For P joints, the elements of $[\boldsymbol{M}(\boldsymbol{q})]$ have units of mass because; for P joints the right hand side will be force and not a moment.

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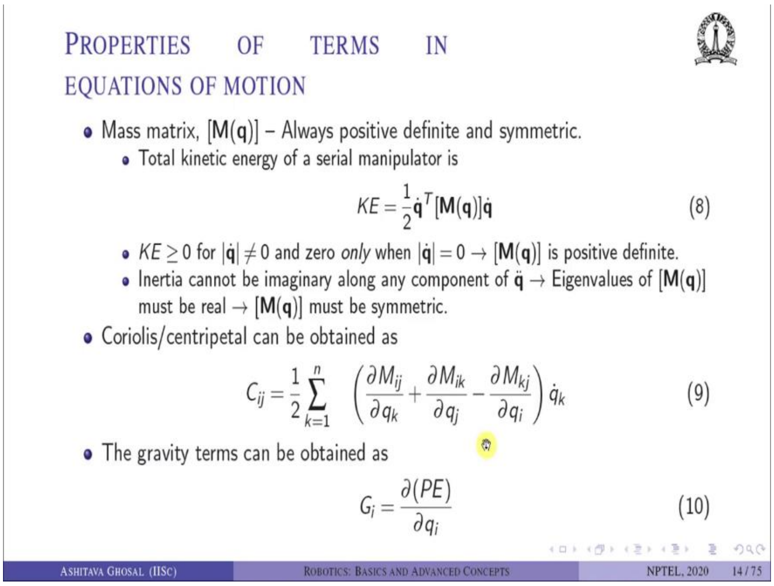

So, the mass matrix $[\boldsymbol{M}(\boldsymbol{q})]$ is always positive definite and symmetric ok. How do we know it is positive definite? Because, you can show that the kinetic energy of the system is nothing but $\frac{1}{2}\dot{\boldsymbol{q}}^T[\boldsymbol{M}(\boldsymbol{q})]\dot{\boldsymbol{q}}$. The kinetic energy is always greater than or equal to 0 for $\dot{\boldsymbol{q}}$ not equal to 0.

And zero only when $\dot{\boldsymbol{q}}$ is equal to 0 so this implies that $[\boldsymbol{M}(\boldsymbol{q})]$ is positive definite ok. Another way of looking at it is, since this is the inertia matrix, inertia cannot be imaginary along any component of $\ddot{\boldsymbol{q}}$. So, hence the eigenvalues of $[\boldsymbol{M}(\boldsymbol{q})]$ must be real and $[\boldsymbol{M}(\boldsymbol{q})]$must be symmetric, ok.

The centripetal and Coriolis term matrix $[\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})]$ the elements being C_{ij} can be derived from the mass matrix by taking this following partial derivative. So, $C_{ij} = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{kj}}{\partial q_i}\right)\dot{q}_k$, this can be proved ok. The gravity term final gravity term G_i it is a vector $n \times 1$ vector, each element is nothing but the potential energy of the manipulator partial derivative with respect to q_i.

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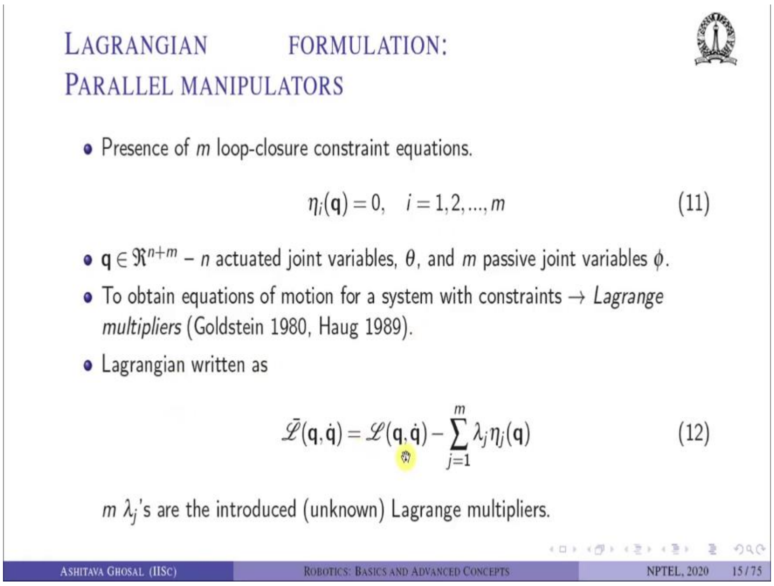

If you have m loop closure equations ok, and we still want to find the apply the Lagrangian formulation to parallel manipulators. So, we have $\eta_i(\boldsymbol{q}) = 0$, i =1 through m. So, now this $\boldsymbol{q} \in \Re^{n+m}$, ok. So, there are n actuated joint variables θ and m passive joint variables ϕ. So, to obtain the equation of motion for a system with constraints, loop closure constraints we use Lagrange multipliers.

So, you can look at this book by Goldstein and Haug. So, the Lagrangian is now written as $\bar{\mathcal{L}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \sum_{j=1}^{m} \lambda_j \eta_j(\boldsymbol{q})$ where j equals 1 through m. So, this λ_j's are the m Lagrange multipliers which have been introduced. So, we need to find them out somehow.

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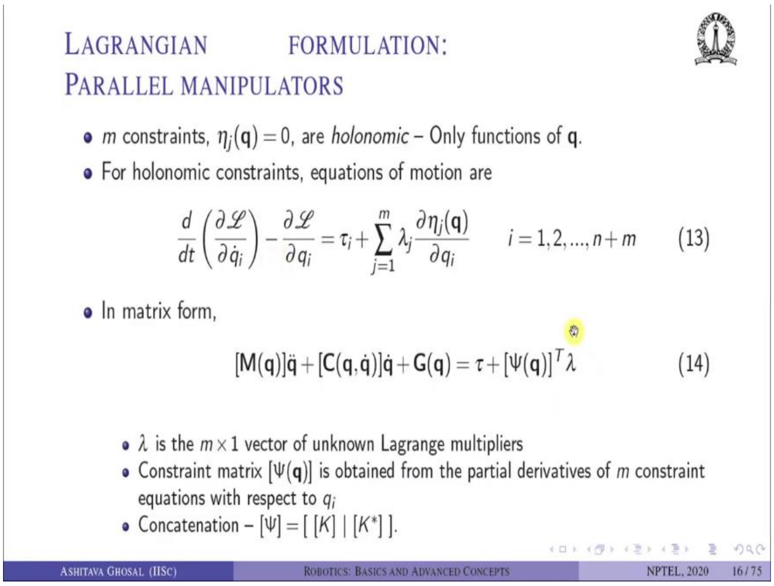

So, if the m constraints $\eta_j(\boldsymbol{q}) = 0$ are holonomic they are only functions of $\boldsymbol{q}$. So, for holonomic constraints the equations of motion can be again derived. So, it is given by $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i + \sum_{j=1}^{m} \frac{\partial \eta_j(\boldsymbol{q})}{\partial q_i}$, we have an additional term $\sum_{j=1}^{m} \frac{\partial \eta_j(\boldsymbol{q})}{\partial q_i}$. So, this is a matrix times λ_j.

So, in matrix form, this will look like mass matrix times $\ddot{\boldsymbol{q}}$, Coriolis term, gravity term equals torque plus some $[\Psi(\boldsymbol{q})]^T \boldsymbol{\lambda}$, where lambdas are the Lagrange multipliers, m of them. This matrix $[\Psi(\boldsymbol{q})]$ is obtained from the partial derivatives of the m constraint equation with respect to q_i ok. So, we have m constraint equation we take partial derivatives with respect to q's meaning partial derivatives with respect to actuated θ and passive ϕ.

And then, we basically remember in parallel robots we found something called $[K]$ matrix and $[K^*]$ matrix. So, $[\Psi(\boldsymbol{q})]$ is nothing but a concatenation of $[K]$ and $[K^*]$. So, remember $[K^*]$ was a square matrix so, partial derivative of constraint equation with passive joint variables and $[K]$ contains partial derivative of the constraint equation with the actuated joint variables.

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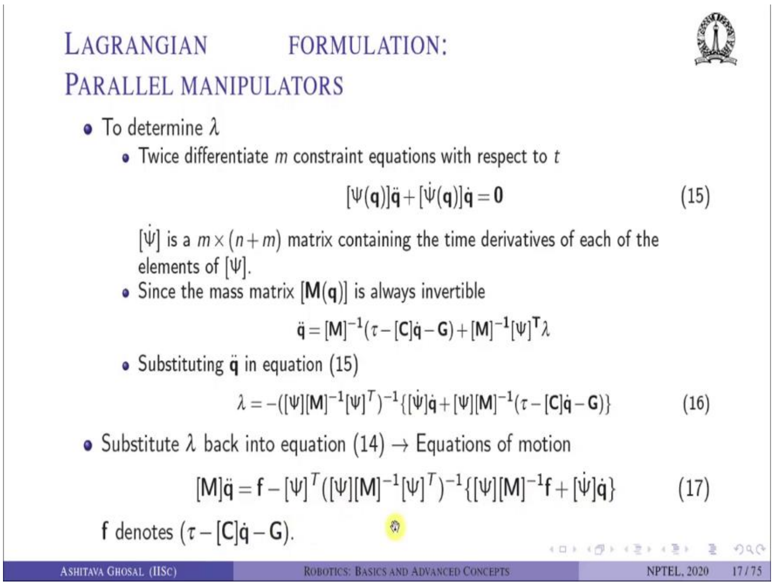

So, to determine λ we twice differentiate the m constraint equation with respect to time. So, if you differentiate $\boldsymbol{\eta}(\boldsymbol{q}) = 0$ with respect to time ok. So, you will get $[\Psi(\boldsymbol{q})]\ddot{\boldsymbol{q}} + [\dot{\Psi}(\boldsymbol{q})]\dot{\boldsymbol{q}} = \mathbf{0}$. First derivative of $\boldsymbol{\eta}(\boldsymbol{q})$ will give you $[\Psi(\boldsymbol{q})]\dot{\boldsymbol{q}} = \mathbf{0}$. Then second time when you again take the derivative you will get $[\Psi(\boldsymbol{q})]\ddot{\boldsymbol{q}} + [\dot{\Psi}(\boldsymbol{q})]\dot{\boldsymbol{q}} = \mathbf{0}$.

And what is $[\dot{\Psi}]$? $[\dot{\Psi}]$ is a $m \times (n+m)$ matrix containing the time derivatives of each of the elements of $[\Psi]$. So, since mass matrix is always invertible I can always write $\ddot{\boldsymbol{q}} = [\boldsymbol{M}]^{-1}(\tau - [\boldsymbol{C}]\dot{\boldsymbol{q}} - \boldsymbol{G}) + [\boldsymbol{M}]^{-1}[\Psi]^T\lambda$. So, the equation of motion contains $[\Psi]^T\lambda$ on the right hand side.

So, we can find $\ddot{\boldsymbol{q}}$. So, why is the mass matrix always invertible? Because, it is symmetric and positive definite ok. It is positive definite that is enough to make it invertible. We can now substitute $\ddot{\boldsymbol{q}}$ which is obtained from this expression back into this expression which we obtained when we differentiated twice the constraint equations ok. So, then you will have $[\Psi]$ times this whole quantity here plus $[\dot{\Psi}]\dot{\boldsymbol{q}}$ equal to 0, ok.

So, what do we have as the unknown in this expression? λ ok, and hence we can find out $\lambda = -([\Psi][\boldsymbol{M}]^{-1}[\Psi]^T)^{-1}\{[\dot{\Psi}]\dot{\boldsymbol{q}} + [\Psi][\boldsymbol{M}]^{-1}(\tau - [\boldsymbol{C}]\dot{\boldsymbol{q}} - \boldsymbol{G})\}$, little bit of algebra will show you that this is indeed correct. And then, we can substitute λ back into this equation ok and we get the equation of motion which is $[\boldsymbol{M}]\ddot{\boldsymbol{q}} = \boldsymbol{f} - [\Psi]^T([\Psi][\boldsymbol{M}]^{-1}[\Psi]^T)^{-1}\{[\Psi(\boldsymbol{q})][\boldsymbol{M}]^{-1}\boldsymbol{f} + [\dot{\Psi}]\dot{\boldsymbol{q}}\}$.

Where $\boldsymbol{f} = \boldsymbol{\tau} - [\boldsymbol{C}]\dot{\boldsymbol{q}} - \boldsymbol{G}$, so what have we done? Lot of math's, but the basic idea is that we have some constraint equations which are m of them which we obtain after twice differentiating the m constraint equation we get this matrix equation. We also have the matrix equation corresponding to the equations of motion. So, what are the unknown's? λ is an unknown ok. And we solve for this λ from the second equation substitute back into this equation and then we can derive the.

So, basically we have eliminated λ. So, now let us take a little bit look at this equation. If it was a serial robot ok then, we had just $[\boldsymbol{M}(\boldsymbol{q})]\ddot{\boldsymbol{q}} = \boldsymbol{\tau} - [\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})]\dot{\boldsymbol{q}} - \boldsymbol{G}(\boldsymbol{q})$. Now, due to all this constraint equation we have these additional complicated terms.

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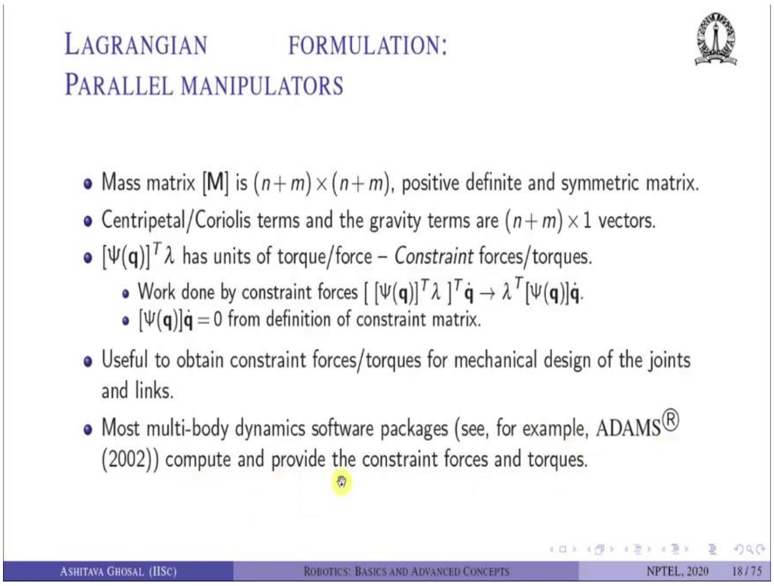

So, the mass matrix $[\boldsymbol{M}(\boldsymbol{q})]$ is now $(n+m) \times (n+m)$, it is a positive definite symmetric matrix. The centripetal and Coriolis terms are also $(n+m) \times 1$ vectors ok. $[\Psi(\boldsymbol{q})]^T \lambda$ has the units of torque or force ok, so that we can see here. So, $[\Psi(\boldsymbol{q})]^T \lambda$ is added to right hand side with torque. So, it will have units of the joint torque or the joint force, which is being applied externally.

You can show that the work done by constraint forces ok, which is $([\Psi(\boldsymbol{q})]^T \lambda)^T \dot{\boldsymbol{q}}$ or basically this constraint forces dot $\dot{\boldsymbol{q}}$ will be this ok, and $[\Psi(\boldsymbol{q})]\dot{\boldsymbol{q}} = 0$. So, $[\Psi(\boldsymbol{q})]\dot{\boldsymbol{q}} = 0$ So, hence the work done by these constraint forces is 0 ok. So, there is a set of forces or torques which are occurring due to the constraint equations and the work done by those is 0.

It is often useful to obtain the constraint forces and torques for mechanical design of the joints ok. So, although the forces are not doing any work constraint forces, but they have to be resisted. So, if you have a bearing ok which is at one of the joints I need to know the radial forces and other forces which are not contributing to the energy, but I still need to design the bearing.

So, I need to know what are the forces. So, the $[\Psi(\boldsymbol{q})]^T\lambda$ are the constraint forces ok, and we can evaluate them to go design later on. Most multi-body dynamics software packages for example, ADAMS will compute and provide the constraint forces and torques ok. Something similar is done in this package called ADAMS. So, not only it will give you the equations of motion ok, it will involve the torques and forces which are causing motion, but it will also tell you what are the constraint forces.

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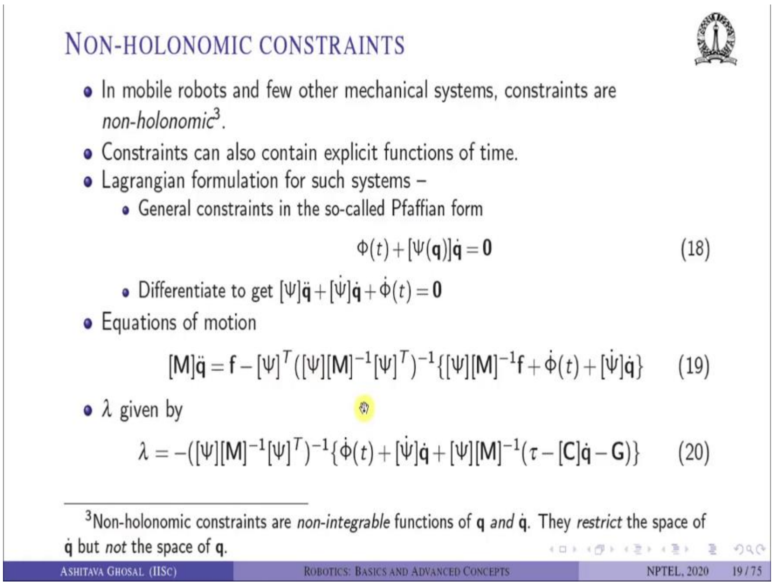

In mobile robots and and few other mechanical systems, the constraints are non-holonomic. So, what is meant by non-holonomic? The constraints are functions of both $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ ok, more importantly these functions are non-integrable so, you cannot get rid of the $\dot{\boldsymbol{q}}$. So, once you have a function of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ equal to 0 ok not $\eta(\boldsymbol{q}) = 0$ that is holonomic constraint.

This is a non-holonomic constraint when you have both $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ occurring and non-integrable. So, what it means is they restrict the space of $\dot{\boldsymbol{q}}$, but not the space of $\boldsymbol{q}$ so;

certain $\dot{\boldsymbol{q}}$ are no longer independent ok. So, now the constraint forces can also be explicit functions of time ok.

The Lagrangian formulation for such systems ok with general constraints in the so-called Pfaffian form $\phi(t) + [\Psi(\boldsymbol{q})]\dot{\boldsymbol{q}} = \boldsymbol{0}$ can also be obtained. So, what do we do? We differentiate once this equation so, you get $[\Psi]\ddot{\boldsymbol{q}} + [\dot{\Psi}]\dot{\boldsymbol{q}} + \dot{\phi}(t) = 0$. So, remember this term is only a function of time this term is a function of $\boldsymbol{q}$'s -- joint variables.

So, now the equations of motion can be derived basically you again find $[\boldsymbol{M}(\boldsymbol{q})]\ddot{\boldsymbol{q}}$ we substitute $\ddot{\boldsymbol{q}}$ here and obtain the Lagrange multipliers, and then resubstitute back the Lagrange multiplier into the equations of motion. And you will get $[\boldsymbol{M}]\ddot{\boldsymbol{q}}$ is equal to $\boldsymbol{f} - [\Psi]^T([\Psi][\boldsymbol{M}]^{-1}[\Psi]^T)^{-1}$ plus this term the only difference is now we have some $\dot{\phi}(t)$, which is coming from this derivative here.

And $\lambda = -([\Psi][\boldsymbol{M}]^{-1}[\Psi]^T)^{-1}\{\dot{\phi}(t) + [\dot{\Psi}]\dot{\boldsymbol{q}} + [\Psi][\boldsymbol{M}]^{-1}(\boldsymbol{\tau} - [\boldsymbol{C}]\dot{\boldsymbol{q}} - \boldsymbol{G})\}$. So, this is the expression for λ. So, once you solve the equations of motion, once you find $\boldsymbol{q}$ as a function of time and $\dot{\boldsymbol{q}}$ as a function of time, and if you are given this ϕ as a function of time ok. Then you can substitute back in this expression and find λ.

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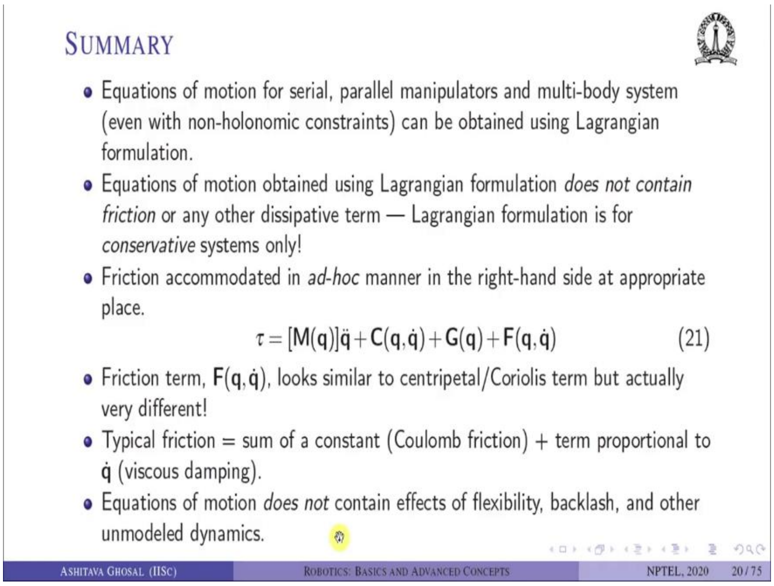

So, in summary the equations of motion for the serial or a parallel manipulator and any multi-body system even with non-holonomic constraints can be obtained using the Lagrangian formulation ok. So, we can compute the kinetic energy, we can always compute the kinetic energy of each moving link, sum it over and get the total kinetic energy we find the potential energy of each link and then sum it over and get the potential energy of the system and then KE – PE is the Lagrangian for the system ok.

The equation of motion obtained using Lagrangian formulation does not contain friction or any other dissipative term ok, remember we do not have anything any formulation for friction and that is because, the Lagrangian formulation originally is only for conservative systems ok. But, we must have or we must need to accommodate friction somehow. So, in the Lagrangian formulation we just add friction in an ad-hoc manner in the right hand side ok.

So, we had $\tau = [\boldsymbol{M}(\boldsymbol{q})]\ddot{\boldsymbol{q}} + [\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})]\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q})$ we add a friction term, which could be a function of both $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$. So, this friction term looks you know concept you know symbolically similar, but they have completely different meaning ok. So, this friction term is a dissipative term ok whereas, this Coriolis term is not a dissipative term. Typically this friction term will be a constant, which is the Coulombs friction and maybe a term proportional to $\dot{\boldsymbol{q}}$ this is called viscous damping ok.

So, we just add it to the right hand side. So, for each equation of motion we have for each joint we add a Coulomb term and a viscous damping term. Equations of motion also do not contain any effect of flexibility backlash or other unmodeled dynamics ok. Remember, in a robot we can have flexible links we can have backlash at the joints because, there is a gearbox and it can bend and it can have some other unmodeled dynamics ok, those are not part of the equations of motion derived using the Lagrangian formulation.

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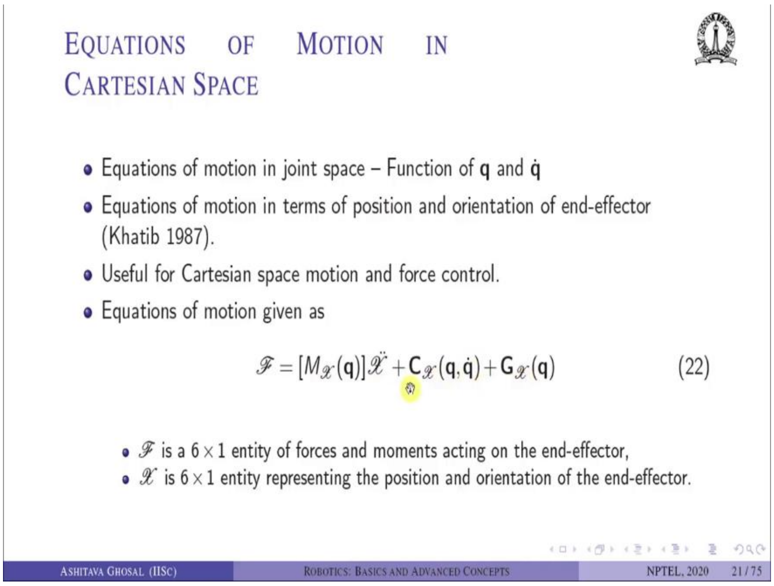

Let us continue, we can also derive the equation of motion in Cartesian space ok. Meaning the currently whatever I have shown you are the equations of motion in joint space are functions of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$. The equations of motion in terms of position and orientation of the end-effector which is the Cartesian coordinates are sometimes needed ok. So, it is useful for Cartesian space motion and force control ok.

So, we need to derive the equations of motion in terms of the end-effector position and orientation and this was done by Khatib in 1987. Equations of motion are given by some mass matrix times $\ddot{\mathcal{X}}$ plus Coriolis term $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$ and a gravity term. So, what I am trying to show here is the left hand side is a 6×1 entity of forces and moments acting on the end-effector ok, it is not a vector it is not the torque vector joint you know joint torques.

It is an external forces and moments which are acting on the end-effector. And this $\mathcal{X}$ is a 6×1 entity representing the position and orientation of the end-effector. So, it could be x, y, z and three Euler angles or some other representation ok. There is $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ here also conceptually we could have removed $\boldsymbol{q}$ and written in terms of $\mathcal{X}$ by inverse kinematics or $\dot{\boldsymbol{q}}$ we could have removed and written in terms of inverse Jacobian, but that is not really required.

Later on, when we do force control or Cartesian space motion control, that is why it is left in this form, but conceptually I can write everything in terms of Cartesian position and orientation of the end-effector ok.

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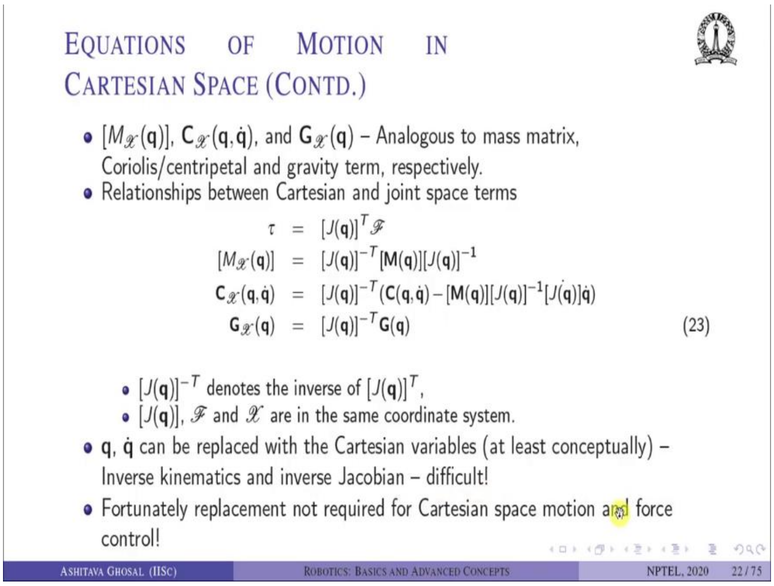

So, this $[\boldsymbol{M}_{\boldsymbol{\mathcal{X}}}(\boldsymbol{q})]$ and $[\boldsymbol{C}_{\boldsymbol{\mathcal{X}}}(\boldsymbol{q},\dot{\boldsymbol{q}})]$ and $\boldsymbol{G}_{\boldsymbol{\mathcal{X}}}(\boldsymbol{q})$ is analogous to the mass matrix, coordinate centripetal and gravity term, respectively. We can also derive the relationship between the Cartesian and joint space terms please think of it as a homework problem. So, we have already derived that the torque is nothing but $[J(\boldsymbol{q})]^T\mathcal{F}$, this was done in statics and the same relationship holds here also.

The Cartesian mass matrix which is on the left hand side is related to the joint space mass matrix by $[J(\boldsymbol{q})]^{-T}$ and $[J(\boldsymbol{q})]^{-1}$, the Coriolis term also now will include $[J(\boldsymbol{q})]^{-T}$, and $[J(\boldsymbol{q})]^{-1}$. And the gravity term is again nothing but $[J(\boldsymbol{q})]^{-T}$ into the gravity term obtained in joint space.

So, $[J(\boldsymbol{q})]^{-T}$ denotes the inverse of $[J(\boldsymbol{q})]^T$. Why do we get $[J(\boldsymbol{q})]^T$? Basically, again remember in the torque and joint torques and joint forces are related by $[J(\boldsymbol{q})]^T$, and that is sort of we carry through when we derive the mass matrix Coriolis term the gravity term in the Cartesian space.

The $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$ as I mentioned can be replaced with Cartesian variables at least conceptually so, we could we could have set inverse kinematics of $\boldsymbol{q}$ is giving $\mathcal{X}$ inverse Jacobian is giving $\dot{\boldsymbol{q}}$, which is difficult but it is not really required also ok. So, this replacement is not required for Cartesian space motion and force control and hence we will leave it in this form.

So, with this we finish the Lagrangian formulation, and how to use the Lagrangian formulation to derive the equations of motion. In the next lecture, we look at Examples of Equations of Motion.

Thank you.