Robotics: Basics and Selected Advanced Concepts

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Lecture - 28
Examples of Equations of Motion

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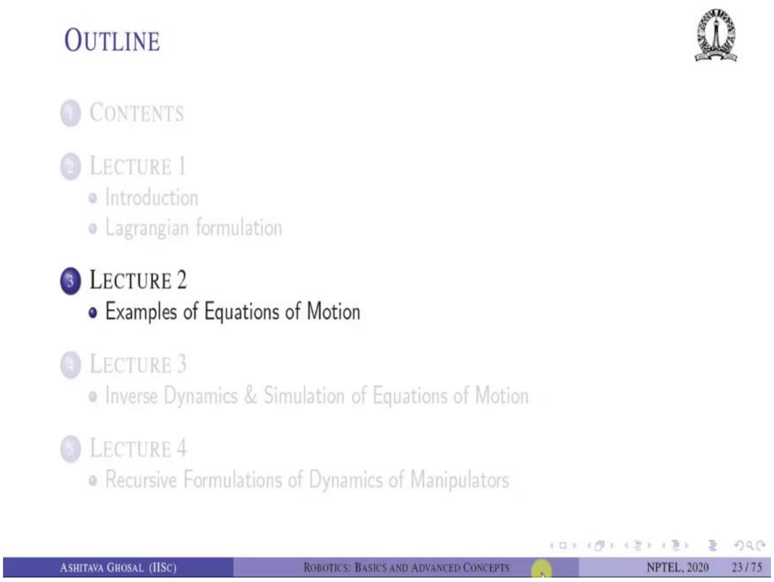

Welcome to this NPTEL lectures on Robotics Basics and Advanced Concepts. In the last lecture, I had looked at the Lagrangian formulation and shown you that to derive the equation of motion, using the Lagrangian formulation we need to calculate the kinetic energy, the potential energy and then using a set of derivatives, ok.

Partial derivatives and time derivatives we can derive the equations of motion. So, in this lecture, we will look at Examples of Equations of Motion, ok. This lecture we will discuss several examples of equations of motion obtained using the Lagrangian formulation.

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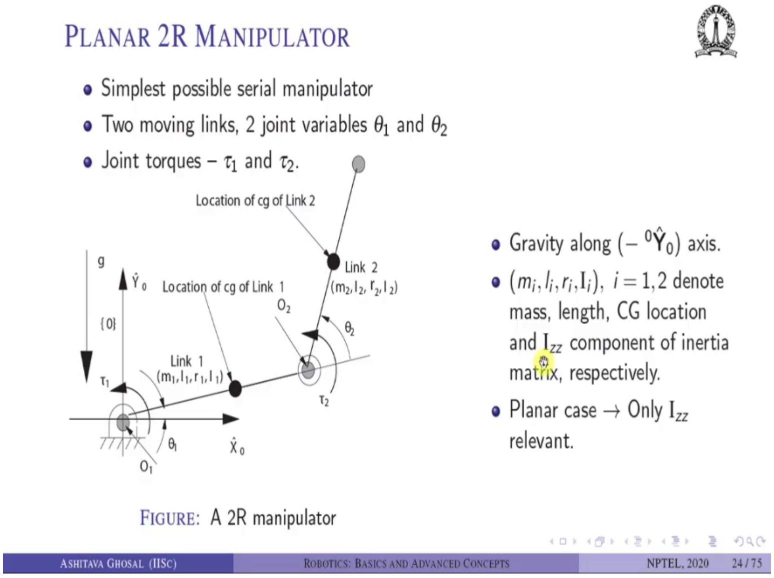

So, let us start with a simple example which is a planar 2R manipulator ok. So, this is the simplest possible serial manipulator. So, it has one rotary joint here, another rotary joint here and then the rotary joint here has an angle θ_1, this is angle θ_2 there is a torque τ_1 which is acting on the first rotary joint and then torque τ_2 which is acting on the second rotary joint. And then we need to define some mass and inertial parameters of these 2 links ok.

So, for link 1 we are going to use the symbols m_1, l_1, r_1 and I_1. So, m_1 is the mass of link 1, l_1 is the length of link 1, r_1 is the location of the CG from the origin of the coordinate system. So, the origin of the first coordinate system is at the rotary joint axis and r_1 is somewhere here ok and I_1 is the moment of inertia, ok. So, we are only interested in the Z component of the moment of inertia since this is a planar motion, ok. So, likewise for link 2 we have m_2, l_2, r_2 and I_2.

So, m_2 is the mass of the second link, l_2 is the link length of the second link r_2 is the location of the CG again from the origin of the second coordinate system and I_2 is the z component of the inertia of this link, ok.

We will assume that the gravity is acting in the minus Y direction as shown in this figure and as I have said earlier m_i, l_i, r_i and I_i are denote the mass, length, CG location and the I_{zz} component of the inertia matrix respectively for link 1 and 2. And because this is a planar motion only the I_{zz} component is relevant, ok.

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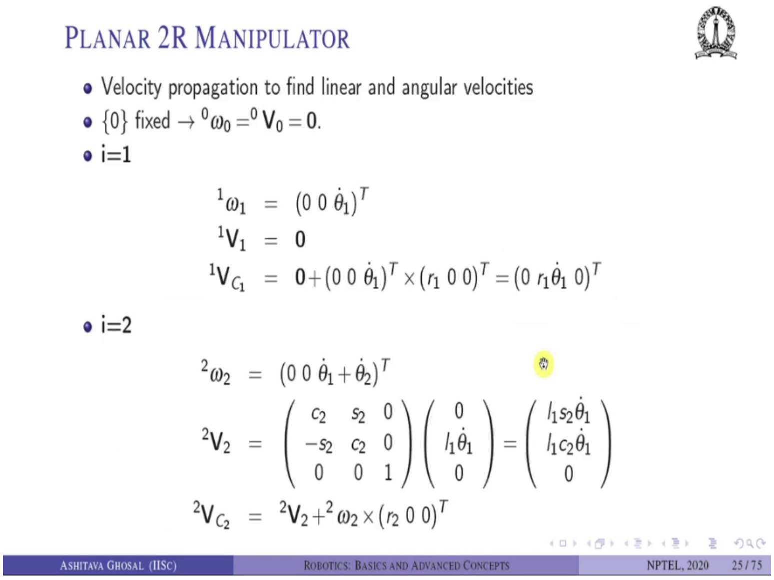

So, we need to find the velocity of each link. So, we are going to use the velocity propagation to find linear and angular velocity. So, since the 0th link is fixed basically it is a fixed link the angular velocity and the linear velocity of the origin of the 0th link is **0**. So, angular velocity of link 0th and the linear velocity of the origin of this 0th link is **0**, then we can substitute $i = 1$ in the propagation equation and then right hand side we will have **0** and we will obtain ${}^1\omega_1$ and ${}^1\boldsymbol{V}_1$.

So, if you go back and see the propagation equations you will see that ${}^1\omega_1 = (0,0,\dot{\theta}_1)^T$. The linear velocity of the origin of the first link is **0** and the linear velocity of the center of gravity of the first link will be the linear velocity of the origin plus some $\omega \times r$, ok. So, location of the CG is r_1 along the x axis, so omega is along the z axis. So, we can take the cross product and we will get $r_1\dot{\theta}_1$ as the y component.

Then for $i = 2$ again we can substitute back in the propagation equation for angular velocity and we will get ${}^2\omega_2 = (0,0,\dot{\theta}_1 + \dot{\theta}_2)^T$ along the z axis, then x and y component still remain 0 because this is basically a planar motion. The linear velocity of the origin of the second link can be obtained again using the propagation equation and it turns out that this is some matrix $\begin{pmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

So, it is a rotation about the z axis times $(0, l_1\dot{\theta}_1, 0)^T$. So, if you carry out this multiplication we will get $(l_1 s_2\dot{\theta}_1, l_1 c_2\dot{\theta}_1, 0)^T$. So, this makes sense why because the origin of the second link is at the beginning of the second link ok. So, only what is happening to the first link which is $\dot{\theta}_1$ will affect the linear velocity. The velocity of the CG now is the origin of the second link plus $\omega \times r$ and r is along the x axis at a distance r_2, r_2 here is the location of the CG ok.

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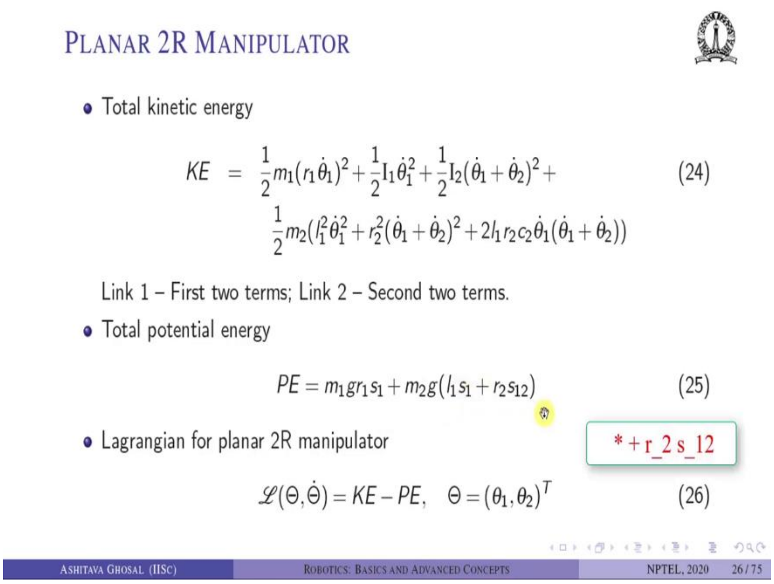

So, we can multiply out and obtain the velocity of the CG, the total kinetic energy is $\frac{1}{2}mV^2 + \frac{1}{2}I\omega^2$ for each link ok. So, we can compute the kinetic energy of link 1 which is $\frac{1}{2}m_1(r_1\dot{\theta}_1)^2 + \frac{1}{2}I_1\dot{\theta}_1{}^2$, for the second link it is $\frac{1}{2}I_2(\dot{\theta}_1 + \dot{\theta}_2)^2$.

And we will also have a term, which is $\frac{1}{2}mV^2$, where V is the center of mass of the second link and it turns out that you will have terms which include $\left(\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2\right) r_2^2$. But also you will get a term which is $2l_1 r_2 c_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)$. So, what it basically means that the kinetic energy of the second link is a function of θ_2, ok does that make sense? Yes.

So, these are like the Coriolis and centripetal term which will show up later. So, as I said link 1 is the first two terms; link 2 are the second two terms, the potential energy is nothing but the mg along the y direction opposite to the gravity. So, the location of the cg is $r_1 s_1$.

So, the potential energy of the first link is $m_1 g\, r_1 s_1$ the location of the CG for the second link is $(l_1 s_1 + r_2 s_{12})$, you can see in the figure.

So, the potential energy is $m_2 g$ times that and since it is opposite to the gravity remember there was a minus sign earlier both the minus signs go away and we will left with one positive quantity. So, the 0 is the 0 of the $\hat{X}_0$ ok the reference is this. So, as you can see this distance is $r_1 s_1$ this distance is $l_1 s_1 + r_2 s_{12}$ that is the y component ok.

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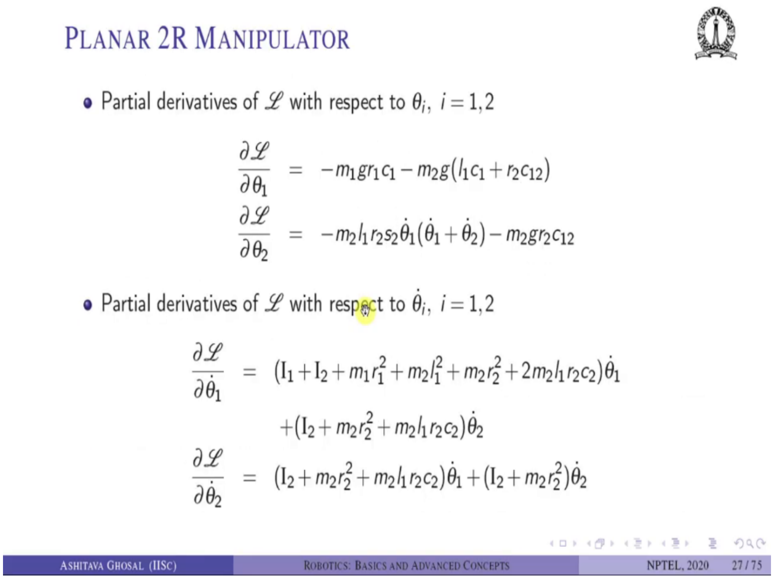

So, we have obtained the potential energy the kinetic energy of both the links and then we can find the Lagrangian for this planar 2R manipulator, ok. So, it is nothing but the kinetic minus the potential energy ok. So, it will be a function of θ_1, θ_2 and also $\dot{\theta}_1$ and $\dot{\theta}_2$. So, as given by the Lagrangian formulation we need to now take partial derivatives of this Lagrangian with respect to θ_i, $i = 1,2$.

So, this Lagrangian with respect to θ_1 will give you $-m_1 g r_1 c_1 - m_2 g(l_1 c_1 + r_2 c_{12})$. The partial of the Lagrangian with respect to θ_2 will also contain $-m_2 g r_2 c_{12}$, but there be also a term, which includes $\dot{\theta}_1$ and $\dot{\theta}_2$ because the kinetic energy is also a function of θ_2. So, we will have some c_2 was there, so partial derivative will be s_2 and we will get the rest of it remaining the same, ok.

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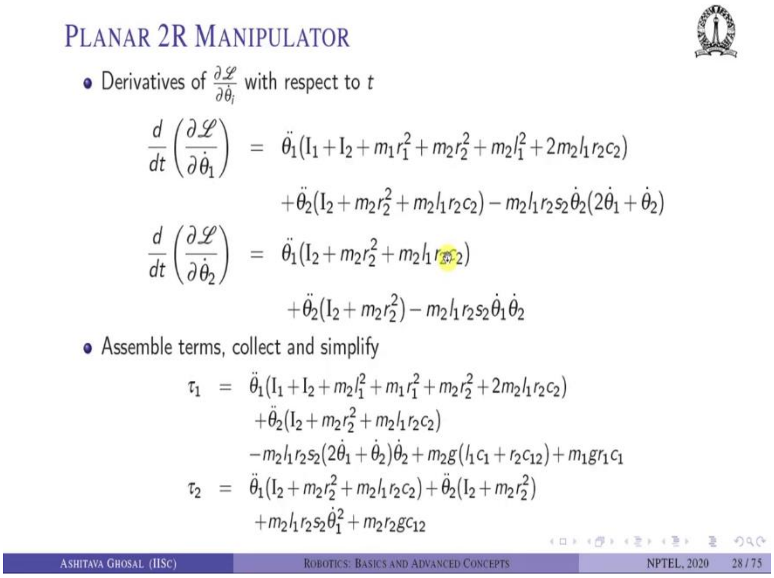

The partial derivative of Lagrangian with respect to $\dot{\theta}_i$ can also be obtained. So, partial this Lagrangian with respect to $\dot{\theta}_1$ will contains this $(I_1 + I_2 + m_1 r_1^2 + m_2 l_1^2 + m_2 r_2^2 + 2m_2 l_1 r_2 c_2)\dot{\theta}_1 + (I_2 + m_2 r_2^2 + m_2 l_1 r_2 c_2)\dot{\theta}_2$.

Likewise the partial derivative of the Lagrangian with respect to $\dot{\theta}_2$ will be given by $(I_2 + m_2 r_2^2 + m_2 l_1 r_2 c_2)\dot{\theta}_1 + (I_2 + m_2 r_2^2)\dot{\theta}_2$. Remember the kinetic energy contains $\dot{\theta}_1{}^2$ and $\dot{\theta}_2{}^2$ and that and it was half. So, we now are left with $\dot{\theta}_1$, $\dot{\theta}_2$ the half has gone away ok.

So, it is very straightforward it is standard taking partial derivatives with respect to $\dot{\theta}_1$ and $\dot{\theta}_2$ nothing new. Then we take the time derivative of this $\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}$ and then we will get $\ddot{\theta}_1$ times some term $\ddot{\theta}_2$ times some term, ok. So, these are the $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right)$, likewise $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right)$.

Partial derivatives will again give you $\ddot{\theta}_1$ into some quantity which is $(I_2 + m_2 r_2^2 + 2m_2 l_1 r_2 c_2)$ and again one more term with $\ddot{\theta}_2$ into this and finally, one more term which is product of $\dot{\theta}_1, \dot{\theta}_2$. So, we can now assemble the expressions which is d/dt of this minus $d\mathcal{L}/d\theta_1$ is equal to τ_1 and $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2}$ will give you τ_2. So, now we have τ_1 equal to this expression, τ_2 is equal to this expression ok.

So, let us quickly just take a brief moment and see what are the term which contains $\dot{\theta}_1$. So now we have $\ddot{\theta}_1$ and inside the bracket we have I_1, I_2, $m_2 l_1^2$, $m_1 r_1^2$, $m_2 r_2^2$. So, you can think of this $I_1 + m_1 r_1^2$ is like the inertia with respect to the origin of the coordinate system.

Similarly $I_2 + m_2 r_2^2$ is again, so, inertia with respect to the CG and we want it with respect to the origin of the coordinate system, so this is sort of expected ok. We have done some kind of transference of the inertia from the CG to the origin, we will also have terms which are $m_2 l_1^2$, but more importantly we will have a term which is $2 m_2 l_1 r_2 c_2$.

So, basically what it is telling you is the inertia as seen by the first joint, θ_1 depends on what is happening to the second joint, what is the angle rotation angle at the second joint is that correct? Yes. Because the second link is rotating with respect to the first link and the first joint will see the inertia due to the second link ok and since it is rotating it will be different at different instant.

So, intuitively if the second link is completely stretched out then the first chain will see a much larger inertia than if it is completely folded in. So, the second joint $\ddot{\theta}_2$ is also multiplied by $I_2 + m_2 r_2^2 + m_2 l_1 r_2 c_2$. So, again the inertia seen by the second joint has sometimes which contains θ_2 and then we have these 2 additional terms which are $(2\dot{\theta}_1 + \dot{\theta}_2)$ and $\dot{\theta}_2^{\,2}$.

So, this is like centripetal term and this is like the coriolis term ok. So, we know that if you have a rigid body which is moving and then the coordinate system is also moving we have this coriolis and centripetal terms and these are those terms. And finally we have a term which is $m_2 g (l_1 c_1 + r_2 c_{12})$ and $m_1 g r_1 c_1$. So, these are the terms which correspond to the torque due to the gravity.

So, the gravity is acting at the CG, so the joint will see some torque ok. We can also see the τ_2 will be related to I_2, $m_2 r_2^2$, $m_2 l_1 r_2 c_2$. So, the first $\ddot{\theta}_1$ times some inertia will come and $\ddot{\theta}_2$ will be just $(I_2 + m_2 r_2^2)$ and again we have a coriolis centripetal term $\dot{\theta}_1^{\,2}$ and $m_2 r_2 g c_{12}$.

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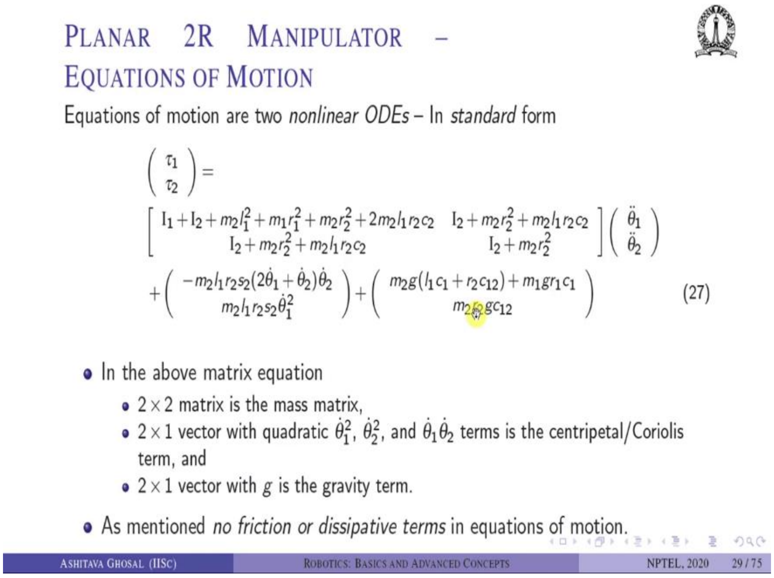

So, both of these 2 equations are non-linear ODEs ok, why? Because they have $\sin\theta_2$, $\cos\theta_2$ and so on and also $\dot{\theta}_1{}^2$ and $\dot{\theta}_1\dot{\theta}_2$. So, in a standard form we can write it as some vector $(\tau_1, \tau_2)^T$ equal to some matrix times theta $(\ddot{\theta}_1, \ddot{\theta}_2)^T$ plus the coriolis term, centripetal term and plus a gravity term.

Remember I had discussed that for any serial robot we can have $\tau = [M]\ddot{\theta} + [C(\theta, \dot{\theta})]\dot{\theta} + G(\theta)$ that is the standard form, here also we can show that we can write it in the standards form, ok. So, the matrix inside the square bracket is the 2×2 mass matrix, ok. The 2×1 vector here contains $\dot{\theta}_1{}^2$, $\dot{\theta}_2{}^2$ and $\dot{\theta}_1\dot{\theta}_2$ terms ok.

So, they are the centripetal and coriolis term and then last 2×1 vector is the gravity term ok. So, remember when we have discussed Lagrangian formulation there is no friction or dissipative terms in the equations of motion at this stage, because the Lagrangian is for conservative system ok.

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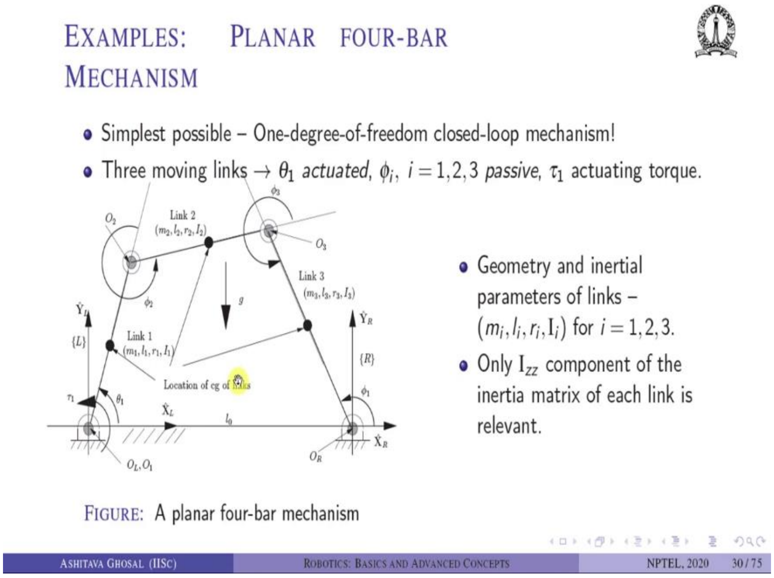

So, $= [M]\ddot{\theta} + [C(\theta, \dot{\theta})]\dot{\theta} + G(\theta)$. So, this $[M]$ is sort of like an inertia matrix, so if you think about it if you have a single rigid body τ will be $I\ddot{\theta}$. So, this is like a generalization of the inertia matrix, this is the generalization of the coriolis centripetal term and this is the gravity term for this 2R manipulator.

Let us look at one more example which is the planar four-bar mechanism ok. So, this is also one of the simplest possible one-degree-of-freedom closed-loop mechanism! you cannot find anything simpler than this. So, this has 3 moving links link 1, link 2, link 3 there is only 1 actuated joint because the four-bar mechanism has one-degree-of-freedom, so θ_1 is actuated ok.

And then ϕ_2, ϕ_3 and ϕ_1 are passive ok, so corresponding to this θ_1 we have a torque which is acting at this joint ok. So, now we need to introduce some geometry and inertial parameters of this link. So, just like the 2R we will assume that the link 1 is m_1, l_1, r_1 and I_1.

So m_i is the mass, l_i is the length of this link, r_i is the location of the CG and I_i is the z component of the inertia of this link ok. Similarly at link 2 we have m_2, l_2, r_2 and I_2 and link 3 is m_3, l_3, r_3 and I_3. So, the gravity is acting in the y direction and as in the planar 2R example only the I_{zz} component of the inertia matrix of each link is relevant.

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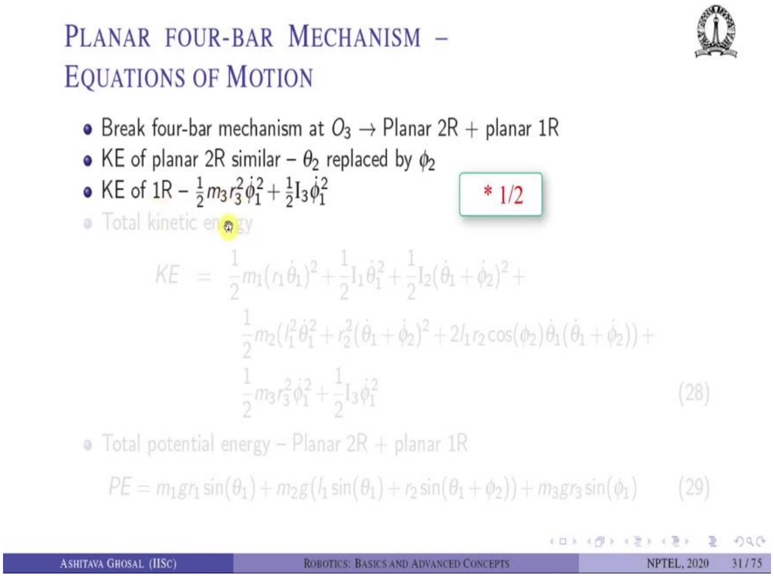

So, we can now break the four-bar mechanism at O_3 we can break it here. So, we have a Planar 2R and a Planar 1R robot ok. So, it is the same formula as what we derived for Planar 2R except now θ_2 is replaced by ϕ_2. The kinetic energy of the 1R is nothing but $\frac{1}{2}m_3 r_3 \dot{\phi_1}^2 + \frac{1}{2}I_3 \dot{\phi_1}^2$. So, the rotation of the last output link third link is given by ϕ_1 and the rate of rotation is $\dot{\phi}_1$, ok.

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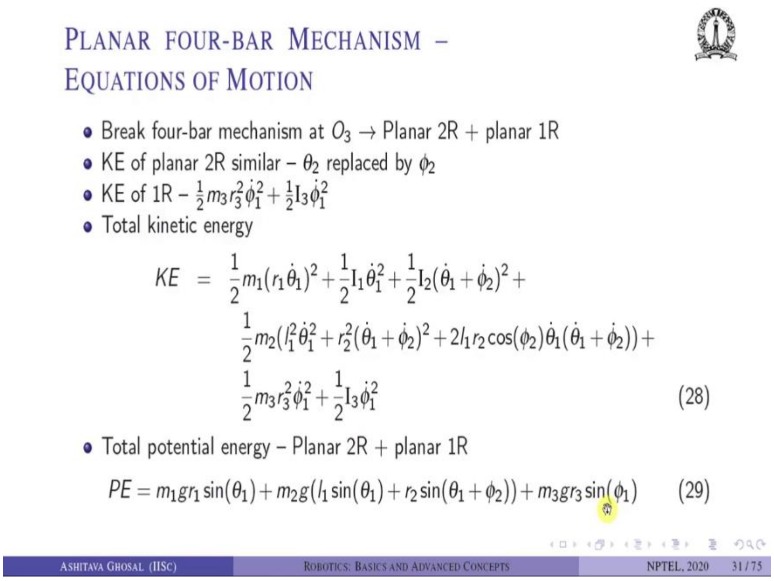

So, the total kinetic energy is nothing but the kinetic energy of the first link which is what we have derived earlier, kinetic energy of the second link these 2 terms and the kinetic energy of the third link, this is the third term for the 1R robot. So, the total potential energy is also the potential energy of the Planar 2R which was derived one term is first link, this is the second link and this is the third link $m_3 g r_3 \sin\phi_1$. It is very straight forward ok the distance from the X axis is r_3 times sin of some angle.

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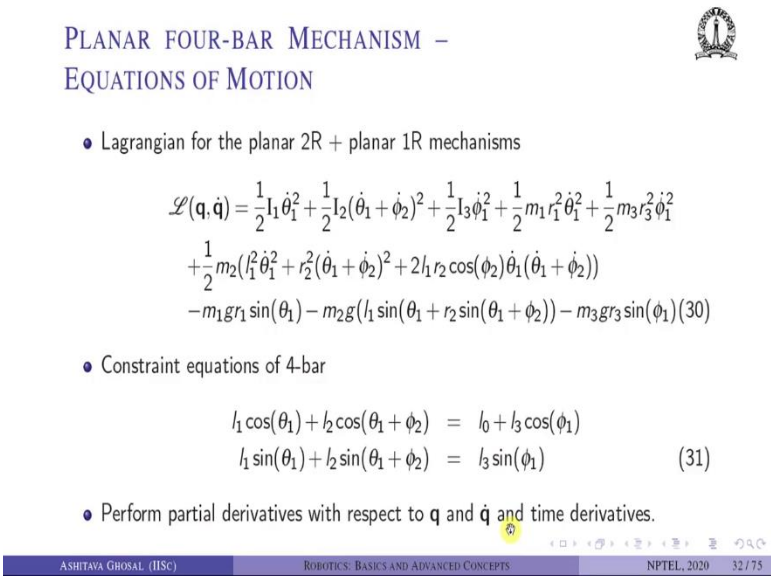

The Lagrangian for the Planar 2R and planar 1R mechanism ok can be obtained as kinetic energy minus the potential energy. So, again kinetic energy has all the 3 kinetic energies and potential energy has all the 3 potential energies, we also have a constraint for the 4 - bar when we break up at the third joint, so the vector from the left origin to the third joint the x component is $l_1\cos\theta_1 + l_2\cos(\theta_1 + \phi_2)$.

And we can reach from the other direction which is $l_0 + l_3\cos\phi_1$. We have discussed this earlier lectures also when we discussed kinematics of parallel robots, y component is $l_1\sin\theta_1 + l_2\sin(\theta_1 + \phi_2) = l_3\sin\phi_1$, there is nothing new we have discussed this earlier.

So, we reach that third joint in two ways. So, we can perform the partial derivatives with respect to $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ just to obtain the equations of motion. So, what is $\boldsymbol{q}$? Here $\boldsymbol{q}$ is θ_1, ϕ_1, ϕ_2. ϕ_3 does not sure, because we have broken up at the third joint ok.

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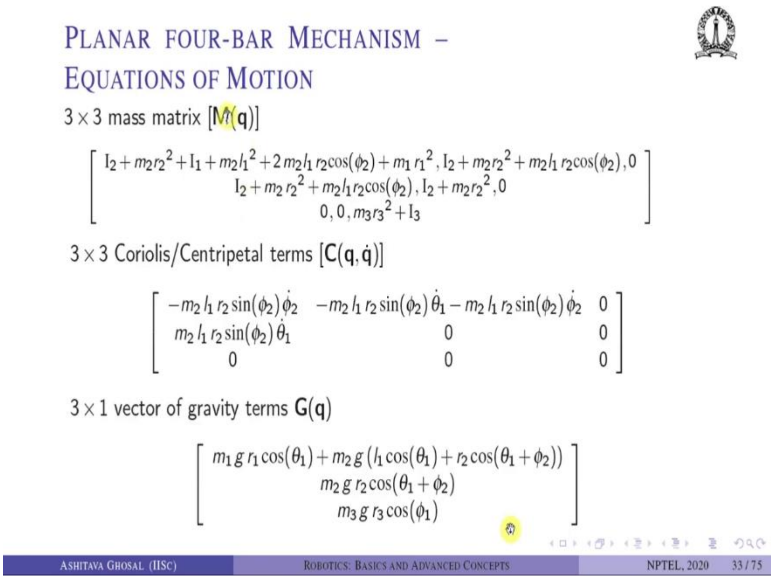

So, we will get a 3 × 3 mass matrix after we take all the partial derivatives and organize into $[\boldsymbol{M}(\boldsymbol{q})]\ddot{\boldsymbol{q}} + [\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})]\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q})$ is equal to torque. So, the mass matrix will have m_{11} will have $(I_2 + m_2 r_2^2 + I_1 + m_2 l_1^2 + 2m_2 l_1 r_2 \cos\phi_2 + m_1 r_1^2)$. So, if you see this is exactly the same as m_{11} which we obtain for the planar 2R. m_{12} is $I_2 + m_2 r_2^2 + m_2 l_1 r_2 \cos\phi_2$ and this term will be 0 ok.

Likewise m_{21} and m_{22} are given by this and m_{23} will be 0 ok, because it is kinetic energy you can show that it will be 0. And the third row of the mass matrix will be 0, 0, $m_3 r_3^2 + I_3$. We can also find this Coriolis and Centripetal terms $[\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})]$ this matrix will

$$\begin{pmatrix} -m_2 l_1 r_2 \sin\phi_2\,\dot{\phi}_2 & -m_2 l_1 r_2 \sin\phi_2\,\dot{\theta}_1 - m_2 l_1 r_2 \sin\phi_2\,\dot{\phi}_2 & 0 \\ m_2 l_1 r_2 \sin\phi_2\,\dot{\theta}_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Finally we can also compute the gravity vector which is 3 × 1 vector. So, it will contain $m_1 g r_1 \cos\theta_1$ some $m_2 g(l_1 \cos\theta_1 + r_2 \cos(\theta_1 + \phi_2))$ and so on. So, it is exactly very similar to the planar 2R. We have a third component of the gravity vector which is coming from the planar 1R. Likewise there will be a third row and a third column in the mass matrix and likewise there will be a third row and a third column in the Coriolis and Centripetal term ok.

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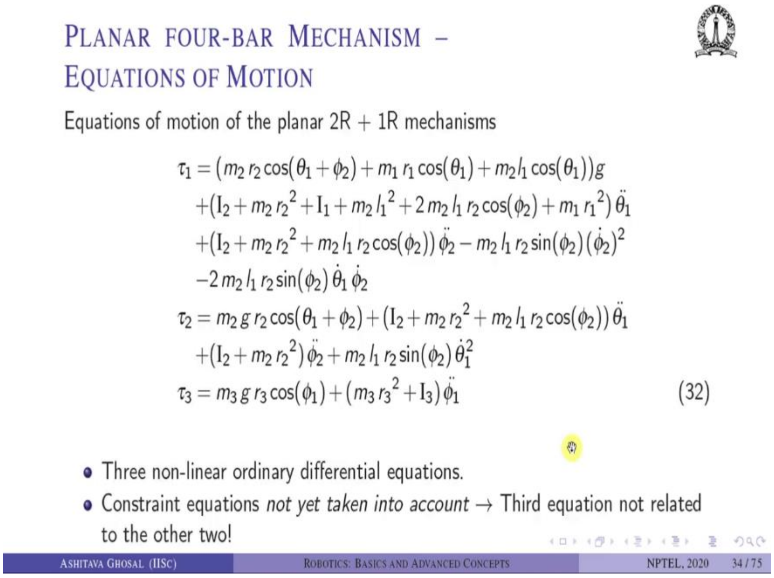

So, we can rewrite the equations of motion as τ_1 which is equal to some matrix which is functions of g then $\ddot{\theta}_1$, then $\ddot{\phi}_2$ and then this coriolis and centripetal time, ok. So, τ_2 can be written as the gravity term, the inertia term, which is $M\ddot{\theta}$ and then the coriolis and centripetal term and τ_3 can also be written in $m_3 g r_3 \cos\phi_1 + (m_3 r_3^2 + I_3)\ddot{\phi}_1$ this is very straightforward because it is a single 1 link robot.

So, these are three non-linear ordinary differential equations ok; however, at this point the constraint equations have not yet been taken into account ok. So, that is the reason why this third equation does not contain θ_1 and ϕ_2 it looks like the third equation is completely independent. But we know that in a four-bar mechanism the all those angles are related to θ_1, they are coupled.

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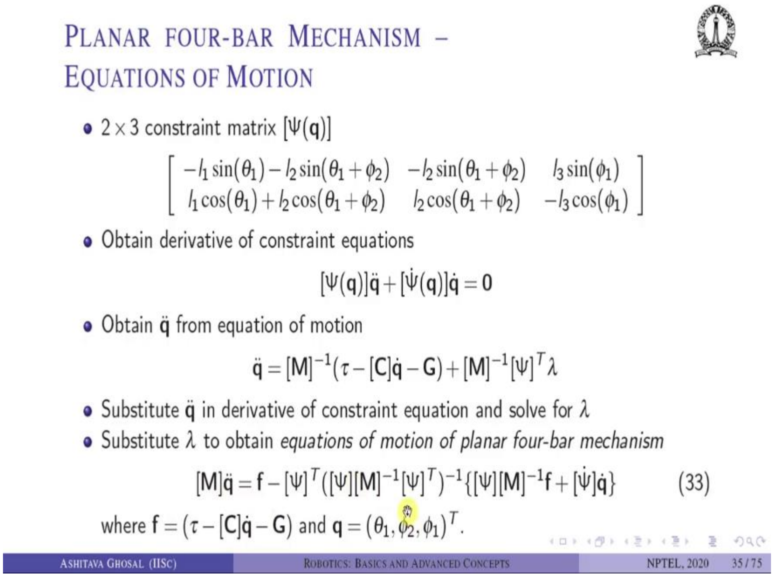

So, we can obtain the constraint matrix which is remember it is $[K]$ and $[K^*]$ and the concatenated. So, I had derived for the four-bar mechanism, what is $[K]$ and $[K^*]$. If you go back and see $[K]$ is this, $[K^*]$ is this ok. So, we put side by side and we get a 2×3 constraint matrix $[\Psi(\boldsymbol{q})]$, we can take the derivative of this constraint equation and we will get $[\Psi(\boldsymbol{q})]\ddot{\boldsymbol{q}} + [\dot{\Psi}(\boldsymbol{q})]\dot{\boldsymbol{q}} = 0$.

So, this is the derivative of the loop closure constraint equations, we organized as $[\Psi(\boldsymbol{q})]\dot{\boldsymbol{q}} = 0$. So, to obtain $\ddot{\boldsymbol{q}}$ from the equation of motion we can write $\ddot{\boldsymbol{q}} = [\boldsymbol{M}]^{-1}(\tau - [\boldsymbol{C}]\dot{\boldsymbol{q}} - \boldsymbol{G}) + [\boldsymbol{M}]^{-1}[\Psi]^T\lambda$, exactly the same what we have discussed when I derive the Lagrangian formulation with constraint.

So, substitute $\ddot{\boldsymbol{q}}$ in the derivative of the constraint equation here ok and solve for λ and then substitute that λ to obtain the equation of motion for the planar 4 bar which is $[\boldsymbol{M}]\ddot{\boldsymbol{q}} = \boldsymbol{f} - [\Psi]^T([\Psi][\boldsymbol{M}]^{-1}[\Psi]^T)^{-1}\{[\Psi][\boldsymbol{M}]^{-1}\boldsymbol{f} + [\dot{\Psi}]\dot{\boldsymbol{q}}\}$

So, $[\Psi]$ here represents the constraint matrix, 2×3 constraint matrix and $\boldsymbol{f} = \tau - [\boldsymbol{C}]\dot{\boldsymbol{q}} - \boldsymbol{G}$. And what is $\boldsymbol{q}$ here? $\boldsymbol{q}$ is the configuration space which is $(\theta_1, \phi_1, \phi_2)^T$, ϕ_3 does not show up because we have broken at third joint ok. So, one last thing before we end here remember I had said in a four-bar mechanism we will have only one actuation which is τ_1.

So, what happens to these equations of motion? Basically τ_2 and τ_3 will be 0. So, we have one equation which is τ_1 which is the given input torque and this will be 0 is equal to this and τ_3 will be 0 is equal to this. So, these are three differential equations. And after eliminating λ the Lagrange multiplier we will have 3 coupled differential equations which are of the form $[\boldsymbol{M}]\ddot{\boldsymbol{q}}$ is this and in this $\boldsymbol{f}$ only τ_1 is non zero, τ_2 and τ_3 are 0.

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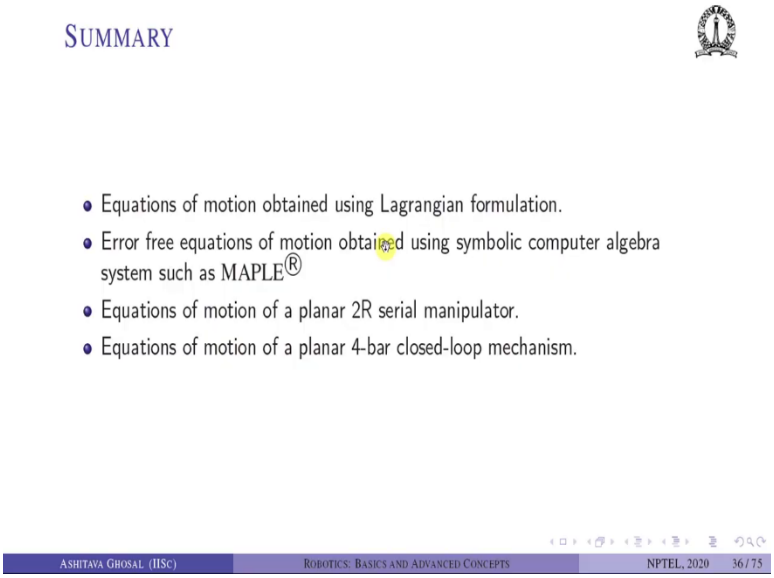

So, in summary the equation of motion can be obtained using the Lagrangian formulation, ok. So, this is very mechanical you find the kinetic energy, you find the potential energy you subtract KE minus PE and then you take this set of derivatives partial derivatives and some time derivatives ok. So, we can obtain error free equations of motion using symbolic computer algebra system such as MAPLE. So, I do not have to do these partial derivatives by hand ok.

So, I showed you that the equation of motion of a planar 2R serial manipulator can be easily obtained. The equation of motion of a planar four-bar mechanism closed loop mechanism can also be obtained. Of course, if you want spatial and multi-degree of freedom serial or parallel manipulator example it will take a lot of time ok. But the basic ideas are there in these 2 examples which I have chosen.

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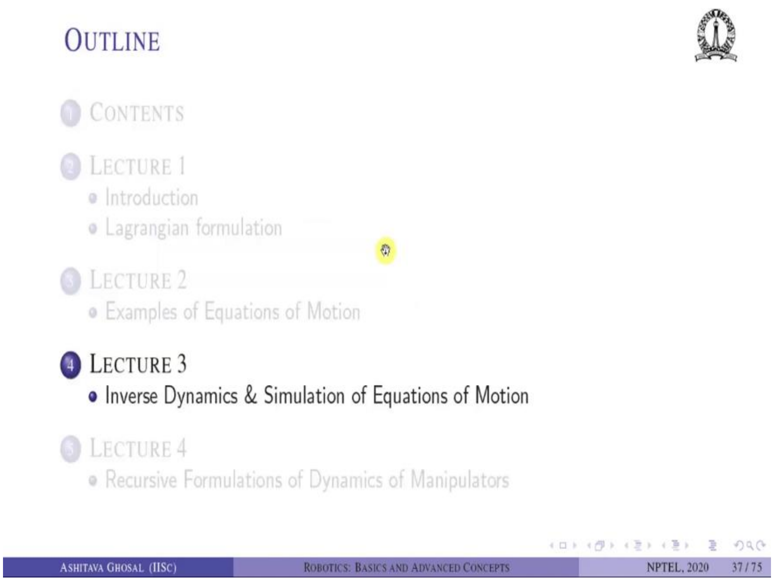

So, in the next lecture, we will look at what to do with these equations of motion. So, we will look at something called the Inverse Dynamics and something called the Simulations of Equations of Motion.