Robotics: Basics and Selected Advanced Concepts
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Lecture - 32
Control of a single link

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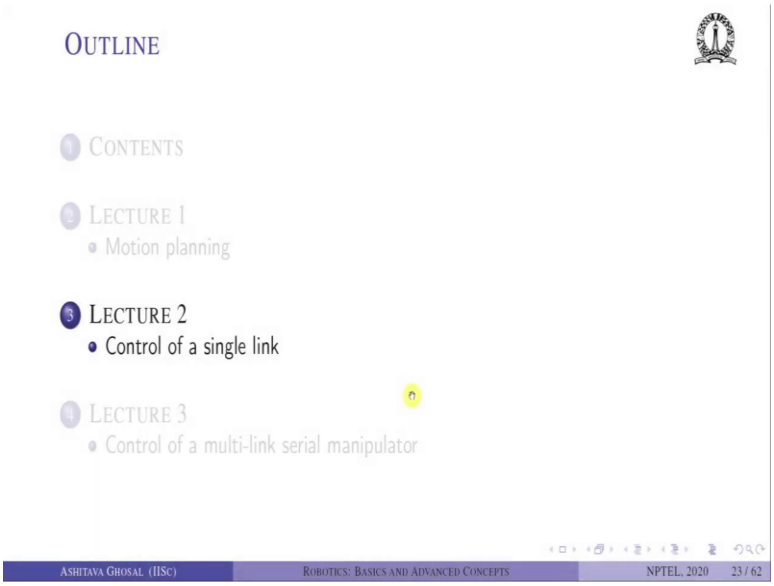

Welcome to these NPTEL lectures on Robotics: Basic and Advance Concepts. In this lecture, we will look at Control of a single link.

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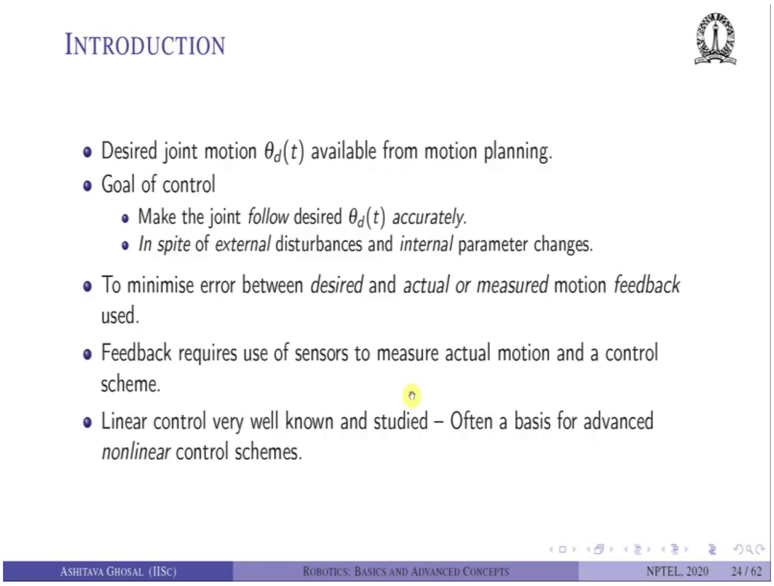

So, what do we have at this point? We know what is the desired trajectory of the single link. We know what is $\theta_d(t)$ and this has been generated in a smooth and continuous manner using a C^2 trajectory and using cubic polynomials. The goal of control is to make the joint follow the desired trajectory accurately.

So, we will see what we mean by follow an accurately little later. More importantly, it should follow the desired trajectory in spite of external disturbances and internal parameter changes ok, and again we will see what is external disturbances and internal parameter change. So, the goal of control is to minimise the error between the desired and actual or measured motion ok. And this requires that we use some feedback.

So, we need to measure the actual motion and then, by using some feedback and then this feedback requires use of sensors to measure the actual motion ok. And we also need to use a control scheme. So, we will look at linear control scheme in this lecture, because it is very well known it is very well studied and more importantly it is also a basis for advanced non-linear control schemes.

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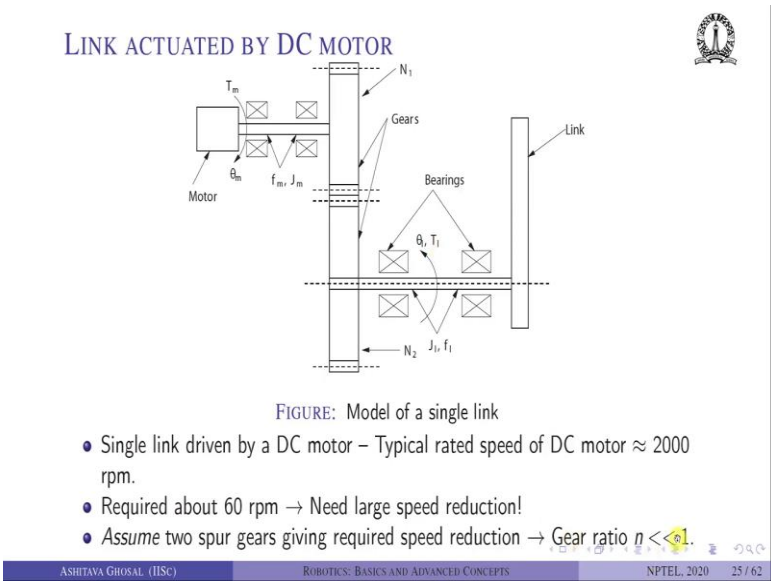

So, let us continue. So, we have a link which is link shown here, it is driven by a DC motor ok. The typical rated speed of a DC motor is like maybe 2000 rpm however, 2000 rpm is too fast. Typical speeds which are required is like 60 rpm ok, or let us say one rotation per second. So, one rotation is like 2π radians per second ok. And if you have a link which is about 1 meter long so, 2π is about 6 let us say so we will get 6 meters per second, which is still very fast.

So, we need something like not more than 60 rpm and hence, we need a large speed reduction. We will assume that this motor is connected to the link by means through a gear box, which achieves this large speed reduction. So, we have a gear ratio which will be of course, much much less than 1, but in this figure I am showing link two spur gears ok, we are not doing gear design. However, we know that if you want a large speed reduction maybe you need multistage gear box.

But, for the purpose of this course or this lecture, we will assume that it is the speed is reduced using one set of spur gears. So, N_1 is one of the gears which is connected to the motor and N_2 is the number of teeth in the output gear, which is connected to the link. Now, when we connect a motor to the link, we also need to assume some friction and the inertia of the motor shaft.

So, this is given by f_m and J_m. So, f_m is the friction in the motor shaft likewise, we will assume that the inertia of the output is J_l and the friction is f_l in the loads shaft. So, we assume that the motor is providing a torque T_m and it is the variable, which is a rotation of the motor is denoted by θ_m.

Likewise, the rotation of the link is denoted by θ_l, but we can also have some external torque, which is acting on this link, which is given by T_l denoted by T_l. So, these are all the parameters that we need to analyze the motion of a single link actuated by a DC motor.

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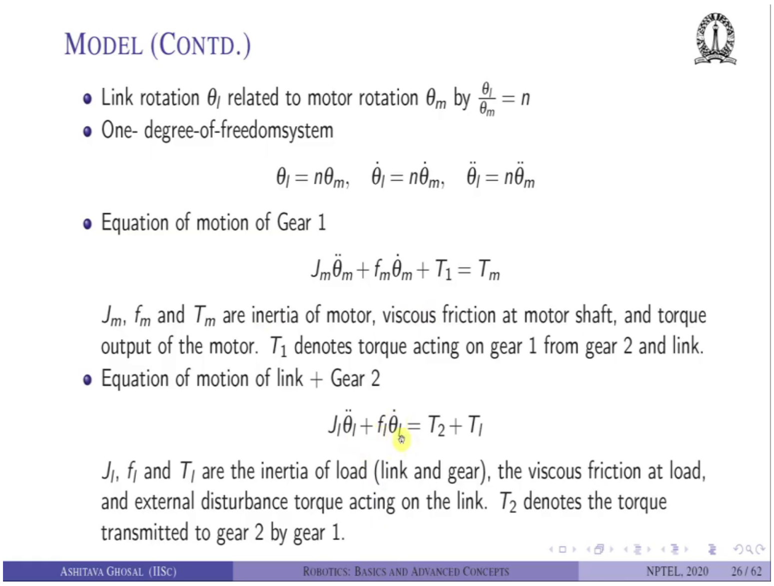

So, the link rotation θ_l is related to the motor rotation θ_m by $\frac{\theta_l}{\theta_m} = n$. So, this is the speed reduction this is what is happening at the gear box. So, n is much much less than 1. We have a one degree of freedom system so, basically $\theta_l = n\theta_m$, $\dot{\theta}_l = n\dot{\theta}_m$, $\ddot{\theta}_l = n\ddot{\theta}_m$.

So, there is only one variable, and we will for the rest of this lecture we will assume θ_m is the independent variable. So, the equation of motion of gear 1, which is basically the motor plus the gear 1 can be written as $J_m\ddot{\theta}_m + f_m\dot{\theta}_m + T_1 = T_m$.

This is straight forward from free body diagram. The equation of link plus gear 2 can be written as $J_l\ddot{\theta}_l + f_l\dot{\theta}_l = T_2 + T_l$. So, this is the T_2 is the torque transmitted to gear 2 by gear 1.

Again this is straight forward free body diagram very simple equations of motion linear.

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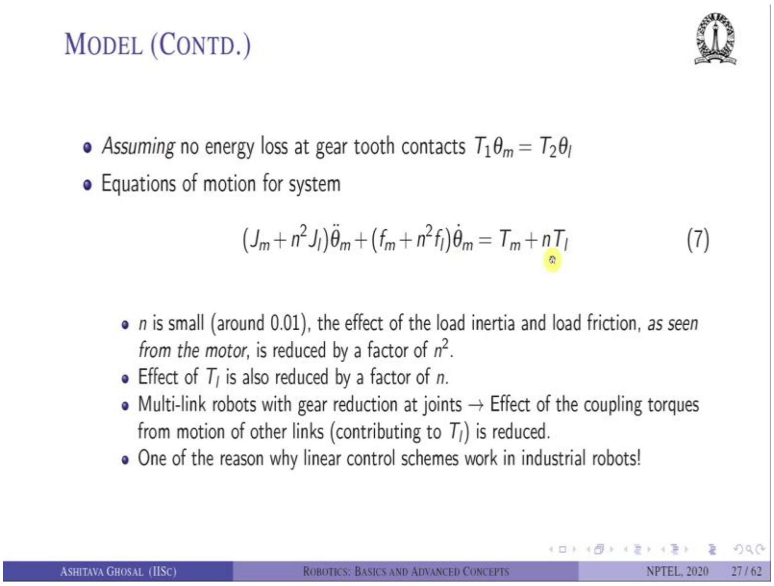

If we assume that there is no energy loss at the gear tooth's then, $T_1\theta_m = T_2\theta_l$. So, the work done by T_1 which is $T_1\theta_m = T_2\theta_l$. The equation of motion for the system which means what? I would like to write everything in terms of θ_m and its derivatives, the input torque from the motor which is T_m and the external torque, which is acting on the link.

So, if you play around with all those equations, if you eliminate θ_l and $\dot{\theta}_l$ and $\ddot{\theta}_l$ and T_1 and T_2 you will end up with this equation $(J_m + n^2 J_l)\ddot{\theta}_m + (f_m + n^2 f_l)\dot{\theta}_m = T_m + nT_l$.

So, what are some of the observations? One is as I said n is small. So, let us say $n = 0.01$. So, which means that n^2 is very very small it is 10^{-4}. So, what is happening? Because, of the gear box the effect of the load inertia and the load friction, as seen from the motor, is reduced by a factor of n^2.

So, the motor basically does not see the load inertia and the friction on the load side. The effect of T_l, this external torque which is acting on the link is also reduced by a factor n. So, multi link robots with gear reduction at the joints in those the effect of coupling torques from one link to the other due to the motion of one link there is a coupling torque, which acts on the previous link ok.

We can sort of think of that as T_l and that is reduced ok. So, this is one of the reason why linear control schemes work in industrial robots because, this is the linear equation ok. Only problem case is this nT_l ok, where T_l is the coupling torque of some other link acting on the link, that which you are trying to control, but the effect is reduced by this factor n ok, and if n is small ok, our life is made easier we can use linear control.

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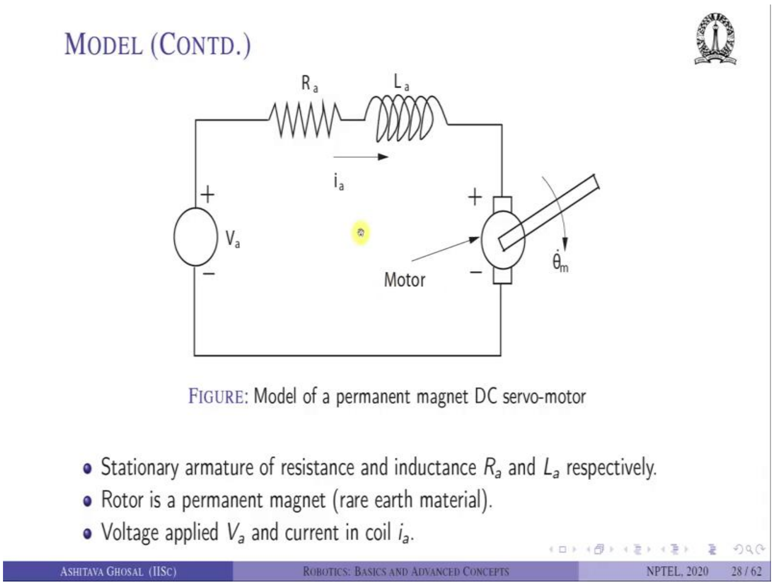

Let us continue, we also need a model of the motor. So, in a motor basically we apply some voltage V_a then, there is an output shaft which is rotating by $\dot{\theta}_m$ the typical DC servo-motors contain a stationary coil ok which is the stator and the rotor is a permanent magnet ok. So, through that coil it has a resistance R_a and inductance L_a, the current flows i_a ok, and it goes to the motor and then because of Faraday's law the rotor will start rotating.

So, typically this stationary armature ok, of resistance and inductance R_a and L_a we have to include. The rotor is a permanent magnet ok, nowadays we get very good permanent magnet motors with very rare earth materials and the voltage V_a is applied and the coil carries a current i_a ok. So, this has all the elements which go into modeling of a DC servo-motor.

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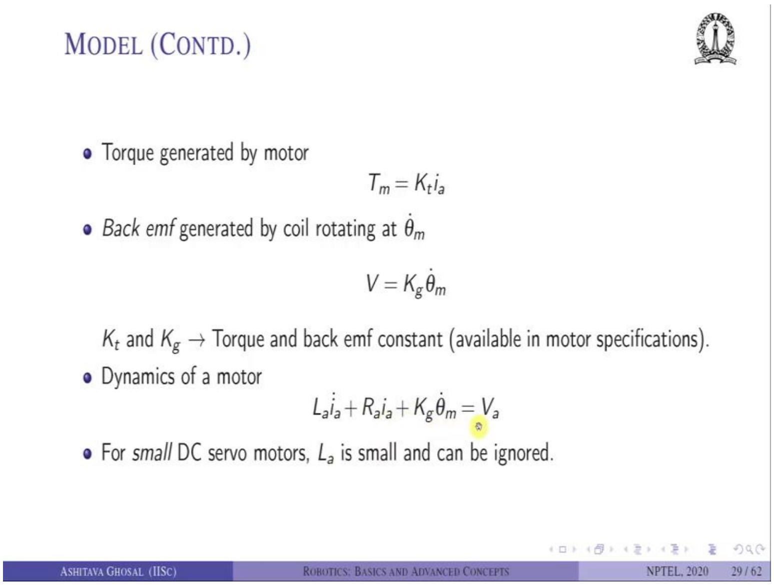

The torque which is generated by the motor can be written in terms of the current which is flowing through the coil stationary coil, and $T_m = K_t i_a$, K_t is called the torque constant, when the rotor starts rotating a back emf is generated ok.

By the motor at in the coil and that is given by $V = K_g \dot{\theta}_m$. So, this is called as the back emf constant K_g, and if we go and buy any motor they will tell you what is the torque constant and what is the back emf constant. The dynamics of a motor can be written in terms of a first order differential equation, which is $L_a \dot{i_a} + R_a i_a + K_g \dot{\theta}_m = V_a$.

So, this is basically the drop of voltage in the inductance which is $L_a \dot{i_a}$ plus the drop across the resistor and the drop due to the back emf must be equal to the applied voltage ok. This is some standard law in motors electric circuits. For small DC servo motor, L_a is small and hence can be ignored. So, what do; what are we left with? $R_a i_a + K_g \dot{\theta}_m = V_a$.

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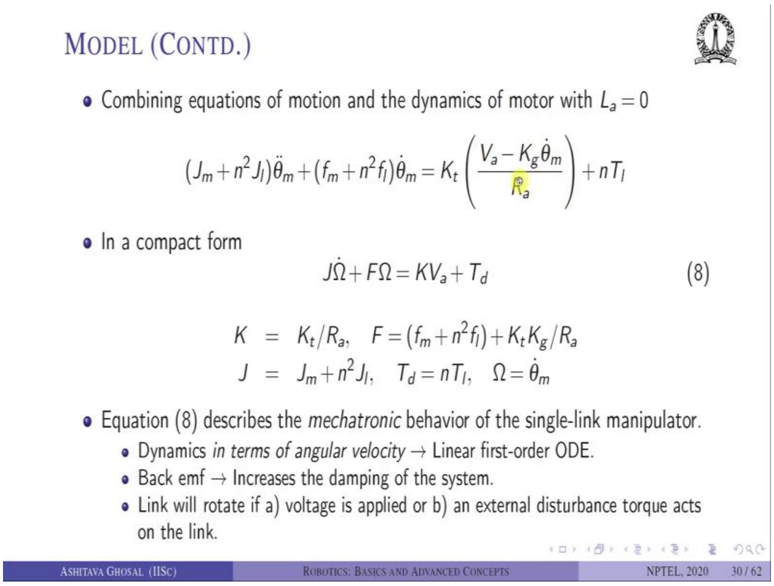

So, now we can solve for i_a from the previous equation which is $\frac{V_a - K_g\dot{\theta}_m}{R_a}$, if you multiply by K_t we will get the torque and hence $T_m + nT_l$ is given by that previous left hand side ok, we are not changing anything in the left hand side. So, in a compact form we can write this second order differential equation as a first order differential equation, which is $J\dot{\Omega} + F\Omega = KV_a + T_d$.

So, you can see $K = K_t/R_a$, $F = (f_m + n^2 f_l) + K_t K_g/R_a$. So, the resistance in the armature or resistance in the coil is like a friction term ok, it adds to the friction. The inertia is $J = J_m + n^2 J_l$, and the disturbance torque or the torque which is acting externally on the link is $T_d = nT_L$ and $\Omega = \dot{\theta}_m$ so, Ω is a speed.

This equation describes the mechatronic behavior of the single-link manipulator ok. What is the word mechatronic means? It defines the mechanical part which is the inertia and the friction and so on. And it also includes the electrical part, which is the voltage applied and the resistance in the coil and so on ok.

So, the dynamics of this system is basically in terms of angular velocity it is a linear first-order ODE, you can see J is constant, F is constant, K is constant ok. So, it is a linear first-order ODE. The back emf increases the damping of the system. So, $-K_t K_g\dot{\theta}_m/R_a$ causes more friction. Now, from this equation we can see that this link will rotate means what it will achieve some Ω ok, if either you apply a voltage or you apply a disturbance torque.

So, you can think of this as like a fan blade of your ceiling fan, if you apply the voltage the blade will rotate or if there is no current flowing through the fan, but if you hit it with some stick the again the blade will rotate. So, that is what we mean by there are two possible inputs one is the voltage and one is the external disturbance.

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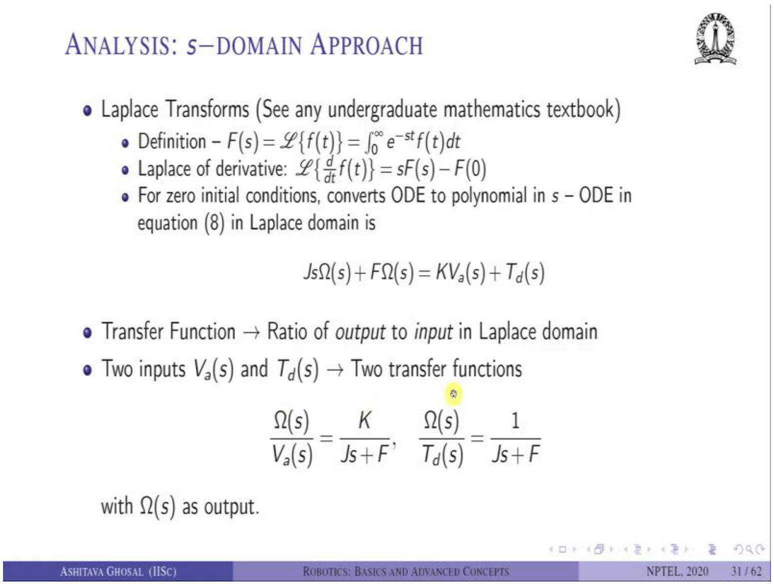

So, how do we analyze these systems? The simplest is to use Laplace transform and this is available with any undergraduate mathematics textbook. So, I am just giving you some of the very basic notions in Laplace transform first thing is if you are given a function in time $f(t)$ the Laplace transform of $f(t)$ is $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$, and this is given by $F(s)$. The derivative of $f(t)$ and if you want to find the Laplace of the derivative, then this is given by $\mathcal{L}\left\{\frac{d}{dt} f(t)\right\} = sF(s) - F(0)$.

So, this is the initial at $t = 0$, for 0 initial conditions, we can convert an ODE to a polynomial in s. So, for example, $\dot{\Omega}$ can be written as $s\Omega(s)$, so $Js\Omega(s) + F\Omega(s) = KV_a(s) + T_d(s)$, remember we have assumed initial conditions as 0. So, in Laplace the differential equation $J\dot{\Omega} + F\Omega = KV_a + T_d$ is written as $Js\Omega(s) + F\Omega(s) = KV_a(s) + T_d(s)$.

So, this is straight forward Laplace transform now, what do we do with this expression, which is obtained after taking the Laplace transform. So, we can define something called as a transfer function, and what is a transfer function it is the ratio of the output to input in

Laplace domain.

So, as I said we can have two inputs $V_a(s)$ and $T_d(s)$ ok, the applied voltage or the external disturbance and the output is clearly the speed $\Omega(s)$. So, we can have two transfer functions one is $\frac{\Omega(s)}{V_a(s)} = \frac{K}{Js+F}$ or we can also write $\frac{\Omega(s)}{T_d(s)} = \frac{1}{Js+F}$.

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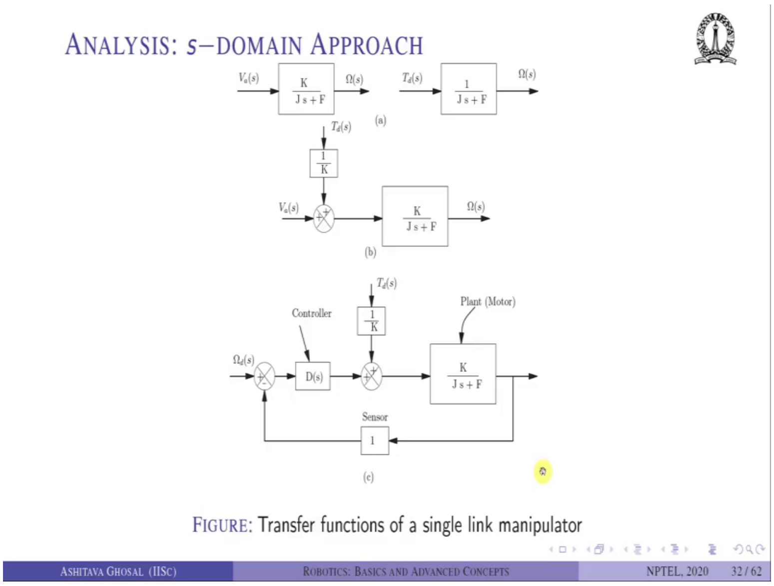

And we can show these transfer functions using block diagram ok. So, this is a block which says $\frac{K}{Js+F}$, input is $V_a(s)$ output is $\Omega(s)$, similarly we can have $T_d(s)$ as the input and the output as $\Omega(s)$ and the block is $\frac{1}{Js+F}$. We can combine these two because this is a linear system. So, we can have input as voltage $V_a(s)$ and $T_d(s)$ also has one more input and then can sum these two and get $\Omega(s)$ and the block is $\frac{K}{Js+F}$.

So, these are basically nothing but the transfer functions. So, these three (a) and (b) so, (a) means one block here one block here, and this block here these are called open loop transfer functions, ok. So, as opposed to something called as the closed loop transfer function in which case what happens is you take this $\Omega(s)$ you measure using a sensor and you feed it back.

So, there is a desired $\Omega_d(s)$ there is a measured $\Omega(s)$. So, and you subtract so this plus and minus sign means you subtract, and you find the difference between $\Omega_d(s)$ and $\Omega(s)$ and you pass it through a controller and then you feed it back into this. So, something like this, where there is a loop where there is a feedback this is called as a closed loop system, ok.

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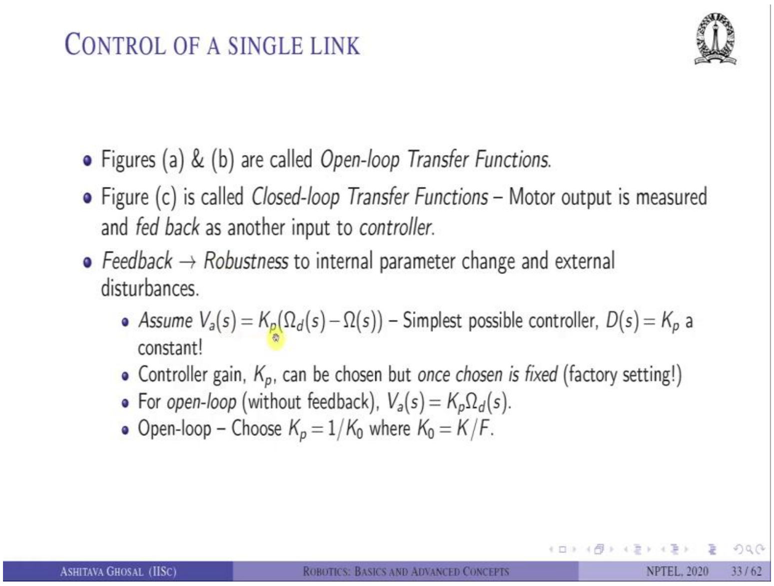

So, as I said figures (a) and (b) are called open-loop transfer function or open loop systems. Figure (c) is called closed-loop can be used to obtain the closed-loop transfer function ok, here the motor output is measured and feedback as another input to controller, ok. Feedback - the robustness to internal parameter changes and external disturbance, we will recall I said we need to use feedback.

So, what happens in feedback, we get something called robustness and that is shown by some simple argument next. So, let us assume that this voltage which you are applying is given by $K_p\big(\Omega_d(s) - \Omega(s)\big)$. So, this block here which I call the controller $D(s)$ the simplest possible thing in this block is a constant K_p. So, at this place we have $\big(\Omega_d(s) - \Omega(s)\big)$, why? because, the sensor has transfer function unity it is not a very serious assumption, ok.

We can see later that this can be easily handled but, in this block we put one single constant which is K_p. So, the output of this block is $K_p\big(\Omega_d(s) - \Omega(s)\big)$. So, voltage applied is $K_p\big(\Omega_d(s) - \Omega(s)\big)$ or the transfer function of this $D(s) = K_p$ a constant. Now, let us assume that this controller gain K_p can be chosen to whatever you want but, once chosen it is fixed. It is like some kind of a factory setting ok, you can do whatever trial and error whatever experiments you want, but once you have done all the experiments you freeze it.

So, let us go back to this previous thing also, if you do not have feedback so, then there is some voltage which is coming in with feedback there is this $\Omega_d(s) - \Omega(s)$ part. So, with open-loop I can choose $K_p = \frac{1}{K_0}$, where, $K_0 = \frac{K}{F}$, let us make a choice of that.

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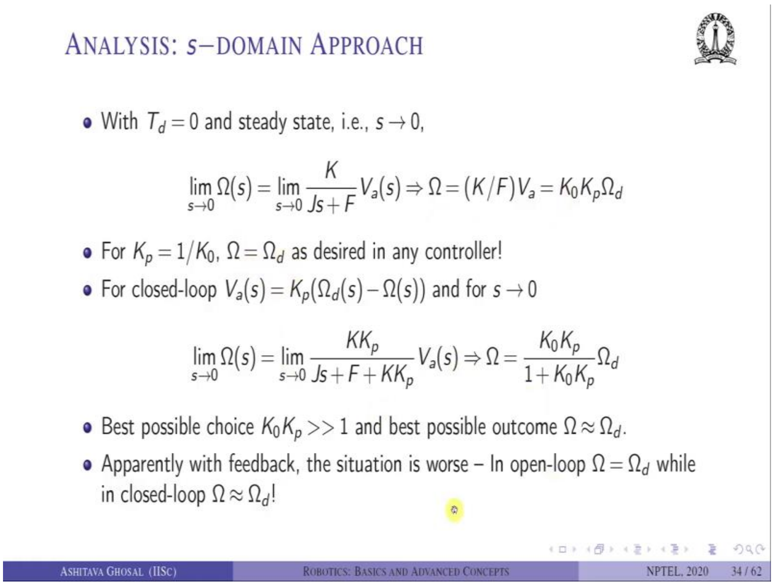

So, then with $T_d = 0$ ok, what is the steady state steady state is when s tends to 0. So, what is the output $\Omega(s)$ as s tends to 0 we have to find the $\lim_{s \to 0} \Omega(s) = \lim_{s \to 0} \frac{K}{Js+F} V_a(s) \Rightarrow \Omega = \frac{K}{F} V_a = K_0 K_p \Omega_d$. The voltage applied, that is nothing but $K_p \Omega_d(s)$ there is no feedback coming back ok.

So, we have $\Omega = \left(\frac{K}{F}\right) V_a = K_0 K_p \Omega_d$, but we have chosen $K_p = 1/K_0$, see this is the choice we have made $K_P = 1/K_0$. So, if you make that choice then you can see that the output omega is exactly equal to Ω_d.

So, this is what we want, we want the output to be the same as what we the desired omega. In case of closed loop we have chosen the output voltage after the controller as $K_p\big(\Omega_d(s) - \Omega(s)\big)$, previously it was $V_a(s) = K_p \Omega_d(s)$. So, now we can find the output as s tends to 0, s tends to 0 means, steady state t tends to infinity we can compute limit s tends to 0 the transfer function with the feedback can be obtained to be $\lim_{s \to 0} \frac{KK_p}{Js+F+KK_p} V_a(s)$.

So, $\Omega = \frac{K_0 K_p}{1+K_0 K_p} \Omega_d$. So, in steady state the output Ω is related to the desired Ω_d by this term

$K_0K_p/(1 + K_0K_p)$. So, what is the best thing that we can do? We choose K_0K_p much greater than 1.

So, let us say we choose K_0K_p as 100. So, what do we have we have 100 divided by 101. So, Ω is approximately equal to Ω_d. So, it looks like with feedback the situation is worse. So, without feedback Ω what was exactly equal to Ω_d whereas, with feedback Ω is approximately equal to Ω_d, so, why should we do feedback?

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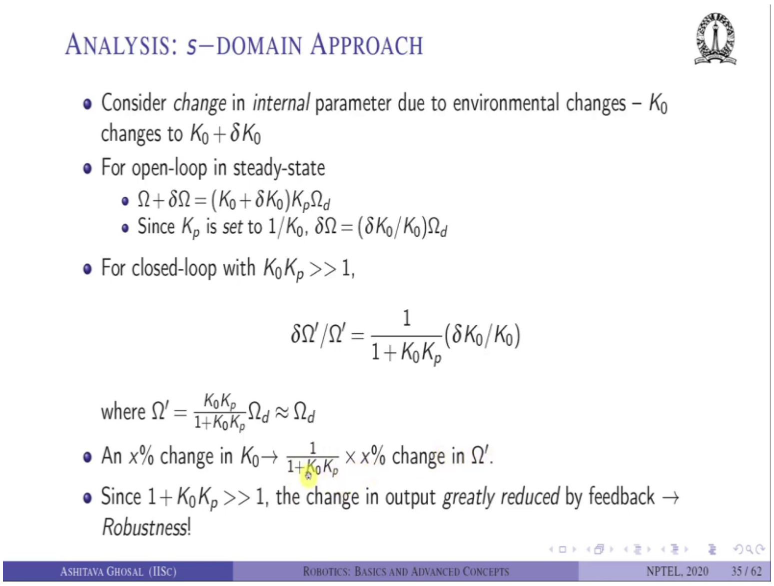

So, now let us look at this whole idea of robustness, which is let us consider the change in internal parameter due to some environmental change ok. So, this motor was designed and we chose and you did all the experiments in Bangalore in a cold place, and then we took it to a desert which is a much hotter place ok.

So, then this internal parameter which is K_0 what is K_0? It contains the resistance it contains the friction and various other things changes from K_0 to $(K_0 + \delta K_0)$. So, for open loop in steady state we will have a change in Ω which is $(\Omega + \delta\Omega)$, and we can show that this leads to $(K_0 + \delta K_0)K_p\Omega_d$ this is what we will get.

Since K_p is set to $1/K_0$ this change in output $\delta\Omega = (\delta K_0/K_0)\Omega_d$. So, if K_0 changes by 5 percent the output will also change by 5 percent however, when K_0K_p is greater than 1 in closed loop system we have chosen K_0K_p much much greater than 1.

So, then you can show that $\frac{\delta\Omega'}{\Omega'} = \frac{1}{1+K_0K_p}\left(\frac{\delta K_0}{K_0}\right)$, where $\Omega' = \frac{K_0K_p}{1+K_0K_p}\Omega_d \approx \Omega_d$. So, hence an x percent change in K_0 leads to $\left(\frac{1}{1+K_0K_p}x\right)$ percent change in Ω', remember K_0K_p was much greater than 1.

So, the example which I gave you was K_0K_p equals 100. So, you have 1 by 101 into x percent. So, what is happening? The change in output is greatly reduced by feedback. So, this is what we mean by robustness. So, I am going to follow the trajectory; desired trajectory of Ω_d in spite of changes in the internal parameters.

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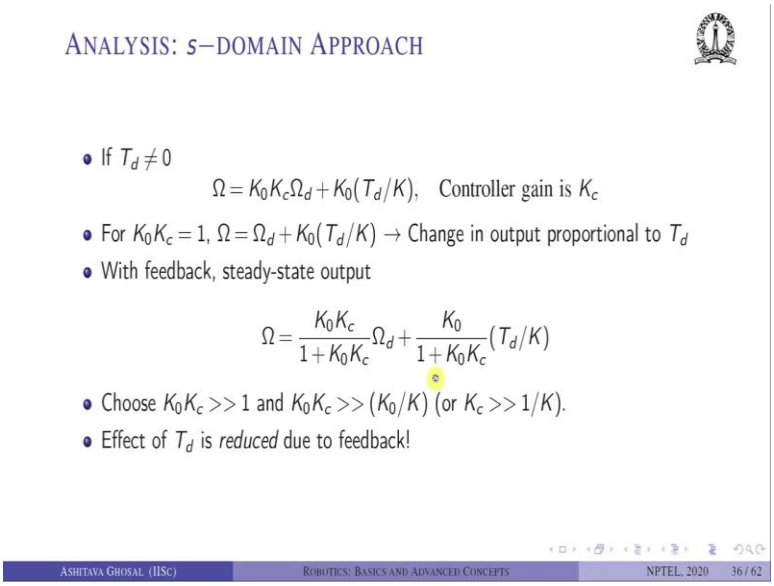

We also said that, it should be able to follow the desired trajectory in spite of changes in the external disturbance. So, if $T_d \neq 0$, T_d is the term which represents the external disturbance then, we have $\Omega = K_0K_c\Omega_d + K_0(T_d/K)$, the controller gain for moment we call it as K_c.

So, for $K_0K_c = 1$ we will have $\Omega = \Omega_d + K_0(T_d/K)$. So, this is for open loop. So, in open look remember $K_0 = 1/K_p$ or $K_p = 1/K_0$. So, if you do the same thing here what happens is, the change in output is proportional to T_d so whatever is happening to T_d, you will see some Ω output.

With feedback the output is given by $\Omega = \frac{K_0K_c}{1+K_0K_c}\Omega_d + \frac{K_0}{1+K_0K_c}(T_d/K)$. So, if you choose

$K_0 K_c$ greater than 1 and $K_0 K_c$ greater than K_0/K or basically K_c much much greater than $1/K$ then the effect of T_d is reduced ok.

So, basically again this is much smaller. So, T_d will be multiplied by a small number. So, what have we showed we are showed that if we use feedback and if you choose proper controller gains, which is the gain or the block diagram $D(s)$ whatever goes into $D(s)$. We can reduce the effect of internal parameter change which is K_0 and also the effect of T_dd, ok.

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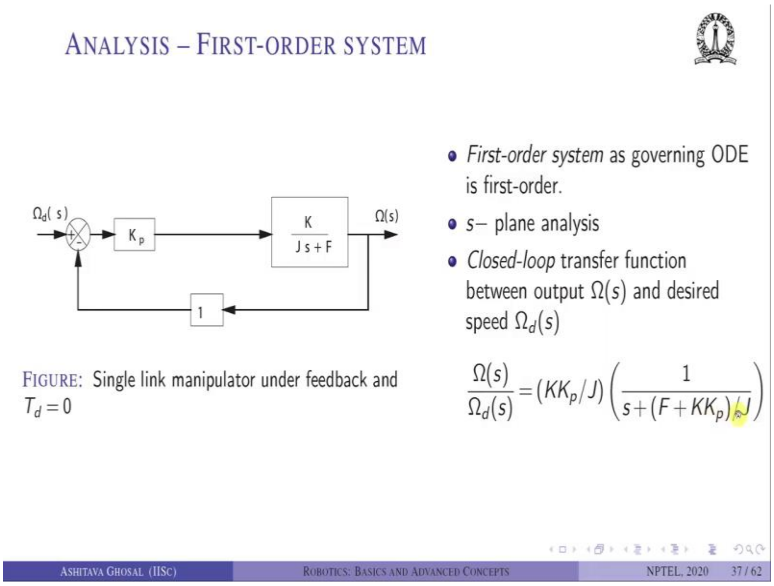

Let us continue with the analysis of the first order system. So, for the moment we will assume that there is no T_d ok. So, what we have? We have a plant which is the model of the motor which is $K/(Js + F)$ there is a voltage which is the input and thus speed of the link which is the output.

And we are going to measure the speed $\Omega(s)$ feed it back using a sensor. So, at this place we have $\Omega_d(s) - \Omega(s)$ and at this place we have the difference between $\big(\Omega_d(s) - \Omega(s)\big)$, which we multiply by K_p and that becomes the voltage which you apply to the plant ok.

So, the closed loop transfer function between $\Omega(s)$ and $\Omega_d(s)$ can be derived and you can show that $\frac{\Omega(s)}{\Omega_d(s)} = \left(\frac{KK_p}{J}\right)\left(\frac{1}{s+(F+KK_p)/J}\right)$. So, this is the first order system. So, those of you who know Laplace transforms you can see that there is only one s plus something term below, ok.

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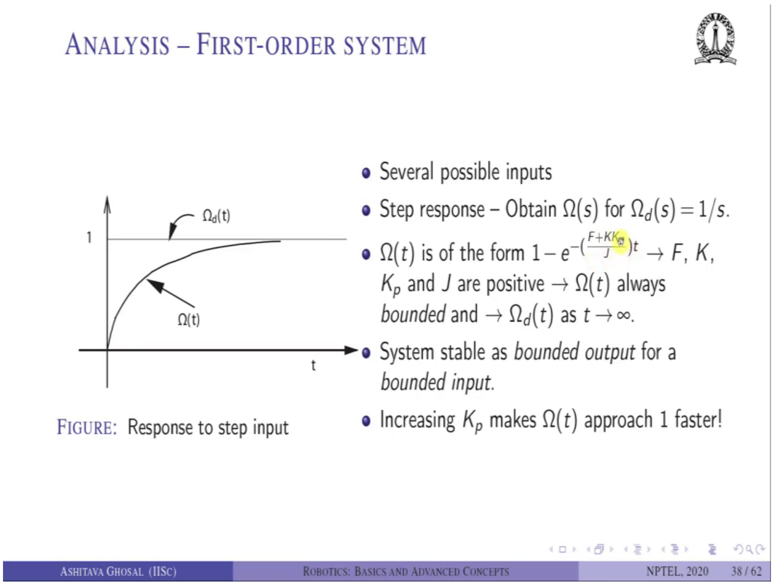

So, $\Omega(s)$ output by $\Omega_d(s)$ input is given by some constant times 1 over s plus something like a this whole thing is a constant. So, there are several possible inputs which you can give, so one of the standard inputs which control theory people use is what is called as the step input.

So, basically at t less than 0 it is 0, at t equal to 0 and t greater than 0 it is 1 unit ok. So, this is the unit step which is applied at t equal 0. So, what do we want to find out, if you apply such a step input what is the output ok. So, the plot of the output will look like this $\Omega(t)$ and we can find out exactly what is the equation for this curve by taking inverse Laplace transform.

So, $\Omega(t) = 1 - e^{-\left(\frac{F+KK_p}{J}\right)t}$. So, F, K, K_p and J are all positive. So, it is like e to the power minus some positive quantity times t. So, as t tends to infinity this term will go to 0 and $\Omega(t)$ will approach 1 ok. So, what are some of the things that we can quickly see that the system is stable meaning what? I have given a bounded input which is Ω_d desired is equal to 1 and the output is also bounded the output does not go to infinitely.

The other important thing is that this e to the power minus something term times t. If K_p were to be increased. So, for example, if $K_p = 1$, I will get e to the power something, but if K_p is let us say 100 I am just taking one small and one large number. So, e to the power minus 100 or something of the order times t, what will happen? This curve will rise much much steeply and it will reach this one much faster, ok.

Because, so this is like e^{-t} or e^{-10t} so, e^{-10t} will reach so this curve will reach 1 much faster ok, e^{-10t} will go to 0 much faster than e^{-t}. So, what is the moral of the story? If I change this controller gain K_p I can make this output curve look different I can reach 1 faster or slower.

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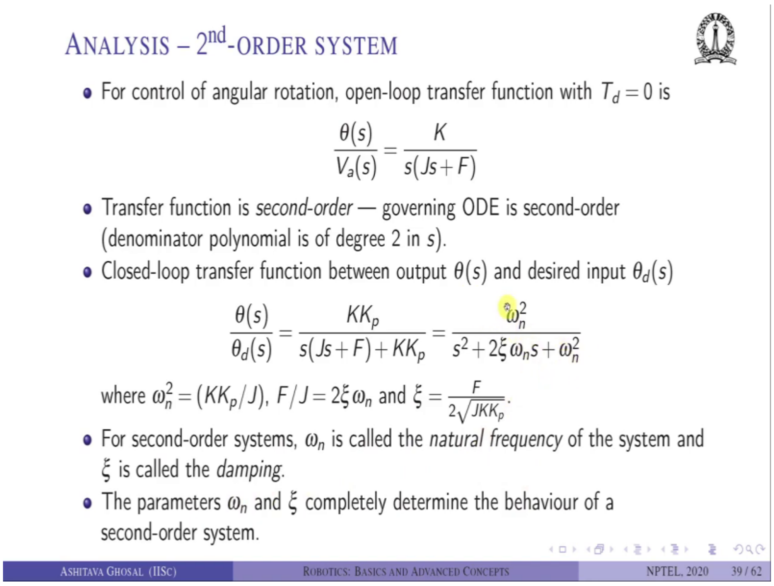

Let us now continue we look at a 2nd order system. So, for the control of angular rotation the open loop transfer function with $T_d = 0$ is given by $\frac{\theta(s)}{V_a(s)} = \frac{K}{s(Js+F)}$. Why did we get this? Because, $s\theta(s) = \Omega(s)$, so $\frac{\Omega(s)}{V_a(s)} = \frac{K}{Js+F}$, but now I want to write it as a problem of controlling the angular rotation not the angular speed.

So, the transfer function in this case is the second-order system because, why we have $s(Js + F)$ so, there is a s^2 term in the denominator. The closed-loop transfer function between $\theta(s)$ and the desired input $\theta_d(s)$ now we have not Ω_d, but θ_d and you can derive the transfer function as $\frac{\theta(s)}{\theta_d(s)} = \frac{KK_p}{s(Js+F)+KK_p}$.

So, this can be written in a generic form which is $\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$. So, where $\omega_n^2 = KK_p/J$,

$F/J = 2\xi\omega_n$ and $\xi = \frac{F}{2\sqrt{JKK_p}}$. So, for second-order system, ω_n is called the natural frequency of the system and ξ is called the damping this is the standard form of any second-order systems.

So, we have ω_n^2 in the numerator some ω_n^2 in the denominator and one damping term $2\xi\omega_n s$. So, both ξ and ω_n completely describe the behavior of a second order system. It also comes from vibration those of you have done a course in vibration we see there is something called natural frequency and damping the same idea is borrowed in control theory.

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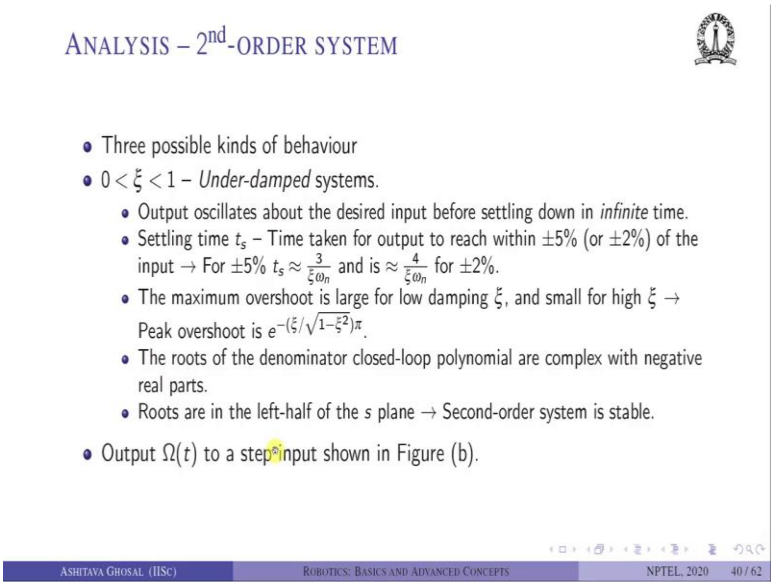

So, there are three possible kinds of behavior one is $0 < \xi < 1$ these are called under-damped systems ok. In an under damped system these output oscillates about the desired input before settling down in infinite time it will go oscillate and slowly come down, but it takes infinite time to reach the output. The settling time t_s is defined as the time taken for the output to reach within ± 5 percent or ± 2 percent of the input ok.

So, for ± 5 percent $t_s \approx \frac{3}{\xi\omega_n}$ and is $t_s \approx \frac{4}{\xi\omega_n}$ for ± 2 percent. So, if you want to reach within the band of 2 percent you have to wait longer. The maximum overshoot is large for low damping and small for high damping ok, the peak overshoot is given by $e^{(-\xi/\sqrt{1-\xi^2})\pi}$.

The roots of the denominator closed loop polynomials are complex with negative real path meaning what? If I were to find the roots of this denominated polynomial $(s^2 + 2\xi\omega_n n + \omega_n^2)$, and ξ lies between 0 and 1 the roots are complex conjugate with negative real parts.

So, the roots are in the left half of the s plane ok, what is the s plane? The X axis is the real and Y axis is the imaginary so, this is called as the s plane and the roots of this denominator polynomial are in the left half as of the s plane ok. What is the output? I will show you what the output looks like in a figure next in the next after two slides.

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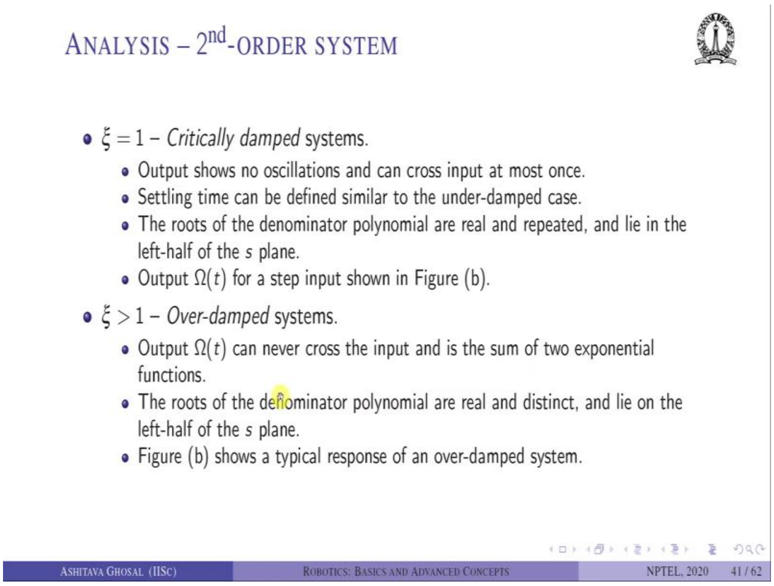

So, if $\xi = 1$ these are called critically damped system the output will show no oscillations, but can cross the input at most one depending and what is the initial velocity ok. The settling time can be defined similar to the under damped case, in this case the roots of the denominator polynomial are real and repeated ok, and lie in the left half of the s plane and the output of $\Omega(t)$ for a step input is again I will show you in the next slide.

If $\xi > 1$ these are called over-damped system. So, in this case the output can never cross the input and it is the sum of two exponential functions ok. The roots are real of the denominator polynomial and if you take Laplace transform you will get $e^{at} + e^{bt}$, with both a and b are negative or they lie in the left half plane.

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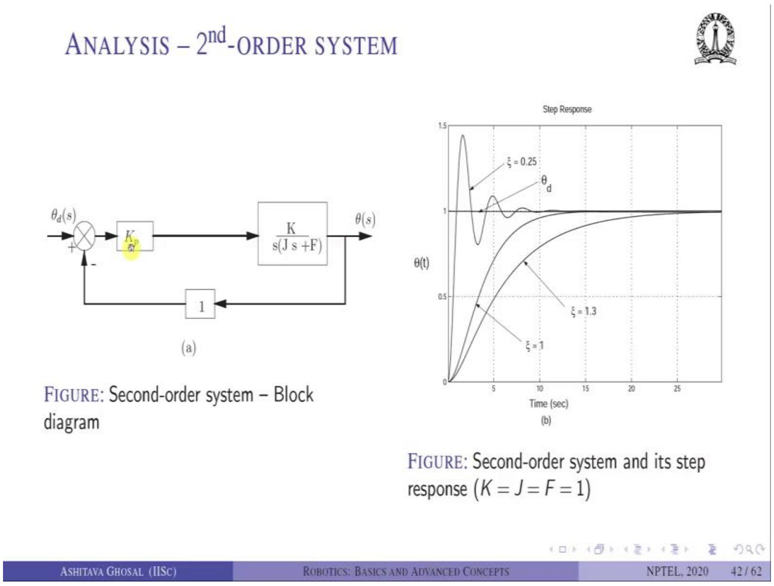

FIGURE: Second-order system – Block diagram

FIGURE: Second-order system and its step response ($K = J = F = 1$)

So, in this figure, I show the typical response for under damped, critically damped, and over damped system ok. So, the first plot here shows if ξ 0.25, θ_d in all these 3 cases is a unit step input so, the output will look like this, and as you can see it will oscillate about the $\theta_d(t) = 1$ and eventually after a long time settle down. If $\xi = 1$, the curve looks like this, so, there are no oscillations and it will reach this $\theta_d(t) = 1$ after infinite time.

Likewise if ξ is over damped which is greater than 1 then the curve looks like this. So, the over damped plot is always lower than the critically damped ok. So, these numerical plots were generated by assuming $K = J = F = 1$. So, we can very easily solve this differential equation in MATLAB ok and plot $\theta(t)$ versus time for different values of K, J, F and K_p.

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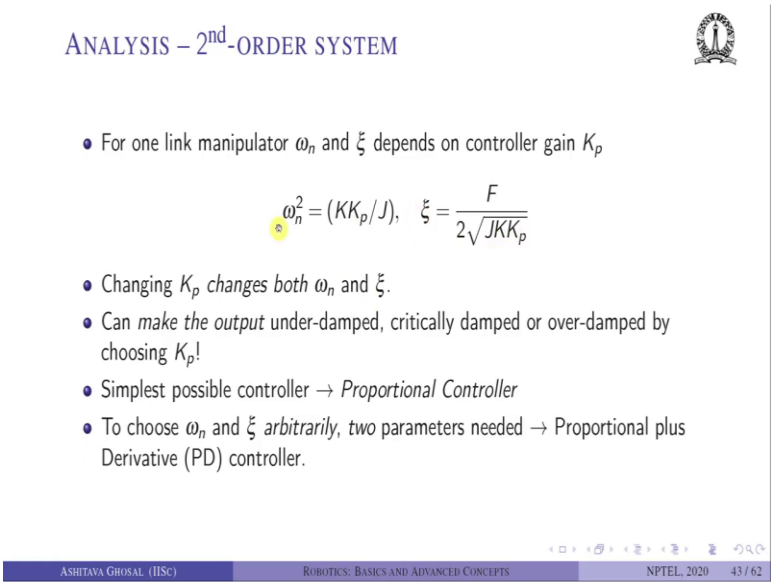

For one link manipulator ω_n and ξ depends on the controller gain K_p ok. So, $\omega_n^2 = KK_p/J$ and $\xi = \frac{F}{2\sqrt{JKK_p}}$ as mentioned earlier. So, if I choose K_p I will get an ω_n, but ξ is automatically fixed ok. So, changing K_p changes both ω_n and ξ. Nevertheless by changing K_p I can change the output, I can make it under damped, I can make it critically damped or I can make it over damped by choosing K_p.

The simplest possible controller is this proportional controller where we choose this gain K_p. If you want to choose ω_n and ξ arbitrarily ok, see here if we choose K_p, ω_n is automatically determined if we choose here K_p both ξ and ω_n is automatically determined. If I want independent ω_n and ξ I cannot do it with the proportional controller I need something called as a proportional plus derivative controller, ok.

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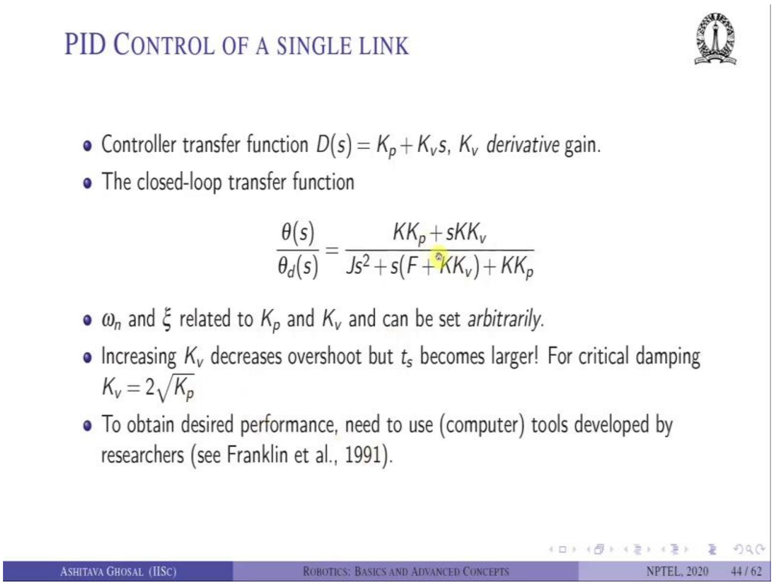

The proportional plus derivative controller transfer function looks like this strictly it is not correct, but nevertheless will let us proceed. So, we have $K_p + K_v s$, where now K_p is the proportional gain and K_v is the derivative gain. The closed-loop transfer function can be again obtained we will get $\frac{\theta(s)}{\theta_d(s)} = \frac{KK_p + sKK_v}{Js^2 + s(F + KK_v) + KK_p}$. So, what you can see here this is like ω_n^2 and this is like $2\xi\omega_n$. So, I can choose K_p which will more or less determined ω_n and then I can choose K_v and I can get different ξ.

So, conceptually we can get arbitrary ω_n and ξ by choosing K_p and K_v. So, if you want to increase K_v that decreases overshoot, but t_s becomes larger ok, so damping decreases the overshoot ok, but it will take much longer to reach your desired output for critical damping $K_v = 2\sqrt{K_p}$.

So, to obtain desired performance we need to play around with this transfer function K_p and K_v, and we need to use a computer as a tool. And there are many many software packages which have been developed over time, where you can play around with this K_p, K_v and we will see later another gain called integral gain such that we get the output in the desired time and in desired manner, ok.

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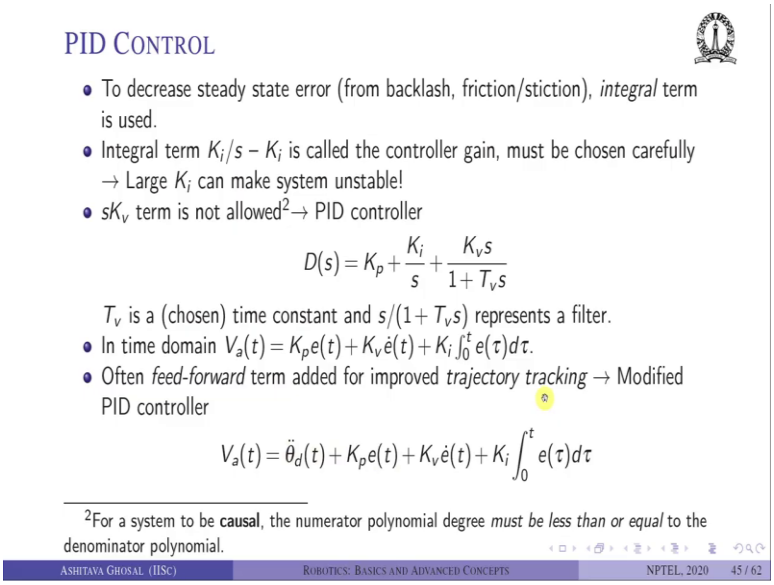

To decrease steady state error; I have not defined what is steady state error; steady state error is basically the fact that you can never ever reach the $\theta_d(t)$ equals to 1 in a real system there will be always be an offset ok. And this comes from things like backlash, friction and stiction at the joint. We cannot reduce steady state error without what is called as an integral term an integral term can be used to reduce the steady state error.

The integral term is K_i/s where K_i is called the controller gain, and we need to choose this very carefully because, when you add a K_i/s into this transfer function it becomes a third order transfer function ok. And third order transfer function can be unstable. So, if you choose a large K_i then the whole system can be unstable.

So, even though the system itself the plant itself is stable, but if you add an integral gain it can cause instability. As I said this $s\omega_n K_v$ is not really a correct way to implement a derivative part, why? Because there is something called as a system to be causal. So, if you want a system to be causal, the numerator polynomial degree must be less than or equal to the denominator polynomial degree.

So, in this case the numerator polynomial is sK_v the denominator is constant. So, that is not allowed in control theory however, we can easily modify it and write $\frac{K_v s}{1+T_v s}$. So, now the numerator polynomial is 1 denominator polynomial is 1 and this is perfectly allowed ok. So, we choose T_v as a time constant. So, T_v is another choice that we have.

So, this $\frac{K_v s}{1+T_v s}$ is basically represents the filter. So, the transfer function of the controller is $\left(K_p + \frac{K_i}{s} + \frac{K_v s}{1+T_v s}\right)$ and we get to play around or choose K_p, K_i, K_v and T_v. So, we need to go to a computer tool play around with this for a given system; such that we get the desired output in the desired time.

In time domain this transfer function can be written as this voltage applied is $V_a(t) = K_p e(t) + K_v \dot{e}(t) + K_i \int_0^t e(\tau)\, d\tau$. So, we have the output of the controller proportional to the error proportional to the derivative of the error and proportion to the integral of the error.

So, this is why it is called PID control P stands for Proportional, I for Integral, D for Derivative ok. In robotics and some other applications also, we often add a feed-forward term ok, so it is it can be shown that this addition of a feed-forward term which is $\ddot{\theta}_d(t)$ improves trajectory tracking.

So, the modified PID controller which is used in most robots is that voltage applied or voltage supplied to the motor is equal to some $\ddot{\theta}_d(t)$ plus this PID term ok. Sometimes there is a constant which multiplies this $\ddot{\theta}_d(t)$ so, in that case we have one more parameter to play with, which is maybe you can call it as K_a which is the acceleration gain.

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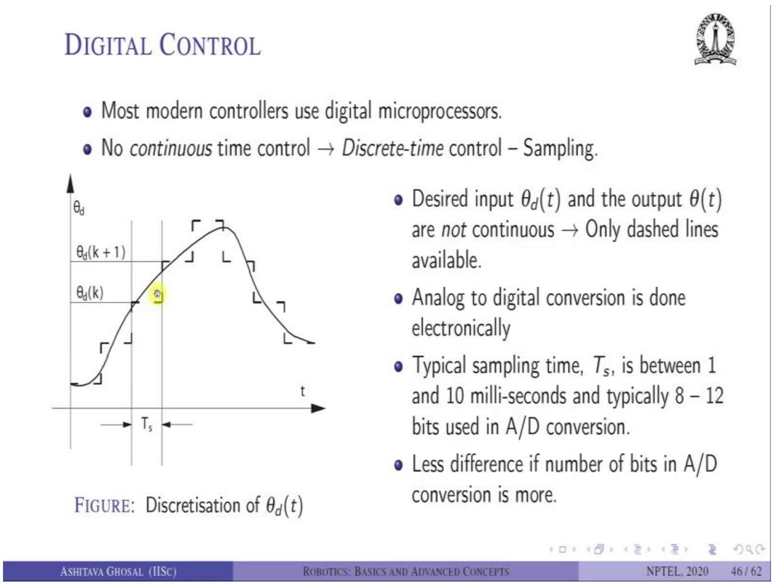

Let us continue, most modern controllers use digital micro processors ok. So, we use digital control and what is the difference, it is not a continuous time control it is a discrete time control and discreet time control important concept is sampling ok. So, for example, if θ_d were to be this solid curve actually we can never achieve the solid curve in a digital control.

We have to discretize this solid curve into steps. So, one step like this, one for a short time then another step, then another step and then another step. So, what the motor will see is an input for a short while then another input, then another input and so on ok, so discrete jumps in inputs $\theta_d(t)$. So, what is the difference between these two jump? In this case it is T_s this is called as the sampling time, and in the y direction in θ_d this is determined by the how much discretization you are doing ok.

So, as I said the desired input $\theta_d(t)$ and the output $\theta(t)$ are not continuous only the dashed lines are available ok. So, analog to digital conversion so, basically instead of this nice analog signal we get this digital thing is done electronically. Typically this sampling time T_s is between 1 and 10 milliseconds, and typically in the y direction 1 unit is divided either by 8 or 12 bit and A/D conversion is used.

So, 1 volt if it is 8 bit is divided by 2^8 which is 256 intervals ok, or if it is a better A to D converter we are using suppose there are 12 bits so, 1 volt is discretized into 2^{12} small steps. So, if you have larger A to D conversion ok the difference between these dashed lines and the solid line is much closer ok. So, we need ideally 12 or even 14 bit A to D converter ok. But they are more expensive. So, it depends on the choice that you have to make.

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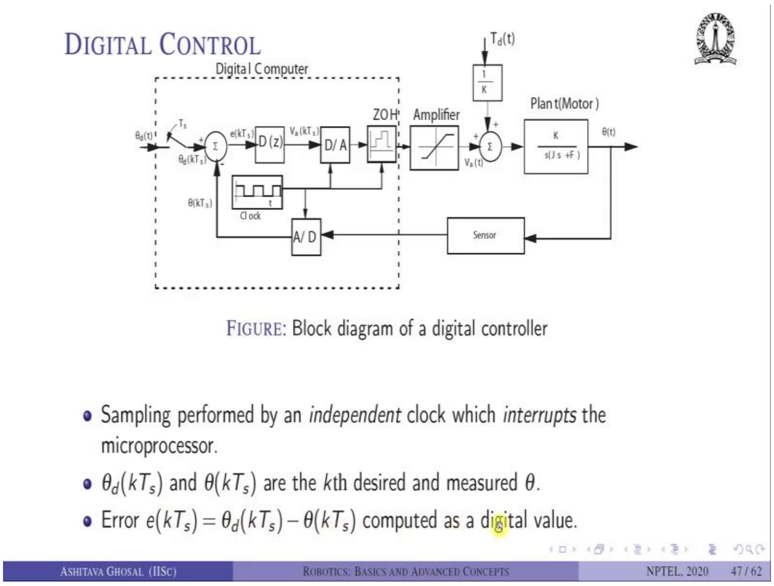

So, a digital controller looks like this, so we have this same plant which is $\frac{K}{Js+F}$ this is the motor with the link, there is a disturbance torque which is coming $1/K$ then there is a voltage which is coming from the controller ok. So, which is $V_a(t)$, we measure the output $\theta(t)$ by a sensor. If this measurement is analog we have to do A to D.

Because, everything inside this dotted block is done in a digital computer ok, this side $\theta(t)$ and this voltage which is coming in is analog it is a smooth continuous signal. So, if you have a sensor which is analog we have to convert it into A to D so, let us assume that we get θ at every sample time which is kT_s.

We also have $\theta_d(t)$ which is coming from motion planning, remember we discussed how to plan the desired trajectory in a smooth and continuous manner, but then we cannot use that smooth and continuous $\theta_d(t)$ we have to sample it. So, we now have $\theta_d(t)$ at every time at every sample so, kT_s. So, first $k = 1$ at one sample, $k = 2$ at the second sample and so on.

The subtraction of these two gives the error ok, this error is spread into this controller we will see what this $D(z)$ means and output is a voltage. So, here also this voltage is discrete ok, but we cannot give discrete voltages to the motor. So, we have to convert it back to analog D to A, this digital is converted back to analog and we have this very important block called zeroth order hold.

So, what this does is when there is so, between two discrete values of voltage what is the value that you need to give to the motor. So, that is that you hold the previous value so, it is in some sense whatever is the previous till the next sample happens we hold the previous value ok.

And finally, this voltage which is obtained after the zeroth order volt is the correct voltage where the current is very small, ok. So, may be milli amperes, you cannot drive a motor in milli amperes with the correct voltage. So, what we need to do is we need to amplify the signal. So, we get the correct voltage and the correct current and in an amplifier we can have saturation.

So, what it; what it is showing here is that it is linear for a while, but then it can saturate we cannot keep on amplifying till whatever amplification you want ok. So, as I said $\theta_d(kT_s)$ and $\theta(kT_s)$ are the k^{th} desired and measured θ. The error at the k^{th} instant is $e(kT_s) = \theta_d(kT_s) - \theta(kT_s)$ and this is computed as a digital value.

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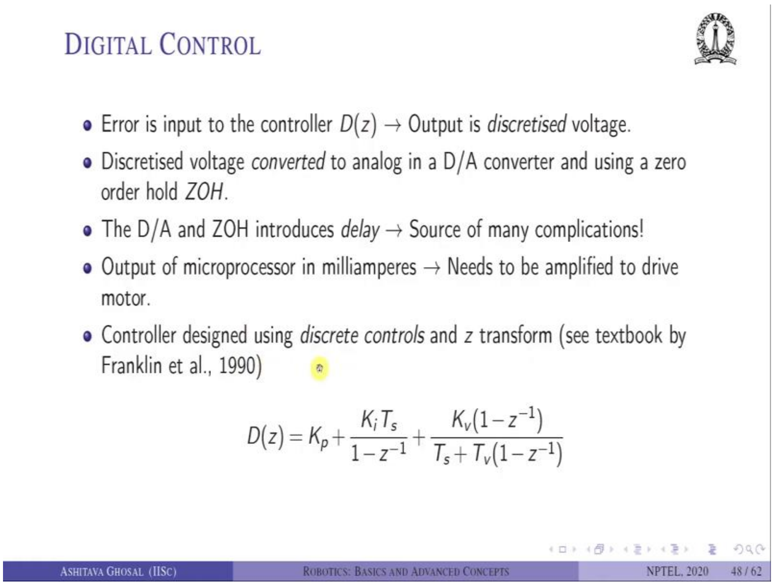

So, the error is input to a controller $D(z)$ output is a discretized voltage. So, instead of $D(s)$ which was in using Laplace we use what is called as the z transform ok. And so, that is why it is called $D(z)$. So, the discretized voltage is converted to analog in a D to A converter and using a zeroth order hold. So, the D to A and the ZOH introduces the delay this is the source of many complications.

Because, it will take some time or it to do this D to A ok. Finally, as I said the output of the micro processor is in milli amperes - need needs to be amplified to drive the motor ok. So, the controller what is in $D(z)$ this is what is called as z transform we are not going to go into details of this digital control using z transform, but a typical $D(z)$ from this book Franklin is given by $K_p + \frac{K_i T_s}{1-z^{-1}} + \frac{K_v(1-z^{-1})}{T_s+T_v(1-z^{-1})}$.

So, as you can see the sampling time gets into the picture, T_v which was there originally that is also there and then you have K_p, but it is not divided by s it is $(1-z^{-1})$ so, it is not K_i/s. So, the integral term is $\frac{K_i T_s}{1-z^{-1}}$. So, not only the sampling is taken into account but, we have a different form ok. So, this is also a very well studied topic which we will not go into this is called digital control using z transforms.

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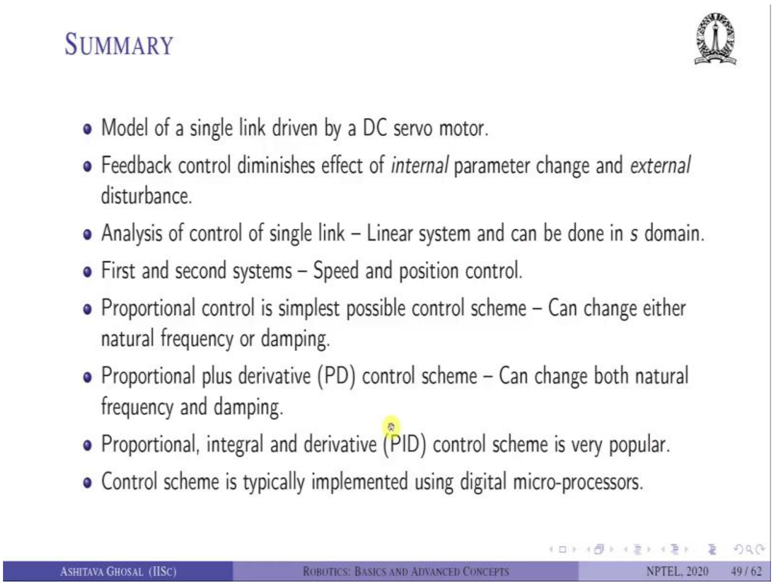

So, in summary we have a model of a single link driven by a DC servo motor. I showed you that feedback control diminishes the effect of internal parameter change and external disturbance. Then we discussed the analysis of control of single link ok. It is a linear system and we can do it in s domain in Laplace domain, I showed you how we can look at a first order system and also as a second order system, as a first order system we were doing speed control, as a second order system we are doing position control.

Proportional controller is the simplest and the simplest possible control scheme, we can either change the natural frequency or the damping when you are looking at as a second order system we cannot do both. If we have proportional plus derivative control scheme also called PD control scheme, we can change both the natural frequency and damping.

And finally, we have this PID control scheme which is very very popular, and the I part can be used to reduce steady state error ok. And finally, all these control schemes are typically implemented using digital micro-processor and we do not have the analysis done using s domain, but we have to use the z transforms ok.

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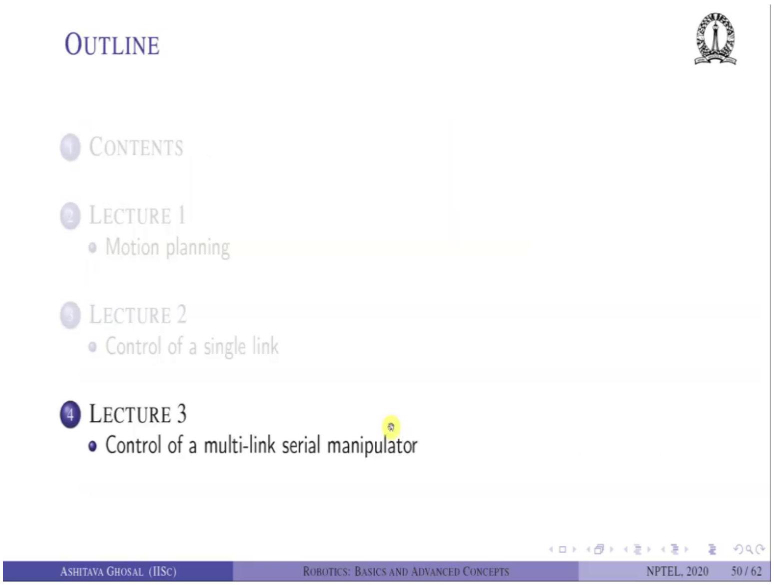

So, with this we will come to end up this lecture, thank you. In the next lecture, we will look at the Control of a multi link serial manipulator.