Robotics: Basics and Selected Advanced Concepts
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Lecture - 37
Advanced topics in non-linear control of manipulators

Welcome to this NPTEL lectures on Robotics: Basic and Advanced Concepts. In the last lecture, we had looked at force control of manipulators and also how to design or implement a hybrid position force control of manipulators ok. So, in this last lecture of this week, we look at some Advanced topics in non-linear control of manipulators.

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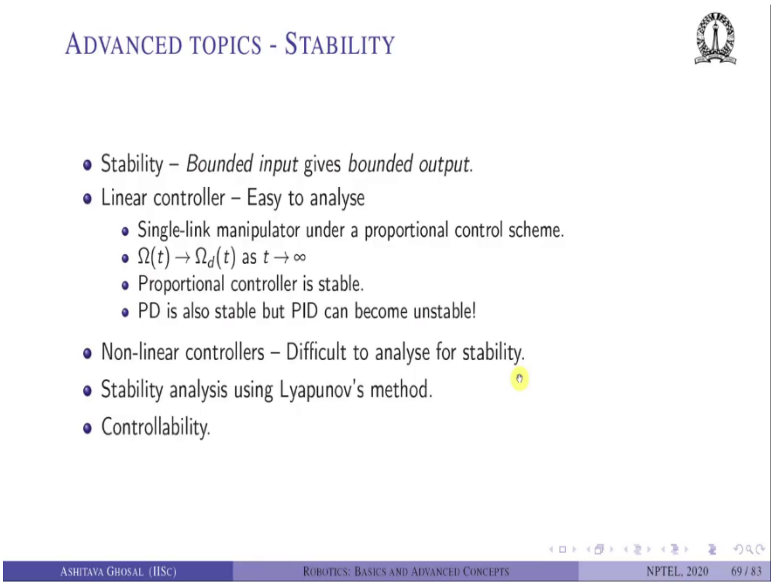

So, the first advanced topic which you will look at is stability ok. A word of warning so, in this last lecture, we will not be doing too much mathematics ok, but I want to introduce you to the some of these advance topics like stability, controllability and so on for manipulators. These are sort of very well understood for linear systems, but what can we say about stability and some of this advanced topic for non-linear robots?

So, let us start with stability. So, first what is the definition of stability? The most widely accepted definition of stability is that if you give a bounded input to a system, the output is bounded ok. If this happens the system is said to be stable ok, in a linear controller this is very easy to analyze ok.

Remember for the single link manipulator under proportional control scheme, I showed you that if I give a step input which is a bounded input the output $\Omega(t)$ which is the speed of the link of the robot approaches $\Omega_d(t)$ as t tends to infinity. It was an exponential curve. It did not go off to infinity, but it approach $\Omega_d(t)$ which was the step input.

So, the proportional controller can be said to be stable. PD controller is also stable ok. We can show that; however, the PID controller is not necessarily stable ok. PID controller can make a stable system unstable why? Because the I introduces a term like K_i/s. So, if you have a second order system and if you have a I part and I controller which you had to it will become a third order system. And third order systems you can show that it can become unstable.

So, basically those who are few who have done a course in controls we can plot the root locus. We have not done this in this robotics course, but we can plot the root locus of a third order system and show that for certain gains the root locus go on to the right half plane ok.

So, PID controller can make a stable system unstable that way, if you do not choose the integral gain properly. Non-linear controllers are very difficult to analyze for stability ok. There are very few general results available.

So, the most well-known way to do stability analysis of a non-linear controller or for that matter any non-linear system is using what is called as Lyapunov's method ok. So, we will look at this Lyapunov's method in a little bit detail. The last topic which we will look at is controllability. So, even if the system is unstable ok, how do we know we can control this non-linear system?

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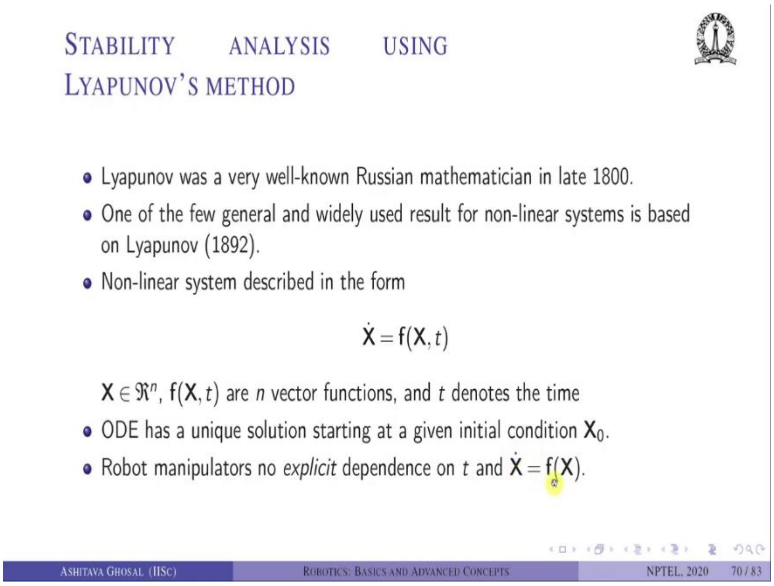

So, let us look at stability analysis using Lyapunov's method, first a little bit of history. Lyapunov was a very well-known Russian mathematician in late 1800s ok. Think of it, more than 130 or 140 years back he wrote, he did his work and he presented one of the few general and widely used results for non-linear systems ok. How to analyze stability for non-linear system; this was presented in 1892.

So, more than 120 years back and there has not been that much improvement of his results ok. So, let us continue a little bit. So, if you have a non-linear system which can be described in this form $\dot{X} = f(X, t)$ and so, X is an element of $\Re^n$ dimensional space, $f(X, t)$ are n vector functions, t denotes time ok.

So, we can only look at systems which can be $\dot{X} = f(X, t)$ there is no nonlinearity in the $\dot{X}$, there is nonlinearity in X. So, $f(X, t)$ could be a non-linear function of X and time ok. So, this ODE has a unique solution starting at a given initial condition ok.

So, this is an ODE, if I give you an initial condition I can at least conceptually integrate definitely, integrate numerically. And find what is X at every time instant ok. So, this is the context of Lyapunov's stability ok. Moreover, in robot manipulators there are no explicit dependence of time. So, we will be left with only $\dot{X} = f(X)$ slightly simpler then $\dot{X} = f(X, t)$.

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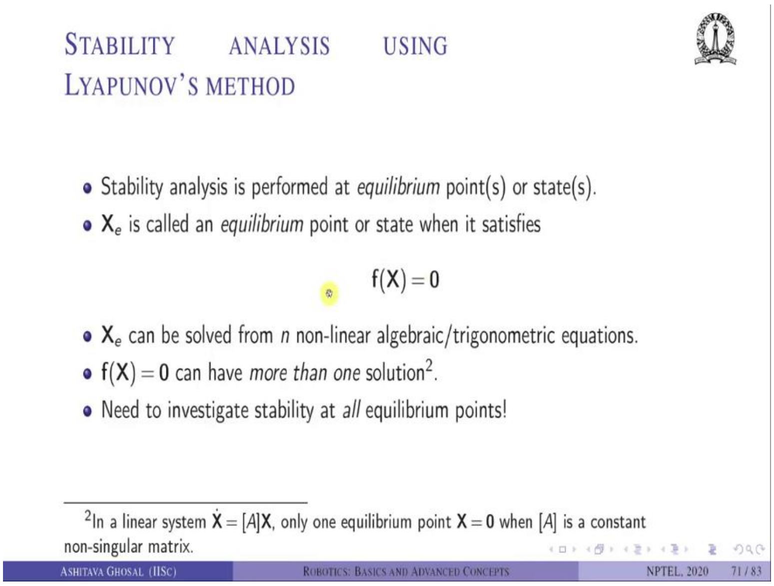

So, what did Lyapunov say? He said that the stability analysis is to be performed at what are called equilibrium points or states ok. So, what is an equilibrium point? X_e is called an equilibrium point or state when it satisfies $f(X) = 0$. So, basically $\dot{X} = 0$. So, we can solve these non-linear algebraic trigonometric equations and obtain the solution. It satisfies $f(X) = 0$, if not analytically, definitely numerically.

The interesting thing is $f(X) = 0$ can have more than one solution ok. It is a non-linear equation ok. So, for example, if you say $\sin(X) = 0$, X could be 0 or $n\pi$. So, n could be 1, 2, and so on. So, it can have more than one solution and what did Lyapunov say he said, you need to investigate stability at all equilibrium point, at all possible solutions of this $f(X) = 0$.

So, in contrast to a non-linear equation, if you have a linear equation which is $\dot{X} = [A]X$, $[A]X$, where now, $[A]$ is a matrix of constant variables it is non-singular. So, what are the equilibrium points for linear system only $X = 0$, because if $[A]$ is regular it is non singular $[A]X = 0$ it satisfy only when $X = 0$ ok. So, there is one equilibrium points for the linear system whereas, for the non-linear system I can have more than one equilibrium point ok.

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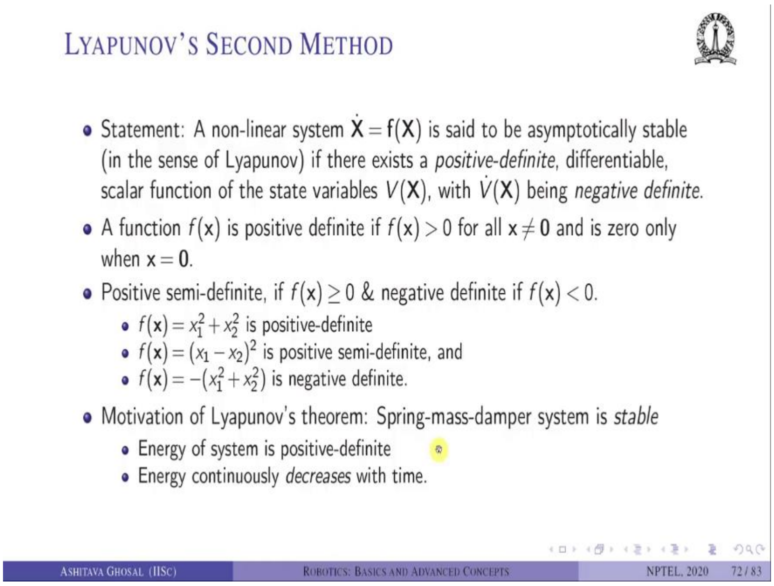

So, now what is Lyapunov second method? So, this is the statement of the Lyapunov second method to obtain the stability of a non-linear system $\dot{X} = f(X)$. So, what is the statement? A non-linear system $\dot{X} = f(X)$ is said to be asymptotically stable in the sense of Lyapunov. If there exists a positive definite differentiable scalar function of the state variables denoted by $V(X)$, with $\dot{V}(X)$ being negative definite ok.

So, let us go over this statement once more. If I have a non-linear system represented by $\dot{X} = f(X)$, we can say that this is asymptotically stable. If there exists a positive definite differentiable scalar function of the state variables X, let us denote that by $V(X)$. And the derivative, $\dot{V}(X)$ with respect to time must be negative definite.

So, this must be positive definite and $\dot{V}(X)$ must be negative definite ok. I will just quickly go through what is a positive definite function and a negative definite function. A function $f(x)$ is positive definite if $f(x) > 0$ for all $x \neq 0$ and is 0 only when $x = 0$, serves typical example.

So, if $f(x) = x_1^2 + x_2^2$, it is positive definite ok, because it will be always be greater than 0 and will be equal to 0 when $x_1 = x_2 = 0$. $f(x) = (x_1 + x_2)^2$ is not positive definite why? Because, this can become 0, when $x_1 = x_2 \neq 0$.

So, for example, x_1 can be 1 and x_2 is also 1. So, X is not 0 nevertheless $f(X) = 0$, but these are called positive semi definite ok. $f(x) = -(x_1^2 + x_2^2)$ is negative definite ok. So, the motivation of Lyapunov's theorem comes from the Spring-mass-damper system.

So, we know that with some damning a spring-mass-damper system is stable why? Because, if you displace the mass after a while due to damping the amplitude will die down ok so, in the case of a spring mass damper system the energy of the system is positive definite ok. The potential energy is $\frac{1}{2}kx^2$ right. The energy also decreases continuously with time ok, when there is damping.

So, that is what E is positive definite, but rate of change of E is negative ok. So, based on this intuition he said that this can be extended to any non-linear system $\dot{X} = f(X)$ and we can find if it is stable if there exist $V(X)$, which is positive definite and $\dot{V}(X)$ which is negative definite ok.

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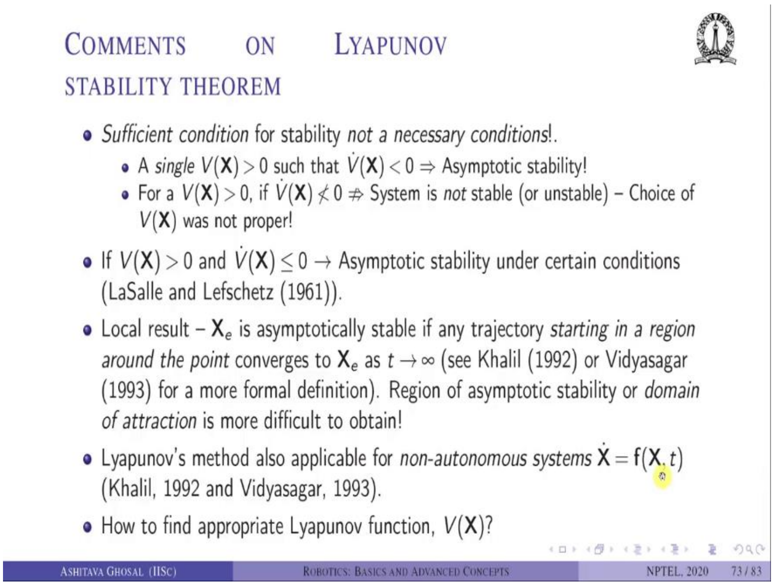

This is one of the very funny theorem, it is that it is a sufficient condition for stability ok not a necessary condition ok. What do I mean by that? If you can somehow find a single $V(X) > 0$; such that the $\dot{V}(X) < 0$, it guaranties asymptotic stability ok.

However, let us assume that you found a $V(X) > 0$, but $\dot{V}(X) \not< 0$. We cannot say that the system is not stable ok. So, the negation of $\dot{V}(X) < 0$ does not imply unstable system. All it means is that you did not make a good choice of $V(X)$. You have to try $V(X)$ some other one ok.

So, the Lyapunov stability theorem is for stability it says nothing about the instability or the unstable nature of a system. So, we can find a system is stable if $V(X) > 0$ and $\dot{V}(X) < 0$, but if we do not find $\dot{V}(X) < 0$ then it does not mean it is unstable ok. That is why it is called as a sufficient condition for stability not a necessary condition ok.

So, if $V(X) > 0$ and $\dot{V}(X) \leq 0$ means that this is not negative definite. We can still show asymptotic stability under certain conditions. I am not going to go into this ok. In the 60's, early 60's these two you know researchers showed that under what condition if it is semi definite what happens? What are some the others features? it is a local result ok.

So, we are going to test for stability at some equilibrium point ok. So, basically if any trajectory starting in a region around this equilibrium point converges to X_e as t tends to infinity. This is known as asymptotic stable ok, but we have no idea of saying how big is this region from where we can start ok. So, this region of asymptotic stability or the domain of attraction is much more difficult to obtain ok.

The next point is this Lyapunov's method is also applicable for non-autonomous system when $\dot{X} = f(X, t)$, but we are not going to go into that. So, we can also determine stability of non-autonomous system. Basically, non-autonomous systems are those where we have explicit dependence one time ok.

So, the bottom line in using Lyapunov stability theorem is how to find appropriate Lyapunov's function $V(X)$. If you find $V(X)$ and its time derivative satisfying less than 0 ok, then our job is done.

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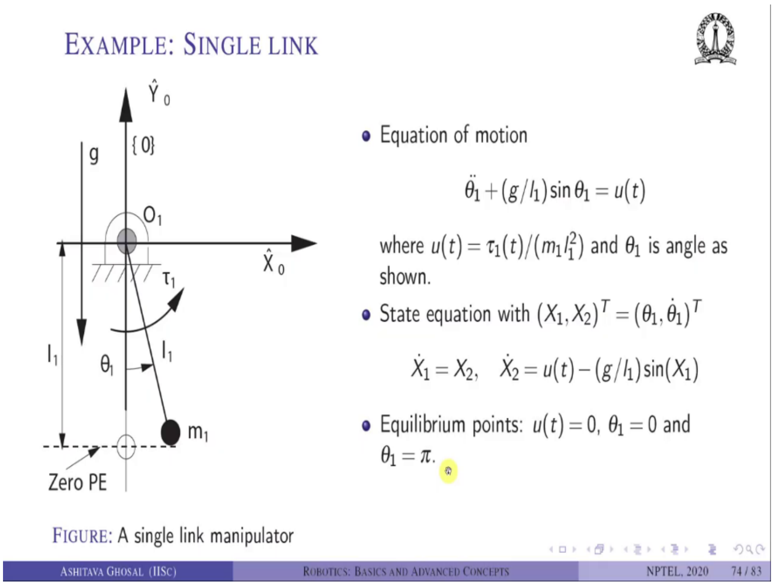

So, let us look at some examples. So, we discussed this single link robot, when we are doing this control. This is even simpler, we do not have I anymore. So, we have a link with a mass at the end, there is a θ_1 rotation, there is a torque ok, gravity is acting this way, this distance is l_1, link length is l_1.

So, the equation of motion of this one link robot can be shown to be $\ddot{\theta}_1 + \left(\frac{g}{l}\right)\sin\theta_1 = u(t)$, θ_1 is measured some here, where $u(t) = \tau_1(t)/(m_1 l_1^2)$. So, we can derive this is not at all hard it is very easy. So, I have to first write this second order differential equation in the state space form.

The state equation in this case can be X_1 and X_2 which are nothing but θ_1 and $\dot{\theta}_1$ and $\dot{X}_1 = X_2$, $\dot{X}_2 = u(t) - \left(\frac{g}{l_1}\right)\sin(X_1)$. So, these are the state base equations. So, what are the equilibrium points of this system? To obtain the equilibrium point we have to set $f(X) = 0$. So, $u(t) = 0$.

So basically, we have $X_2 = 0$ and $(g/l_1)\sin X_1 = 0$. So, basically $\left(\frac{g}{l_1}\right)\sin X_1 = 0$ means, $X_1 = \theta_1$. So, $\theta_1 = 0$ or $\theta_1 = \pi$ and $\dot{\theta}_1 = 0$. So, we have two equilibrium points which is $(0,0)$ which means this is coming all the way down and then $(\pi, 0)$ when the link is vertically upwards ok. This is vertically downwards $(0,0)$, vertically upwards $(\pi, 0)$.

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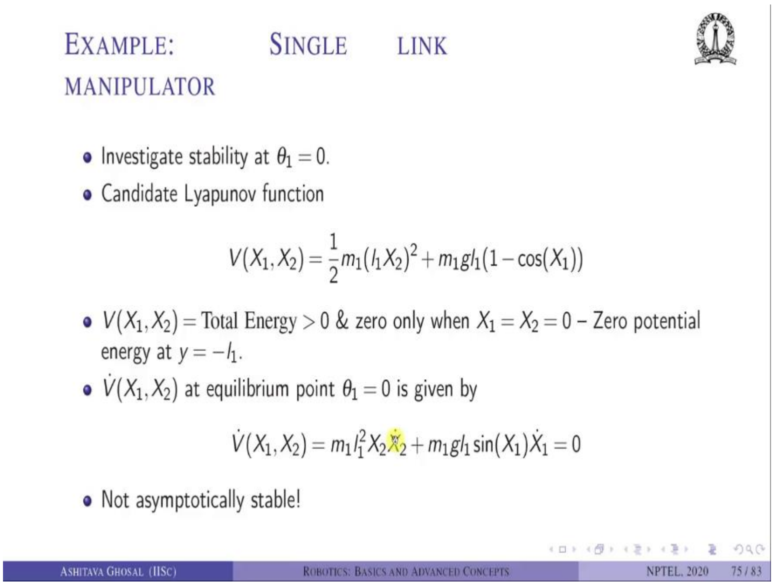

So, let us investigate the stability at $\theta_1 = 0$ and $\dot{\theta}_1 = 0$. So, the one possible candidate Lyapunov's function is the kinetic energy of this system. So, $V(X_1, X_2) = \frac{1}{2} m_1 (l_1 X_2)^2 + m_1 g l_1 (1 - \cos X_1)$. So, the candidate Lyapunov's function is the total energy of the system, kinetic plus potential energy ok.

So, we know that the total energy will always be greater than 0 and it is zero only when $X_1 = X_2 = 0$. So, it is a positive definite function. The derivative of this $V(X_1, X_2)$ at $\theta_1 = 0$ can be obtained and we can see it is equal to 0 ok. So, you will get some $m_1 l_1^2 X_2 \dot{X}_2 + m_1 g l_1 \sin(X_1) \dot{X}_1$ and what do we do with $\dot{X}_1$ and $\dot{X}_2$?

We use the state equations and we substitute the state equation here. So, the derivative is not less than 0. So, strictly speaking it is not asymptotically stable ok. Does it make sense yes. So, if I put up this bob by some angle and leave it will keep on oscillating, it will not go to the equilibrium point which is vertically downwards ok.

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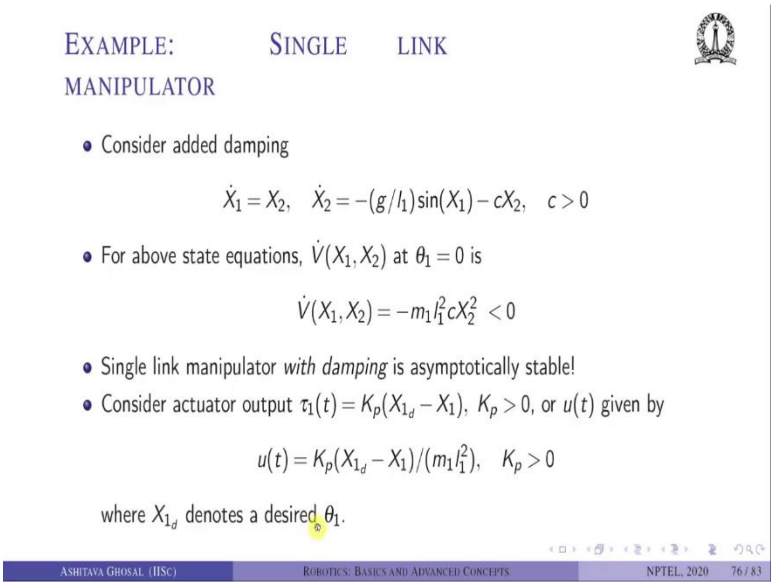

Now, let us consider we add some damping at the joint. So, basically we say that there is a damping which is like $c\dot{\theta}$. So, which is like $-cX_2$. So, now, my state equations are $\dot{X}_1 = X_2$ and $\dot{X}_2 = -\left(\frac{g}{l_1}\right)\sin(X_1) - cX_2$. So, for the above state equations again we can use the same V and we can obtain $\dot{V}$ and we can see that the $\dot{V}(X_1, X_2) = -m_1 l_1^2 c X_2^2$.

So, this is the square. So, this is less than 0 ok. So, the single link manipulated with damping is asymptotically stable ok. Let us continue. Now, let us assume that we are applying a torque to this robot and this is the proportional control torque. So, this torque is equal to $\tau_1(t) = K_p(X_{1_d} - X_1)$, ok and $K_p > 0$, where X_{1_d} is some desired θ_1. So, we can show that $u(t) = K_p(X_{1_d} - X_1)/(m_1 l_1^2)$.

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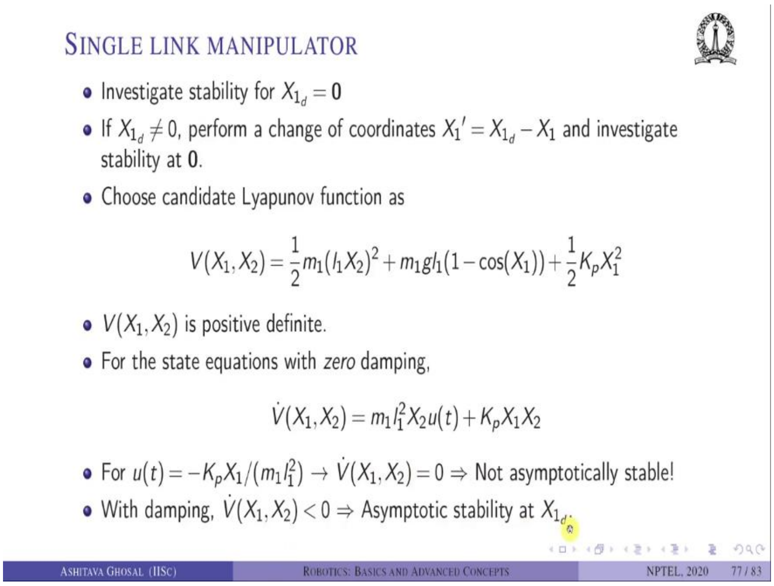

So, for simplicity we investigate the stability at $X_{1_d} = 0$. Is that very serious? No, if X_{1_d} or θ_1 was some other angle 45 degrees, we can do a change of coordinates ok and then investigate the stability at that point, but for simplicity we will assume θ_{1_d} is 0. So, it is lying vertically down ok.

So, as I said if $X_{1_d} \neq 0$ we can perform a change of coordinates, $X_1' = X_{1_d} - X_1$ and investigate again the stability at $X_1' = 0$. So, we choose a candidate Lyapunov function and again as the kinetic energy plus potential energy, but let us add one more term which is $\frac{1}{2} K_p X_1^2$ where K_p is the proportional gain ok.

So, this term is positive, this was also positive. So, the sum of these two functions is positive also. So, it is positive definite ok. So, again with state equations with 0 damping, we can show that $\dot{V}(X_1, X_2) = m_1 l_1^2 X_2 u(t) + K_p X_1 X_2$ and for $u(t) = -K_p X_1/(m_1 l_1^2)$, $\dot{V}(X_1, X_2) = 0$. So, we are using proportional controller and $\dot{V}(X_1, X_2) = 0$.

So, again it is not asymptotically stable. So, what it is telling you is even with a proportion and controller if I give some input it will keep on oscillating about that desired quantity ok. If you add damping which is like $-cX_2$ again, we can show that it is an asymptotically stable at $X_{1_d} = 0$. So, proportional controller with damping will result in asymptotic stability ok.

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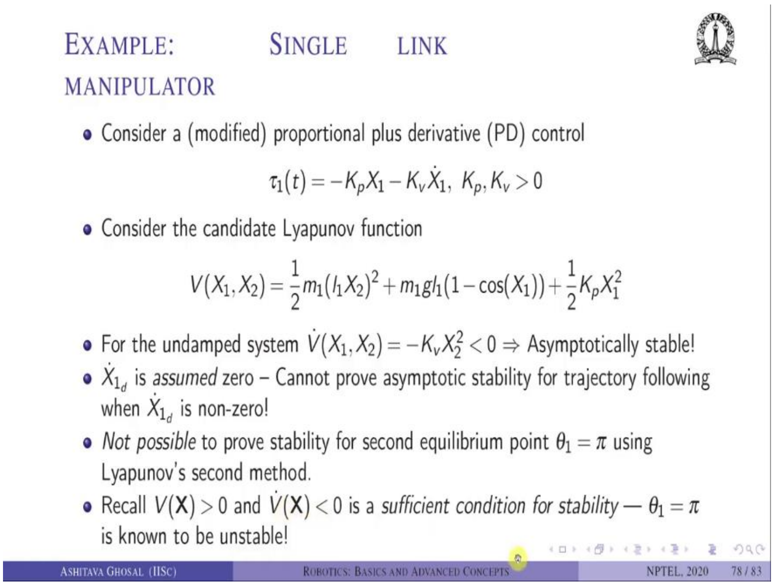

So, now, let us consider the modified proportional plus derivative controller ok. So, we have $\tau_1(t) = -K_p X_1 - K_v \dot{X}_1$. Why am I calling it modified? Because there is no X_{1_d} and there is no $\dot{X}_{1_d}$. So, $X_{1_d} = 0$ is not a problem, we can do change of coordinates, but $\dot{X}_{1_d}$ is set to be 0.

In fact, there are no, we will see later that there are no known results when you have a desired rate of change of X_1. So, now let us consider again candidate Lyapunov function which is the kinetic energy plus potential energy plus this $\frac{1}{2} K_p X_1^2$. So, for the undamped system now, we can see that $\dot{V}(X_1, X_2) = -K_v X_2^2$ which is less than 0 ok.

So, there is no damping in the system, but there is a derivative controller, there is a derivative part of the control and due to this derivative control which is $-K_v \dot{X}_1$ we get asymptotic stability ok. So, as I said, $\dot{X}_{1_d}$ is assumed to be zero. We cannot prove asymptotic stability for trajectory following when $\dot{X}_{1_d}$ is non-zero.

So, if I tell you that the X_{1_d} could also have some time derivative, desired time derivative at each instants of time, there is no way to prove asymptotic stability ok. So, it is not possible to prove stability for second equilibrium point. Remember, we said Lyapunov said that we need to investigate stability at equilibrium point when the link is hanging down and when the link is pointing upwards ok.

There is no way or no way to choose $V(X)$ such that $\dot{V}(X) < 0$. We cannot prove the second equilibrium point to be stable using Lyapunov's method why? Because you know that it is not stable ok and Lyapunov's result deals only with stability ok.

So, when the link is pointing vertically upwards, any small perturbation to the link will go away from that $(\pi, 0)$. It is an unstable point ok. From, if you linearize it we can show that the poles of this linearized system are on the right half plane ok. So, those of you have done a course in controls you are looked at this inverted pendulum. And we know that the inverted pendulum is unstable ok.

Recall that $V(X) > 0$ and $\dot{V}(X) < 0$ is a sufficient condition for stability ok and $\theta_1 = \pi$ is known to be unstable. Lyapunov's theorem has nothing to do with instability. This is a big thing; many people do not understand that what is meant by a sufficient condition.

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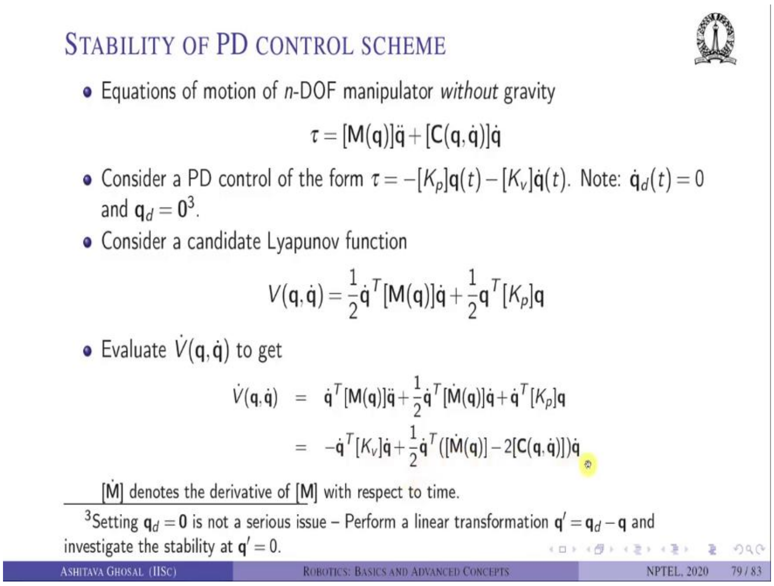

How about stability of a PD control scheme for a general serial robot with n degrees of freedom? So, if you do not have gravity, we can write torque $\tau = [M(q)]\ddot{q} + [C(q,\dot{q})]\dot{q}$. So, we can have a PD controller of this form $\tau = -[K_p]q(t) - [K_v]\dot{q}(t)$. Note again, that $\dot{q}_d(t) = 0$, $q_d(t) = 0$ is again not a serious problem, because we can again do a change of coordinates.

Let us consider a the Lyapunov function which is nothing but $V(q,\dot{q}) = \frac{1}{2}\dot{q}^T[M(q)]\dot{q} + \frac{1}{2}q^T[K_p]q$. What does this? If you recall, if you think back a little bit this is nothing but the kinetic energy of the system, kinetic energy of the robot manipulator. And what is this? This is like $\frac{1}{2}K_p\theta^2$. This is coming from the controller, there is no gravity so there is no question of potential energy.

We can evaluate the derivative of this $V(X_1, X_2)$ and it can be shown by using chain rule that this is $\dot{q}^T[M(q)]\ddot{q} + \frac{1}{2}\dot{q}^T[\dot{M}(q)]\dot{q} + \dot{q}^T[K_p]q$ and it can be written as $-\dot{q}^T[K_v]\dot{q} + \frac{1}{2}\dot{q}^T([\dot{M}(q)] - 2[C(q,\dot{q})])\dot{q}$. So, this term is negative, but we do not know; what is this? This is the derivative of the mass matrix minus twice the Coriolis matrix ok.

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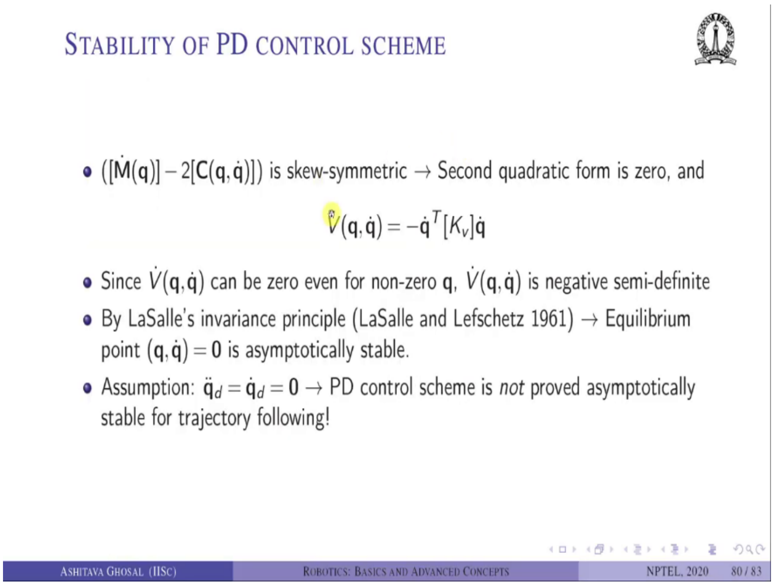

So, it turns out that this $([\dot{M}(q)] - 2[C(q,\dot{q})])$ is skew-symmetric ok. So, skew symmetric means the diagonal terms are 0 ok. We can prove this, you can take this as a homework. You can try it yourself. If you have a skew symmetric matrix the second quadratic form is zero. What do you mean by this? $\frac{1}{2}\dot{q}^T$times some skew symmetric matrix times $\dot{q}$ will be 0 ok.

And hence, $\dot{V}(q,\dot{q}) = -\dot{q}^T[K_v]\dot{q}$, but since $\dot{V}(q,\dot{q})$ can be zero even for non-zero q so, it is not negative definite ok. So, this is negative semi-definite. So, this is like that $(q_1 - q_2)^2$ that similar to that case. So, the function can be 0 even when q_1 and q_2 are not 0 ok.

However, you can show by this LaSalle Lefschetz invariance principle that the equilibrium point q and $\dot{q}$ is asymptotically stable ok. So, what have we shown? We have shown for a general n degree of freedom manipulator without gravity, without a desired rate of change of the desired trajectory ok, this manipulator under PD control is stable ok.

As soon as you add $\ddot{q}_d$ for $\dot{q_d} \neq 0$, we cannot prove an asymptotic stability ok. So, this is a very unsolved problem. So, if you have trajectory tracking or trajectory following that the desired derivative of the joint is not 0, we cannot prove asymptotic stability ok. There are no known methods or results for asymptotic stability.

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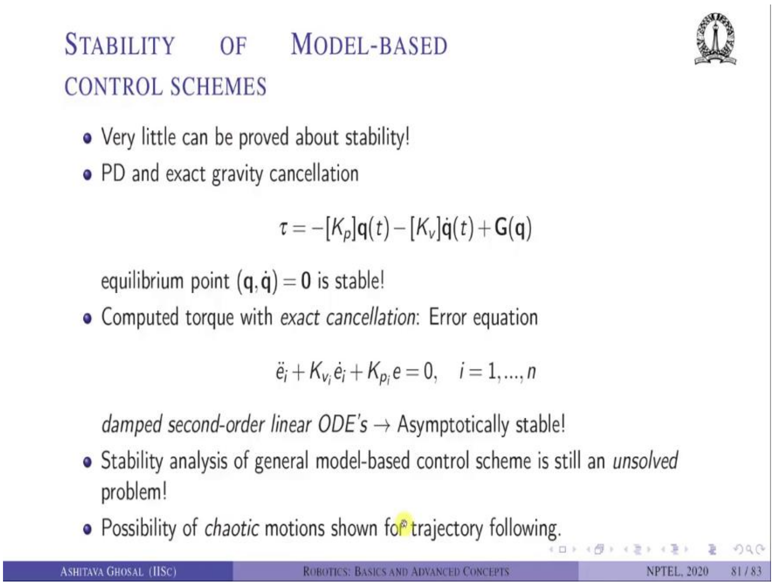

So, how about stability of model based control schemes? Again, very little can be proved about stability ok. So, as I said PD and exact gravity cancellation so, $\tau = -[K_p]q(t) - [K_v]\dot{q}(t) + G(q)$ this can be shown to be stable why? Because the gravity is going away and we are left with the equation $\tau = [M(q)]\ddot{q} + [C(q, \dot{q})]\dot{q}$ that we show is stable.

How about computed torque with exact cancellation? Again, that can be proven to be stable why? Because the error equation is damped linear second ordered ODE's ok. And depending in choice of K_p and K_v we know that the errors will go down ok to 0. So, this is asymptotically stable.

So, the stability analysis of general model based control schemes where the mass matrix is not exactly known, where the gravity is not exactly known, is still an unsolved problem ok and it turns out that such systems can exhibit chaotic motions ok. So, there are results which show that the model-based control schemes where the mass matrix and the coriolis term or the gravity term is not known exactly we can exhibit chaotic motions ok. So, that is clearly not asymptotic stability.

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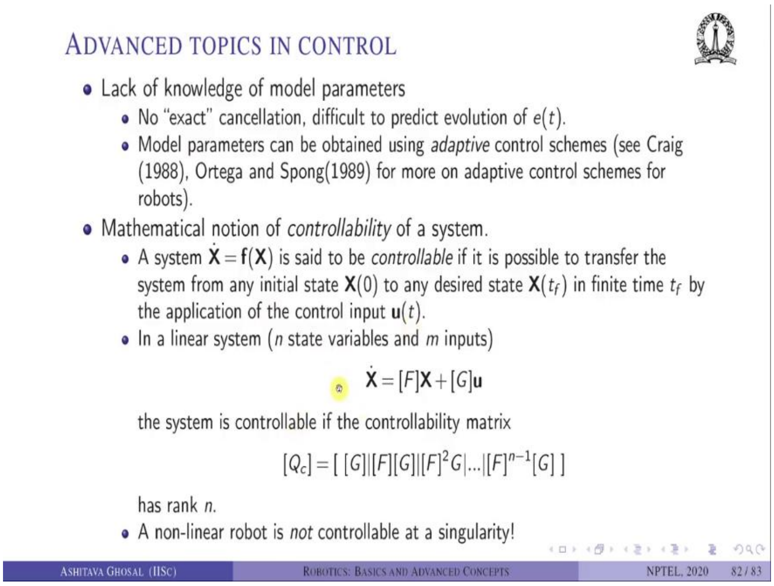

Let us continue, we also discuss that there is this lack of knowledge of model parameters in most practical robots ok. And hence, no exact cancellation is possible and it is difficult to predict the evolution of t. So, remember when you had inexact models, we had linear second order system, but the right hand side was a complex quantity ok. There was a forcing which was not 0 ok.

So, we do not know what $e(t)$ will look like ok. So, it is a linear second order ODE, but with a non-linear right hand side. We do not know what it can do depending on the non-linearity. We have seen that if the estimates are good then the right hand side is small and the $e(t)$ and $\dot{e}(t)$ will go to 0 ok or become small.

But there are no general results of what is this range of estimates or how good their estimates should be for $e(t)$ and $\dot{e}(t)$ to go to 0. The other issue is if you do not know the model parameters exactly can we find it out? Ok, yes, the answer is yes, the models parameters can be obtained using adaptive control schemes ok.

So, we do not want to get into adaptive control schemes. At basically, we can look at the errors and then we can have some adaptive control laws which go and change the model parameter such that the errors go to 0 over time ok. And at the end of this adaptation period, we will know exactly what are the model parameters. The last thing that we want to discuss, I want to discuss is this controllability of a system ok.

So, the mathematical notion of a controllability of system is the following ok. A system $\dot{X} = f(X)$ is said to be controllable if it is possible to transfer the system from any initial state $X(0)$ to any desired state $X(t_f)$, in finite time t_f by application of the control input $u(t)$. So, I have given you some mathematical system.

How do I know I can take this system from a given initial state to a final desired state by applying torques or applying the external input ok. In a linear system this is well known. People know exactly what is the result, people can predict what happens or under what conditions the system is controllable. So, in a linear system we have $\dot{X} = [F]X + [G]u$. $[F]$ is a constant matrix, $[G]$ is some another constant matrix.

So, n state variables with m inputs. The system is controllable if you find the matrix which is $[G]$ concatenated with $[F][G]$ and $[F]^2[G]$ and all way till $[F]^{n-1}[G]$. So, first so, if $[G]$ is one dimensional so if column first column is $[G]$, $[F][G]$ will be a second column and so on. So, there will be n columns ok.

So, this matrix if it has rank n then this system $\dot{X} = [F]X + [G]u$ is controllable, then what about non-linear robots? There are no known results or controllability.

So, we cannot say that this robot or this robot mathematically, I can take it from an initial θ and $\dot{\theta}$, initial state to a final θ and $\dot{\theta}$ state using the torque ok. What do we know about controllability? The only thing we know about controllability and which can be proved is that the non-linear robot is not controllable at a singularity ok, why?

Because at a singularity you either loss the degree of freedom or you gain a degree of freedom ok and those loss and gain degrees of freedom ok, we cannot move perpendicular to the direction or along the loss of degree of freedom and similarly, problems arise when you gain a degree of freedom ok.

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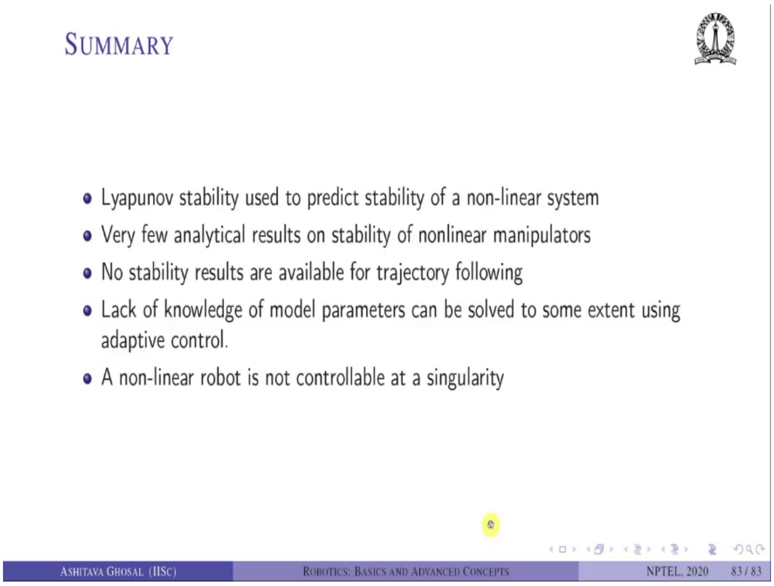

So, in summary we have discussed Lyapunov's stability and we have used Lyapunov's stability to predict the stability of a non-linear systems ok. So, there are very few analytical results on non-linear robots and manipulators ok. So, no stability results are available for trajectory following when derivative of θ_d, first or second derivative is non-zero ok.

The lack of knowledge of model parameters can be solved to some extent using adaptive control ok and finally, a non-linear robot is not controllable at the singularity ok. So, that is another reason why we need to know where the singularities of a robot are, because we cannot control the motion ok. So, with this we come to a stop of this week ok. Next week, we move on to a different topic.

Thank you.