Robotics: Basics and Selected Advanced Concepts

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Lecture - 04
Mathematical Preliminaries, D-H Convention & Examples

Welcome to this NPTEL lectures on Robotics: Basic and Advanced Concepts. In this week there will be three lectures, the first lecture deals with mathematical preliminaries required to model and analyze robots, it will also include the very important notion of something called homogeneous transformation matrix.

In the second lecture we will look at how to model and represent joints and links of a robot and in the third lecture we will look at examples of D-H parameters and link transformation matrices.

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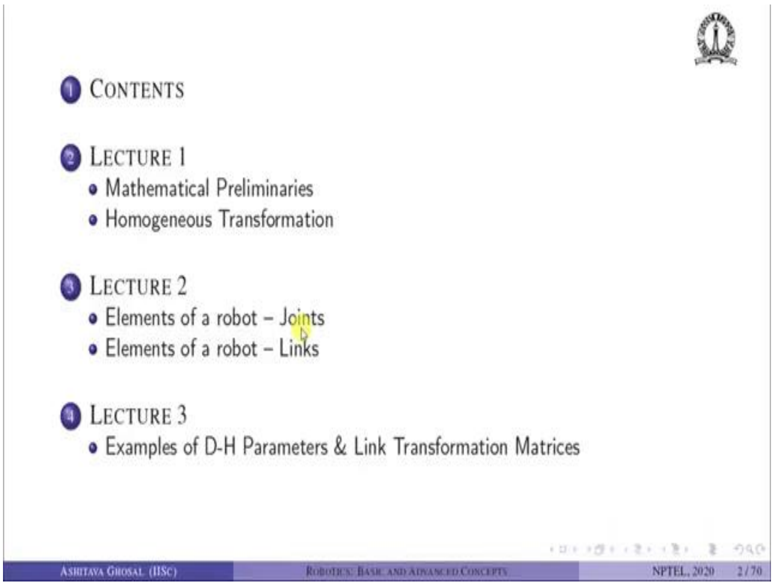

So, the first lecture is on Mathematical Preliminaries.

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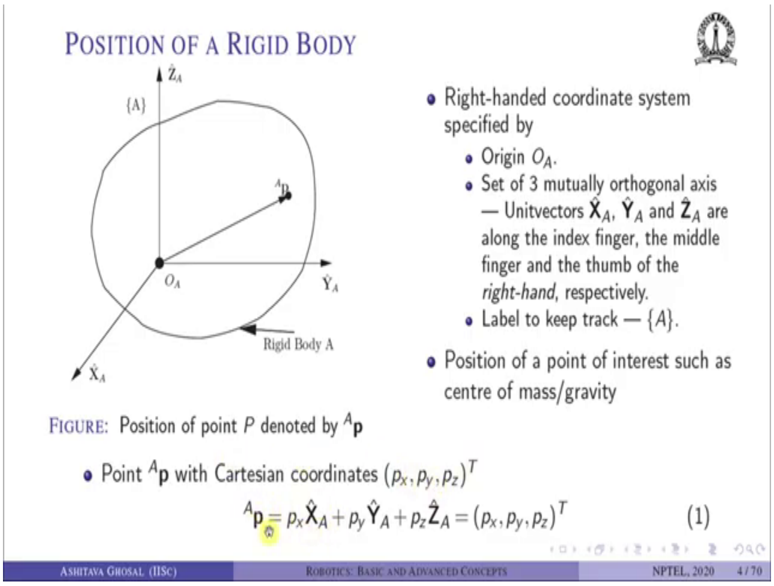

So, what do we need to know or how do we proceed? So, we recognise or as mentioned earlier, the links of a robot will be assumed to be rigid bodies ok. So, the first important thing that we need to understand is, how do we represent the rigid body which models the links of a robot.

So, to represent a rigid body in 3D space, we look at the position and orientation of the rigid body. To represent the position of a rigid body basically we need a right-handed coordinate system to start with, we need a reference coordinate system ok. A reference coordinate system or any right-handed coordinate system is specified by an origin O_A, X axis, Y axis and Z axis we are going to use several coordinate systems. So, we will keep track of by labelling the coordinate system with $\{A\}$ $\{B\}$ and so, on.

So, in this case we have an X_A axis, a Y_A axis, a Z_A axis and an origin O_A. So, the rigid body is represented by potato shaped contour ok. So, the right-handed coordinate system is specified by an origin O_A, a set of three mutually orthogonal axis unit vectors X_A, Y_A and Z_A. You can think of these along the index finger, the middle finger and the thumb of the right hand respectively and as mentioned we will label each coordinate system and in this case it is labelled as curly bracket A ok.

So, as I said we want to represent the position of a rigid body in 3 D space. So, the first natural question is position of what. So, basically we want to find the position of a point of interest on the rigid body and what could be the point of interest? It could be say the centre of mass or the centre of gravity or even maybe location of a sensor or some special point on the rigid body.

So, this point P which is shown in this figure can be represented using a vector drawn from the origin O_A to that point and we are going to keep track of in which coordinate system this vector is shown by this superscript leading superscript A ok. So, this vector $^A p$ can be represent that using three Cartesian coordinates p_x, p_y, p_z. So, this is the column vector. So, p_x, p_y, p_z transpose mean it is a column vector.

So, what are these p_x, p_y, p_z? They are nothing but the projection of this vector $^A p$ along that X_A axis, Y_A axis and Z_A axis. So, this $^A p$ can be written as p_x into X_A, p_y into Y_A plus p_z into Z_A. X_A axis is nothing but (1 0 0) as a column vector. Y_A axis is (0 1 0) and Z_A axis is (0 0 1). So, hence $^A p$ is nothing but p_x, p_y, p_z as a column vector.

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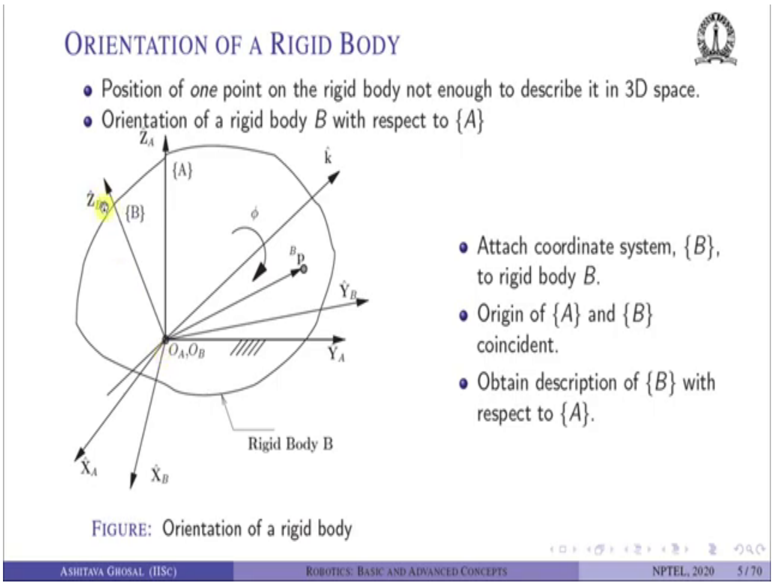

Position of one point on the rigid body is not enough to describe it in 3D space. So, you can think of a let us say you are holding a cube. If I give you the corner of the cube that is not enough because based from that one point which is the corner of the cube I can rotate the cube about some different directions and it will look very different to different people when you rotate it.

So, the moral of the story is the position of one point on the rigid body is not enough to describe it in 3D space. We need something called as the orientation of a rigid body with respect to the reference coordinate system ok. So, how do we describe the orientation of a rigid body B?

So, first is we attach a coordinate system B to the rigid body B. So, this is the rigid body B. So, we have a reference coordinate system X_A, Y_A, Z_A and an origin O_A we attach another coordinate system X_B, Y_B, Z_B with the origin O_B on the rigid body.

Now, we are only interested in the orientation right now not in the translation, we will come to that later. So, then the origin of O_A and O_B can be at the same place. So, if I can describe to you or specify to you the X_B axis, the Y_B axis and the Z_B axis with respect to the A coordinate system or another way of saying if I can describe the coordinate system B with respect to A, then I could very completely describe the orientation of the rigid body in A ok.

So, again the example of a cube if I tell you that the three edges of the cube let us say labelled as *X*, *Y* and *Z*, if I tell you which way or how to draw this axis on the cube then completely describes the orientation of the cube ok. So, in this figure we are I am also showing some vector *k* and *phi* we will come to this later ok. So, remember there is an axis *k* and an angle *phi* on the rigid body which is useful to represent the orientation of the rigid body.

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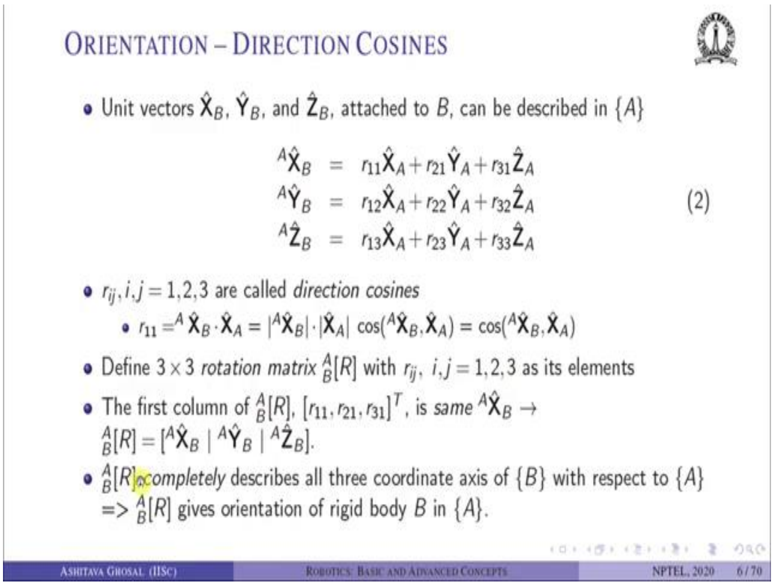

So, let us proceed. So, the unit vectors X_B, Y_B and Z_B are attached to the body B and they can be described in the {A} coordinate system just like the position vector of point P in {A} coordinate system. So, X_B is nothing but the point p. So, it can be written in terms of X_A, Y_A, Z_A. So, project this X_B vector unit vector along X_A, Y_A, Z_A and call the coordinates as r_{11}, r_{21}, r_{31} ok.

So, what are we doing? So, we take a point one unit away along the X_B axis. So, this is the vector ${}^AX_B$, we project on X_A, we project on Y_A and we project on Z_A and when we project the coordinates are r_{11}, r_{21}, r_{31} ok. So, why is it written in r_{11}, r_{21}, r_{31}? Basically ${}^AX_B$ is a column vector ok. So, r_{11} into (1 0 0), r_{21} into (0 1 0), r_{31} into (0 0 1). So, it will form a column vector.

Likewise, I can take a point on the Y_B axis, one unit away from the origin and again project onto X_A, Y_A, Z_A axis and let us call the coordinates r_{12}, r_{22} and r_{32} and finally, we can also take a point on the Z_B axis one unit away and project onto X_A axis, Y_A axis Z_A axis and the coordinates are r_{13}, r_{23}, r_{33} ok. So, these r_{ij}*'s* are called the direction cosines you must have heard of this before, but let us see why are they call direction cosines.

So, let us compute what is r_{11}. So, how do I compute r_{11}? So, we take this first equation and dot product the left and the right-hand side with X_A. So, ${}^AX_B$ dot X_A will be nothing, but magnitude of ${}^AX_B$, magnitude of X_A and the cosine of the angle between ${}^AX_B$ and X_A ok.

So, since these are unit vectors we are only left with the cosine of the angle between ${}^AX_B$ and X_A and this is r_{11}. So, r_{11} is nothing but cosine of an angle between two axis in 3D space. So, we can define this 3 by 3 matrix containing rij and this is called as a rotation matrix ok. So, there are 9 elements r_{11}, r_{21}, r_{31} first column r_{12}, r_{22}, r_{33} second column, r_{13}, r_{23}, r_{33} third column and this is again repeated again.

So, the first column of this rotation matrix is r_{11}, r_{21}, r_{31} and it is the same as the ${}^AX_B$ vector ok. So, effectively what is this matrix ${}^A_B[R]$? It is first column is ${}^AX_B$, second column is ${}^AY_B$ and the third column is ${}^AZ_B$ ok. So, what does this ${}^A_B[R]$ means? It is the rotation matrix describing the rigid body B with respect to the reference coordinate system A.

${}^A_B[R]$ completely describes all the three coordinate of axis of B with respect to A. Because if I know this rotation matrix I know ${}^AX_B$, ${}^AY_B$, ${}^AZ_B$ these are numbers. So, I can draw these three unit vectors X_B, Y_B, Z_B and hence it gives you the orientation of the rigid body B in

A ok. So, the rotation matrix containing the column vectors ${}^{A}X_{B}$, ${}^{A}Y_{B}$, ${}^{A}Z_{B}$ completely describes the orientation of rigid body with respect to the A coordinate system.

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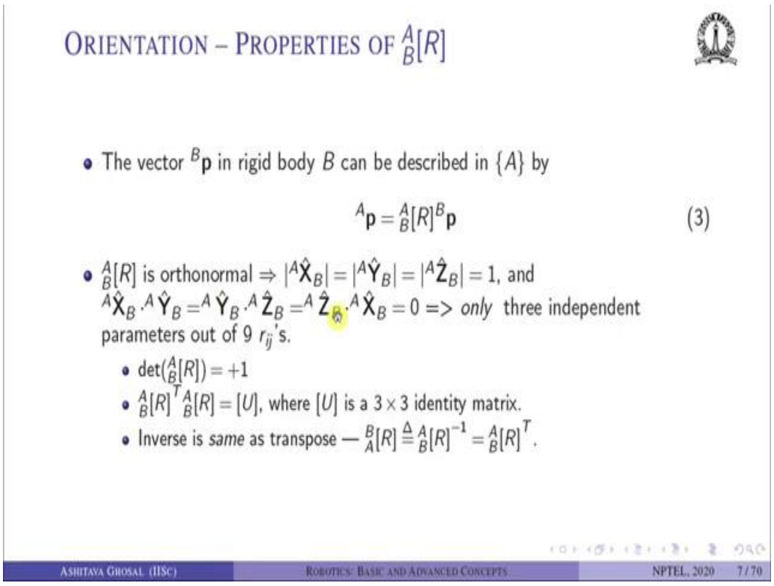

The vector p if it is fixed in the {B} coordinate system can be described in the {A} coordinate system just by this simple pre-multiplication by a rotation matrix. Everybody knows this, we have seen this in many places in mechanics.

So, the rotation matrix one of the property is that it converts a vector in one coordinate system into another coordinate system. A vector in the {B} coordinate system ${}^{B}p$ can be described in the {A} coordinate system by pre-multiplying by the rotation matrix.

The other important property of this ${}^{A}_{B}[R]$, this rotation matrix is that it is orthonormal ok. So, what is an orthonormal matrix? The column vectors are all unit vectors ok. So, the magnitude of the column vectors is 1, and the two column vectors ${}^{A}X_{B}$ and ${}^{A}Y_{B}$ likewise ${}^{A}Y_{B}$ and ${}^{A}Z_{B}$ and likewise ${}^{A}Z_{B}$ and ${}^{A}X_{B}$ are perpendicular to each other.

So, we have three constraints here. So, $(r_{11}^{2} + r_{21}^{2} + r_{31}^{2}\,)$ which represents the column vector ${}^{A}X_{B}$ is equal to 1, it is a unit vector.

Likewise, ${}^{A}X_{B}$ is perpendicular to ${}^{A}Y_{B}$. So, we have six constraint equations three unit vector constraint equations and three perpendicular constraint equations. So, hence out of the 9 r_{ij}'s only 3 are independent ok. So, this is a very important result that in a rotation matrix there are only three independent parameters.

So, remember rotation matrix represents the orientation of a rigid body in the {A} coordinate system, a rotation matrix has three independent parameters, hence the orientation of a rigid body in 3D space can be represented by three independent parameters.

So, let us continue - this ${}^{A}_{B}[R]$ is orthonormal. because of this property determinant of ${}^{A}_{B}[R]$ is always +1 ok. Secondly, we know ${}^{A}_{B}[R]$ inverse into ${}^{A}_{B}[R]$ will be identity matrix; however, if ${}^{A}_{B}[R]$ is orthonormal, the inverse is the same as the transpose ok.

So, ${}^{A}_{B}[R]^T\,{}^{A}_{B}[R]$would be identity matrix or the inverse is the same as the transpose. And what is ${}^{A}_{B}[R]^{-1}$? You know conceptually, ${}^{A}_{B}[R]$ was the orientation of {B} with respect to {A}; ${}^{A}_{B}[R]$ inverse is nothing, but the orientation of {A} with respect to {B} that is what is means by ${}^{A}_{B}[R]^{-1}$.

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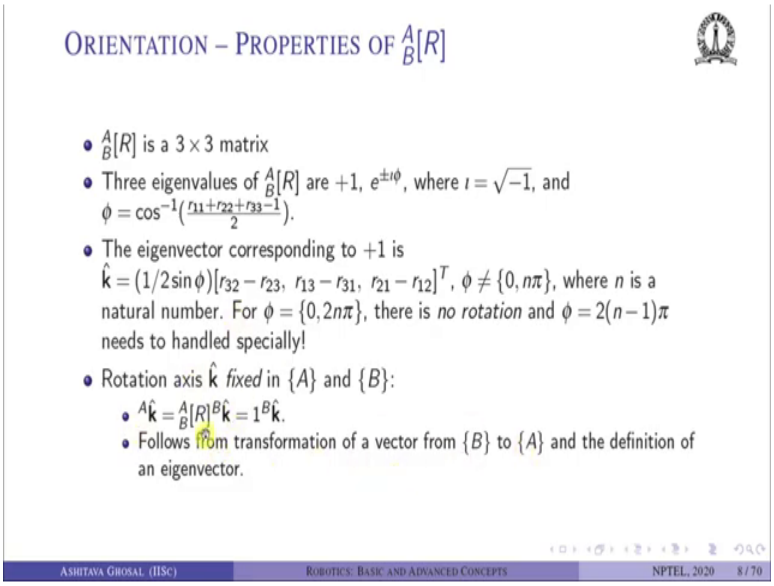

Let us continue - this ${}^{A}_{B}[R]$ is a 3 by 3 matrix ok. So, if it is a 3 by 3 matrix it must have three eigenvalues. You can show that the eigenvalues are plus 1 and *e* to the power $i\ \phi$ where *i* is this imaginary number square root of minus 1. So, what is e to the power $i\ \phi$? We can expand it as $\cos\phi \pm i \sin\phi$ ok.

So, this matrix has one real and two imaginary eigenvalues where this ϕ can be obtained as cos inverse of $(r_{11} + r_{22} + r_{33} - 1)$ divided by 2. So, this will be a homework problem you can derive and show that the three eigenvalues are 1, *e* to the power $\pm\, i\ \phi$ and where ϕ is given by this expression.

Now, corresponding to this real eigenvalue 1 we can also find an eigenvector ok. So, the eigen vector is given by k which is 1 by 2 sin ϕ into this column vector. So, the first element of the column vector is (r_{32} - r_{23}), the second element is (r_{13} - r_{31}) and the third element is (r_{21} - r_{22}).

So, as long as ϕ is not 0 and nπ where n is some natural number, we are not dividing by 0 and we can determine this k vector ok. If ϕ is 0 or 2nπ then actually there is no rotation. So, there is no notion of a rotation matrix and hence there is no notion of k ok. If ϕ is (2n - 1)π then we need to worry about it a little bit; if ϕ is (2n - 1)π, then you can show that there are three real eigenvalues.

The next important property of this axes k is eigenvector k this that this axis k is fixed in {A} and {B}. How do we prove this? We have a vector k in the {B} coordinate system if we pre multiply by ${}^{A}_{B}[R]$, we will get the vector in the {A} coordinate system just like any other property of a rotation matrix it transforms the ${}^{B}k$ into ${}^{A}k$.

Now, if you look at the right-hand side ${}^{A}_{B}[R]$ into ${}^{B}k$ is also equal to 1 into ${}^{B}k$ this is the eigenvalue problem. So, hence ${}^{A}k$ is same as ${}^{B}k$ ok. So, this follows from the definition of a eigenvector and also for the definition of transformation of vector from B to A ok. So, what does this mean? It means that this eigenvector which we have found out k is same either in the {A} coordinate system or {B} coordinate system ok.

So, hence when I orient the object or rotate the objects from {A} to {B} there is a vector k which is remaining fixed and that is what was shown in the original figure here that whenever you have a {A} coordinate system and a {B} coordinate system, we can have a vector k which will take this A coordinate system to the B coordinate system. So, what do we have to do? We have to rotate about this k axis by an angle ϕ. So, this axis k is fixed in both {A} and the {B} coordinate system.

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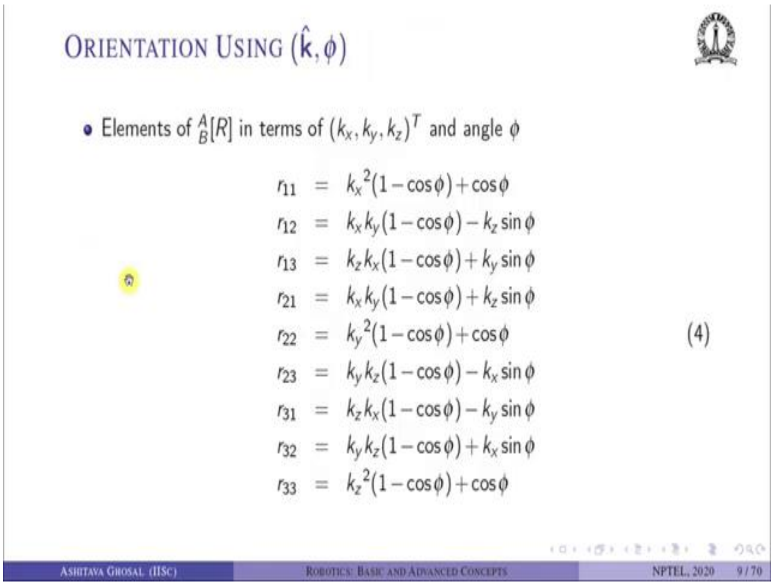

So, we can find k and ϕ given the rotation matrix ${}^A_B[R]$ ok. So, k is the eigenvector corresponding to the real eigenvalue 1 and ϕ is this expression which is cos inverse of (r_{11} + r_{22} + r_{33} - 1) divided by 2.

So, ϕ and k can be obtained from the elements r_{ij} of this rotation matrix. Can we do the opposite? Yes. So, it turns out that if I give you k and ϕ, k as a unit vector meaning (k_x^2 + k_y^2 + k_z^2) is 1 and an angle ϕ; can I find the elements of the rotation matrix?

Yes. So, r_{11} is given as $k_x^2(1 - \cos\phi) + \cos\phi$. So, for example, r_{21} is $k_x k_y (1 - \cos\phi) + k_z \sin\phi$ and so on. So, this will also be a homework problem which you can attempt yourself and derive this.

So, what have we done? Given a rotation matrix, I can find k and ϕ, given k and ϕ, I can find the elements rotation matrix.

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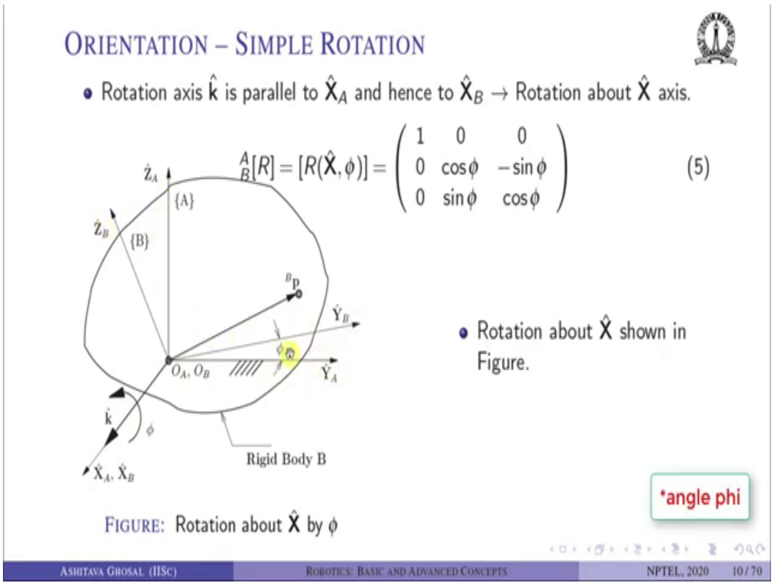

There is also a very useful notion of or extension or use of k and ϕ ok. So, this is let us consider the rigid body B and the rotation axis k is parallel to the X_A axis. So, both X_A and X_B are at the same place. So, that is the fixed axis.

So, now if I rotate about this X_A or X_B axis by ϕ, I can find the rotation matrix by substituting in this previous expression k_x is 1, k_y is 0 and k_z is 0, and rotation angle is ϕ. If you substitute k_x as 1. So, what is r_{11}? We will get 1 – (1- cos ϕ) + cos ϕ

What is r_{23}? ky , k_z is 0. So, we will be left with sin ϕ, k_x is 1, - sin ϕ. So, we have 1 0 0, 1 0 0, cos ϕ - sin ϕ sin ϕ cos ϕ. So, this is the rotation matrix obtained when the fixed axis is along the X axis. So, we are going to use this symbol rotation about X by an angle ϕ will give a rotation matrix ${}^{A}_{B}[R]$ which is 1 0 0, 1 0 0, cos ϕ -sin ϕ, sin ϕ cos ϕ and this is what is shown in this figure.

So, basically, X_A and X_B are at the same place - this is the k axis we are rotating by an angle ϕ about this axis. So, Y_A will go to Y_B, Z_A will go to Z_B, and angle between Y_A and Y_B is this angle.

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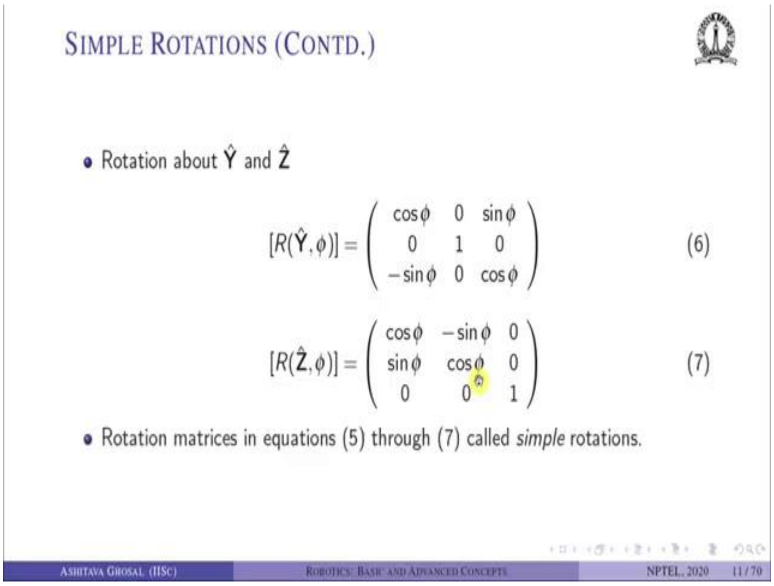

The rotation about Y and Z axis can also be similarly obtained. All we need to do in that general expressions for r_{ij} as a function of k_x, k_y, k_z and ϕ we have to say now that the k axis is 0 1 0, for Z the k axis is 0 0 1 and we can obtain cos ϕ 0 sin ϕ 0 1 0 and so, on.

Similarly, for the Z the last column will be 0 0 1, last row will be 0 0 1 and this is cos ϕ - sin ϕ sin ϕ cos ϕ ok. You may have seen this rotation matrices when we rotate about X, Y and Z axis. So, these rotation matrices are called simple rotations because they are the simplest possible rotations -- we are rotating about the X axis, Y axis and Z axis.

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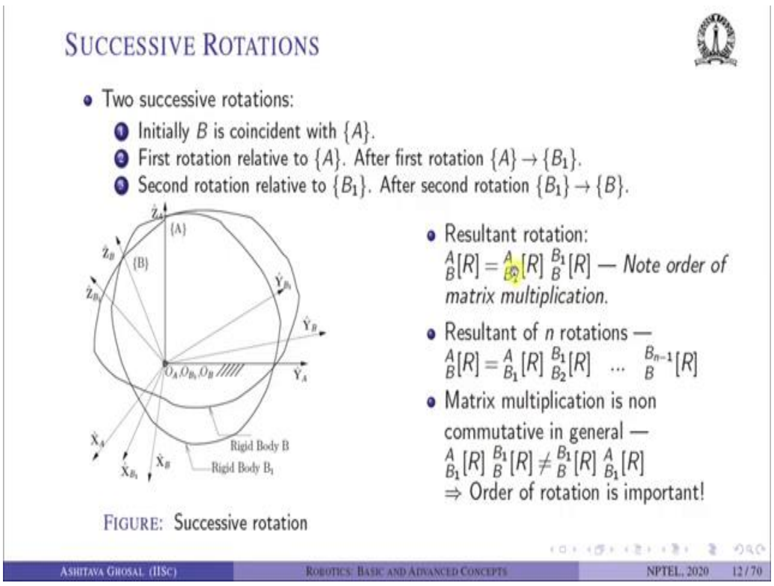

Now, let us look at one more important concept which is of two successive rotations. What do we mean by 2 successive rotations? So, initially we have a rigid body B which is coincident with the {A} coordinate system ok. So, the first rotation is relative to {A} ok. So, after the first rotation the coordinate system {A} goes to {B_1}.

The second rotation is relative to {B_1}, the new coordinate system {B_1} not with respect to the original {A} coordinate system. So, after the second rotation {B_1} goes to {B}. As its shown here we have X_A, Y_A, Z_A the first rotation it goes to X_{B1}, Y_{B1}, Z_{B1} and after the second rotation which is with respect to X_{B1}, Y_{B1}, Z_{B1} we get X_B, Y_B, Z_B

So, the question is what is the resultant rotation after these two successive rotations? Ok. So, the answer is ${}^{A}_{B}[R]$ is ${}^{A}_{B1}[R]{}^{B1}_{B}[R]$ ok. So, we multiply the rotation matrices in the order of the two successive rotations. That is important and this is true whenever we are rotating about axis fixed to the moving rigid body. Here we are rotating about axis which are fixed to the moving body -- it is about the moved coordinate system the second rotation ok.

Generalising we can also show that the resultant of *n* such successive rotations about axis fix to the moving body is product of multiplications of matrices ${}^{A}_{B1}[R]\,{}^{B1}_{B2}[R]]$ all the way till ${}^{B(n-1)}_{B}[R]$ ok. So, again just to stress the matrix multiplication is noncommutative in general. So, ${}^{A}_{B}[R] = {}^{A}_{B1}[R]{}^{B1}_{B}[R]$ which is not ${}^{B1}_{B}[R]{}^{A}_{B1}[R]$.

So, if you switch the order then you do not get the correct rotation matrix. And in our notation where we have leading superscript A which represents the reference coordinate system and B which is the final orientation of the rigid body, the final coordinate system,

.

We can see that this B_1 and B_1 will cancel intermediate coordinate system and we are left with ${}^{A}_{B}[R]$This is just a way to remember what we are doing ok. So, in summary the order of the rotation is important. So, we will multiply the rotation matrix in the order of the rotations.

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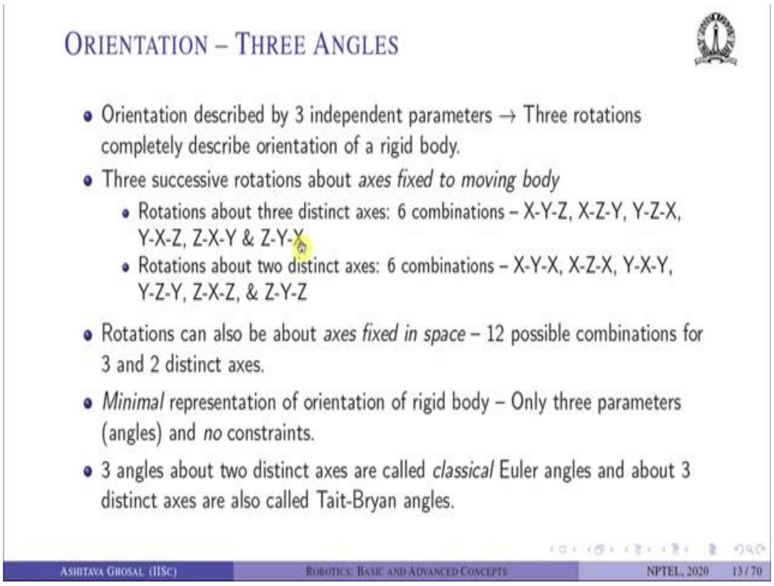

Now we can have enough material to look at orientation in terms of three angles. We have the tools and techniques to describe orientation of a rigid body in terms of three angles.

So, we know that orientation described by 3 independent parameters remember in r_{ij} there were nine r_{ij}'s, but then there was the 6 constraints. So, there were actually only 3 independent parameters ok. So, can we obtain the rotation matrix in terms of 3 independent parameters?

Answer is yes we can do three successive rotations about axes fixed to the moving body. We can have rotations about X, Y, Z and since the order of multiplication is important we could have got another rotation matrix if we did X, Z, Y or Y, Z, X. So, there are six different possible combinations of rotations about three distinct axes.

We can also represent the orientation of a rigid body by rotation about two distinct axes we are also 6 combinations. So, we can do first rotation about X, then a rotation about Y and then again a rotation about X. We could have also done Z, Y, Z or Z, X, Z. So, that are 12 possible combinations of rotations about three axes -- in one case 3 distinct axes and one case 2 distinct axes and we will get three parameters in each one of them.

So, these three rotations completely describe the orientation of the rigid body. We can also do rotations about axes fixed in space we are not going to discuss it, but we can see that there are 12 possible combinations for 3 and 2 distinct axes where we rotate about the original fixed axes in space the X_A, Y_A and Z_A axis.

So, as you can see we have exactly three rotations θ_1 about X, θ_2 about Y and θ_3 about Z or some three angles about say let us say two axes X, Y and X and we can describe the orientation of the rigid body. So, this is a minimal representation of orientation of a rigid body only three parameters three angles and no constraints are used ok.

So, historically three angles about two distinct axes are called classical Euler angles and about three distinct axes are also called Tait-Bryan angles. In many textbooks all of these are called Euler angles ok. So, this minimal representation of orientation using three successive rotations are these three Euler angles.

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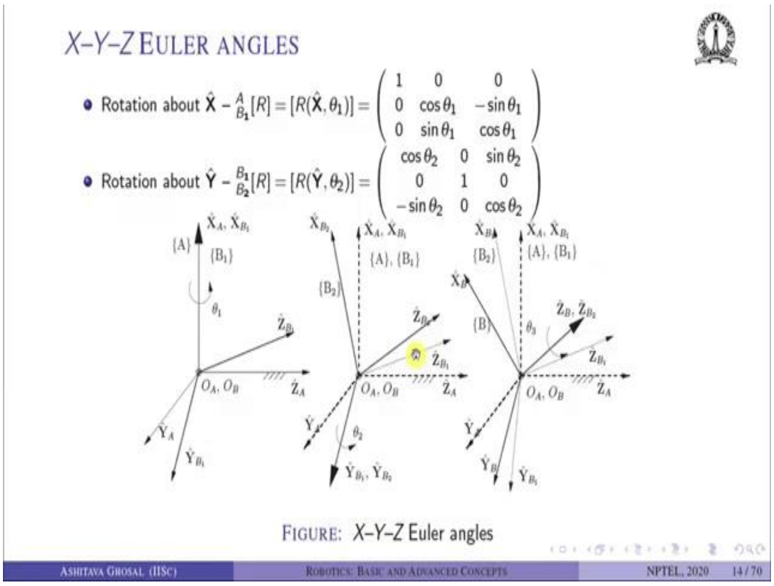

So, let us take an example if you have X, Y, Z Euler angles what does it mean? We are first going to rotate about X, then we are going to rotate about the moved Y and then third we are going to rotate about the moved Z ok. So, the rotation about X is ${}_{B1}^{A}[R]$, this is rotation of X by an angle θ_1, we can find this rotation matrix it is a simple rotation about by θ_1 about the X axis. So, you will get 1 0 0, 1 0 0, cos θ_1 -sin θ_1, sin θ_1 cos θ_1.

So, what is happening? In this figure here. So, we have X_A and X_{B1} aligned Y_A will go to Y_{B1}, Z_A will go to Z_{B1} and the rotation is angle θ_1. The second rotation is about the moved

Y axis. So, what do we mean by the moved Y axis? About Y_{B1}. So, Y_{B1} and Y_{B2} are at the same place.

So, Z_{B1} will go to Z_{B2} and X_{B1} will go to X_{B2}. So, this is the $\{B_2\}$ coordinate system. We are discussing orientation and rotations, so, the origins of all the coordinate systems are at the same place. So, the second rotation is about Y axis by θ_2. So, we can again get this rotation matrix which will be cos θ_2 0 sin θ_2 0 1 0 - sin θ_2 0 cos θ_2 ok.

The third rotation is about the moved Z axis that is why it is called X, Y, Z Euler angles. So, pictorially what is happening? So, if you can see that the third rotation is Z_{B2}. So, Z_{B2} and Z_B are at the same place we are rotating by an angle θ_3. So, hence X_{B2} will go to X_B, Y_{B2} will go to Y_B, Y_{B1} and Y_{B2} were at the same place and Z_B and Z_{B2} are at the same place. So, the final coordinate system is X_B, Y_B, Z_B ok.

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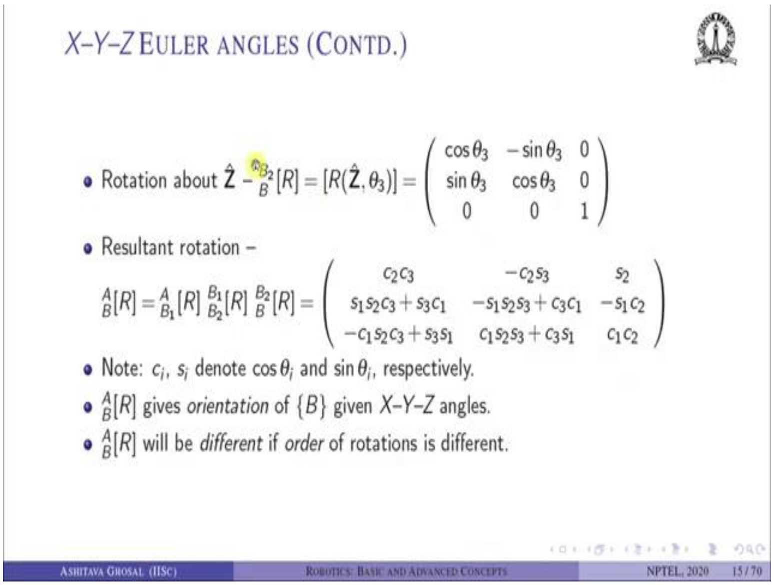

And the rotation matrix obtained by rotating about the Z axis which will take from B_2 to B is given by cos θ_3 -sin θ_3 0, sin θ_3 cos θ_3 0, 0 0 1. So, what is the resultant rotation from A to B which is the product of the three rotation matrices in the order that you did the rotation? So, we went from A to B_1, B_1 to B_2, B_2 to B.

So, again in our notation you can see that this B_1 and B_1 sort of cancels out in your head B_2 and B_2 will cancels out and we have left with A and B. So, if you multiply these three rotation matrices, we will get the rotation matrix which contains θ_1, θ_2 and θ_3 ok. So, in this matrix c_2 means cos θ_2, c_3 means cos θ_3, short form. s_1 means sin θ_1. So, c_i and s_i denote cos θ_i and sin θ_i, respectively.

So, this matrix c_2c_3 - c_2s_3 s_2, $-s_1c_2$ c_1c_2 etcetera gives the rotation matrix which describes the orientation of B with respect to A obtained after three Euler angle rotations and that to about X, Y and Z axis. So, if you were to do X, Y, X or Z, Y, X some other sequence, then the rotation matrix will be different -- because why matrix multiplication is not commutative.

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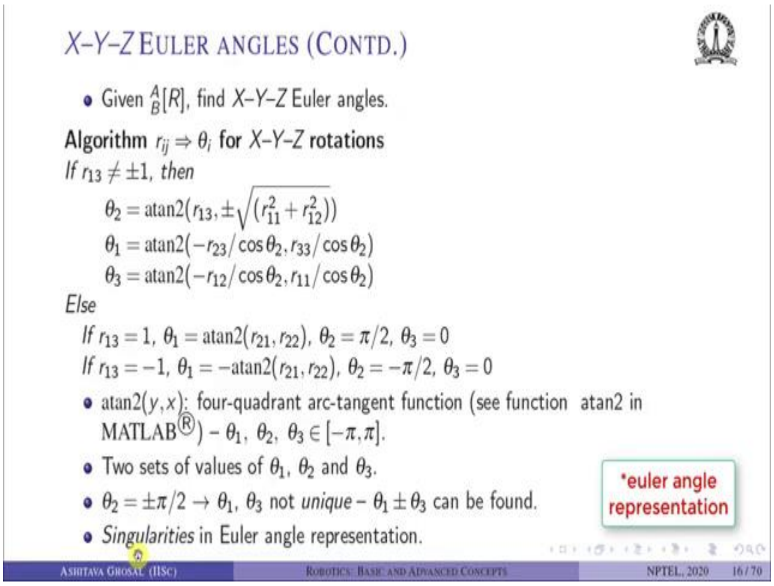

So, I have shown you that if I tell you what is the sequence of rotation about X, Y, Z and rotation is θ_1, θ_2 and θ_3 about this X, Y and Z axis, I can find the rotation matrix. Can I do the reverse? Yes. So, if I give you a rotation matrix with numbers ${}^A_B[R]$ following the properties of the rotation matrix, meaning unit vectors and orthogonality and so on, can I find the X, Y, Z Euler angles? The answer is yes.

So, how do we do that? We look at this rotation matrix which we have obtained for X, Y, Z Euler angles. So, you can see here this term r_{13} is sin θ_2 ok. So, if I give you some number, θ_2 is sin inverse of that number correct. But instead of doing sin inverse what we will do is we will look at both of these r_{11}, r_{12} and r_{13}.

So, if I take the square of these two and take the square root we will be left with cos θ_2. So, we will have sin θ_2 and cos θ_2 and then we can use something like tan inverse ok. So, this is what is explained here. So, θ_2 is atan2 (r_{13}, $\pm \sqrt{(r_{11}^2 + r_{12}^2)}$) .

Why do we use atan2? Because this function atan2 (y, x) it takes the y coordinate and the x coordinate, it gives you the angle in the correct quadrant ok. So, it is basically doing tan inverse y by x; however, it looks at the sign of y and x. So, for example, if this was -1 and -1 then we will get in the third quadrant, but tan inverse y by x of both y and x is -1 will give you still 45 degrees. So, it will give you the angle in the correct quadrant.

So, once we know θ_2 let us go back and see. So, now, I know θ_2, I can take these two terms which is r_{23} and r_{33} divide by cos θ_2. I can again do atan2 of this term and this term. This is a minus sign. So, we will take care of the minus sign ok.

So, atan2 of r_{23} and r_{33} of course, with a minus sign. So, that is what is shown here -- θ_1 is atan2 ($- r_{23}/\cos\theta_2$, $r_{33}/\cos\theta_2$) and likewise if θ_2 is known we can now find θ_3 using again atan2 of these two terms r_{11} and r_{12} which is what is exactly shown here ($- r_{12}/\cos\theta_2$, $r_{11}/\cos\theta_2$).

Now, there is a small problem -- which is that if this r_{13} is ± 1 Meaning what? θ_2 is $\pm\pi/2$ ok. So, if $\theta_2 = \pm\pi/2$ cos θ_2 will be 0 and we cannot divide this term by cos θ_2 which is what we were doing. We cannot divide both of these terms by cos θ_2.

So, if r_{13} is 1 which is θ_2 is $\pi/2$, then we can just do atan2 (r_{21}, r_{22}) what do we do? We look at these two terms this one and this one this one and this one ok. So, if θ_2 is $\pi/2$, you can see that this is sin ($\theta_1 + \theta_3$) and cos ($\theta_1 + \theta_3$) we will get. So, we assume or make a convention that θ_3 is 0 and θ_1 is atan2 (r_{21}, r_{22}). If r_{13} were -1 then we have to suitably change. So, θ_1 is - atan2 (r_{21}, r_{22}), θ_2 is still $-\pi/2$ and θ_3 is 0.

So, what does this algorithm tell you that there are two sets of values of θ_1, θ_2, t θ_3 for a given ${}^A_B[R]$ ok. If θ_2 is $\pm\pi/2$, θ_1 and θ_3 is not unique we can only find $\theta_1 \pm \theta_3$ ok.

We make the convention that in either case θ_3 is assumed to be 0 and we obtain the unique θ_1. This condition θ_2 is $\pm\pi/2$ are called singularities. So, there always are singularities in any Euler angle representation where we cannot find the angles uniquely.

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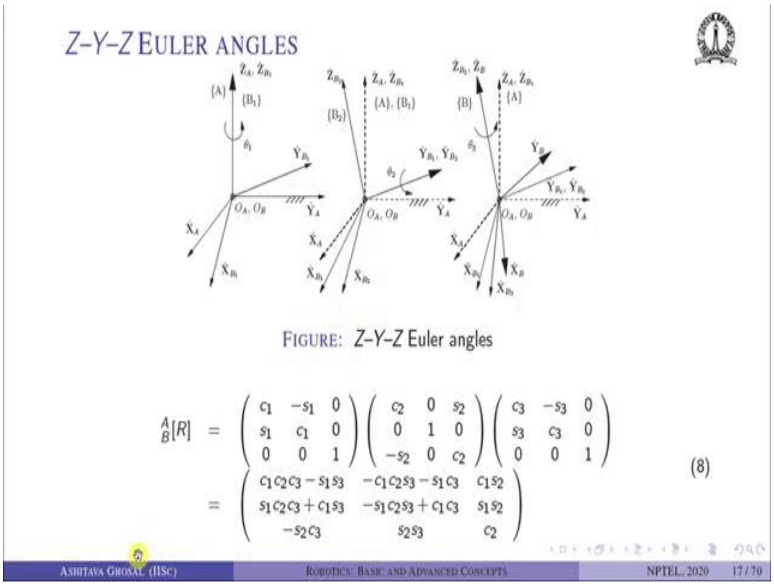

Let us look at another Euler angle which is the Z, Y, Z Euler angles. So, in this case what is happening? The first rotation is about Z axis. So, Z_A and Z_{B1} are at the same place, X_A goes to X_{B1}, Y_A goes to Y_{B1} and the rotation is θ_1. The next rotation is about Y axis.

So, if I rotate about the moved Y axis which is Y_{B1} and Y_{B2} are at the same place the rotation is θ_2, the Z axis will go to Z_{B2} and X axis X_{B1} will go to X_{B2} . And the last rotation is about the Z axis of the moved coordinate system again. So, Z_{B2} and Z_B are at the same place. So, this is the rotation by θ_3.

So, now X_{B2} will go to X_B and Y_{B2} will go to Y_B and Z_B remain same. So, after this three successive rotations the {A} coordinate system will go to {B} coordinate system which is X_B, Y_B, Z_B and what is the resultant rotation matrix which is nothing, but multiplication of three simple rotations about Z, Y and Z Euler angles ok. So, we will get some terms which are c_1, c_2, c_3, $-s_1$, s_3 and so, on the r_{33} term will be c_2 ok.

Why are we looking at Z, Y, Z Euler angles? We will see later that this occurs in many manipulators where we have intersecting wrist. The last three joint axes in a robot are intersecting that is called as an intersecting wrist and that can be very easily modelled by Z, Y, Z Euler angles ok.

So, what have we done we have shown that if you rotate about Z, Y, Z by θ_1, θ_2, θ_3 respectively I can obtain a rotation matrix of this form. So, r_{33} again is c_2. So, if I give you some number here ok, so, what is that θ_2 that is cosine inverse of that number ok.

Likewise, once I know θ_2, I can divide these two expressions by sin θ_2 and do atan2 and find θ_1. Likewise, I can divide by here and find atan2 and find θ_3. So, that is the basic idea - very similar to what we have done before.

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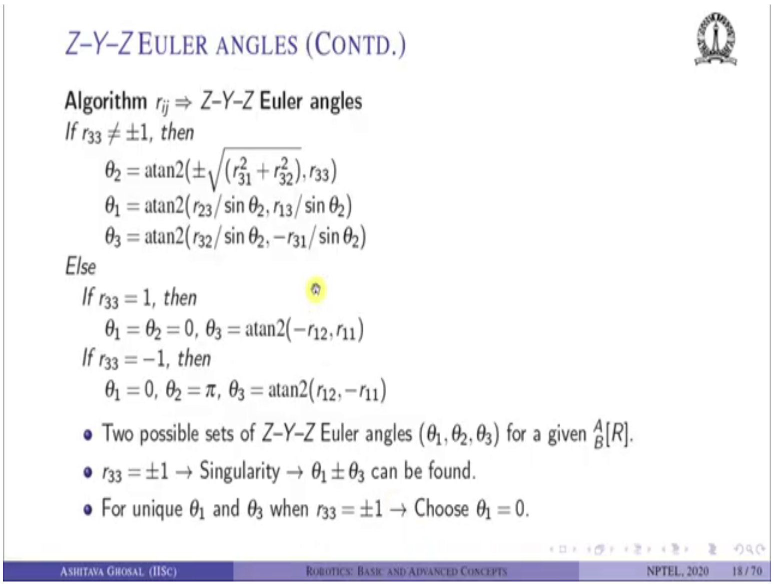

So, we can find an algorithm that if I give you r_{ij}'s what are the Z, Y, Z Euler angles? So, if $r_{33} \neq \pm 1$. cos $\theta_2 \neq$ to 0 or 180, then θ_2 is atan2 ($\sqrt{(r_{31}^2 + r_{32}^2)}$, r_{33}) is that correct?

Yes because I can take the square of these two and we will be left with sin θ_2, ± because square root is there and we will get cos θ_2 and hence we can find θ_2. We can then divide by sin θ_2, these r_{23} and r_{13} and find θ_1 and then we can divide r_{32} by sin θ_2 and - r_{31} by sin θ_2 and find θ_3.

So, given r_{ij}, the elements of this rotation matrix, I can find θ_2, θ_1, θ_3, if $r_{33} \neq \pm 1$. Again, if r_{33} is equal to 1 or -1, I cannot divide these quantities. So, sin θ_2 will be 0. So, again we have a convention which says θ_1, θ_2 is equal to 0 when r_{33} is equal to 1 and θ_3 is atan2 (-r_{12}, r_{11}). If r_{33} is -1, we have a convention that θ_1 is 0, θ_2 is π and θ_3 is atan2 (r_{12}, - r_{11}) ok.

So, again given a rotation matrix r_{ij}, we can get two possible sets of Z-Y-Z Euler angles, these two possible sets comes from here ±. If r_{33} is ±1, we have a singularity, only $\theta_1 \pm \theta_3$ can be found and for unique θ_1 and θ_3, we choose θ_1 to be 0 -- this is by convention.

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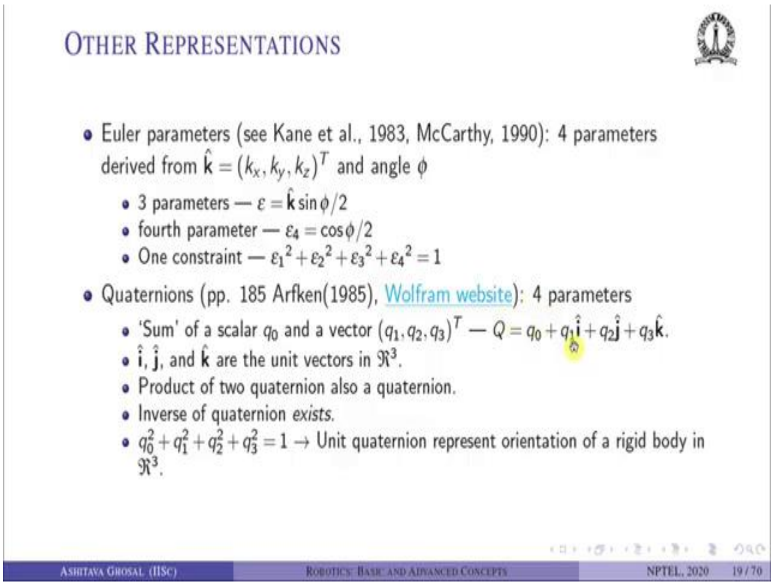

Now, let us look at some other representation of orientation. I have shown you rotation matrix with r_{ij}, I have shown you k and ϕ and I have shown you Euler angles - three Euler angles we can also have something called as a Euler parameter. We can look at these books by Kane and McCarthy.

So, these are four parameters which are derived from k and ϕ they are not exactly equal to k and ϕ the three parameters are k into sin $\phi/2$ and the fourth parameter is cos $\phi/2$ ok.

We have we still have one constraint $\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1$. So, ε here is a vector with ε_1, ε_2, ε_3. The ε_1, ε_2 and ε_3 and ε_4 are called the Euler parameters. They are derived from k and ϕ, but not exactly same as k and ϕ and it turns out to have some small advantage because it is sin $\phi/2$. So, we do not have problems of π basically.

So, angle ϕ is π, sin $\phi/2$ is still well defined it is 1 and we can define and derive and various things with ϕ as π.

We can also have something called quaternions. You can look at this book by Arfken, this is also 4 parameters these are typically labelled as q_1, q_2, q_3 which is a vector and a scalar q_0. It is in some sense sum of a scalar and a vector. So, it is neither a scalar or a vector, but it is often written as $q_0 + q_1\, i + q_2\, j + q_3\, k$ where *i*, *j* and *k* are the unit vectors along the X, Y and Z axis.

It turns out that if you have two quaternions Q_1 and Q_2 the product of two quaternions is also a quaternion. The inverse of a quaternion also exists and the square of q_0, sum of the squares of q_0, q_1, q_2, q_3. So, $q_0^2 + q_1^2 + q_2^2 + q_3^2$ square is 1 and this is called as a unit quaternion and unit quaternion represents the orientation of a rigid body in 3D space ok.

There are certain advantages in using a quaternion specifically this is extensively used is spacecraft dynamics orientation of a spacecraft when it is orbiting the earth or some other place. The Euler parameters are also used extensively in motion planning, we will see later they have certain advantages of relating Euler parameters and angular velocities.

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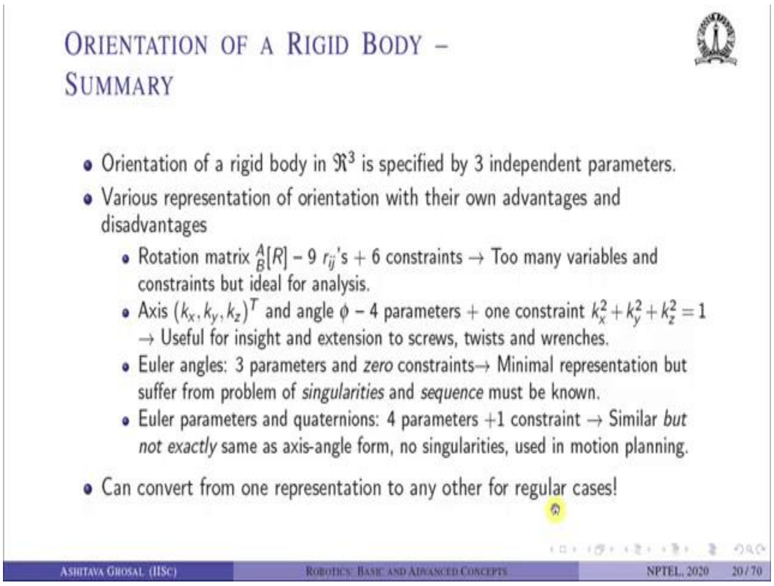

In summary the orientation of a rigid body is specified by 3 independent parameters in 3D space. There are various representation of orientation with their own advantages and disadvantages. So, the rotation matrix ${}^A_B[R]$ contains 9 r_{ij} 's plus 6 constraints. It is not really useful to carry along 9 variables and 6 constraints for any computation work, but it is ideal for analysis.

We can also have axis k_x, k_y, k_z components and angle ϕ. This is a 4 parameter representation of orientation with one constraint. The $k_x^2 + k_y^2 + k_z^2$ is 1, *k* is a unit vector. This is very useful for insight and extension to screws, twists and wrenches. So, as I showed you can think of rotating a rigid body from {A} coordinate system to a {B} coordinate system by rotating about this k axis by an angle ϕ and later on we will see that this can be extended to rotation and translation.

We can also have Euler angles which are 3 parameters and zero constraints. So, this is a minimal representation of orientation; however, it suffers from the problem of singularities. Any set of three Euler angles we will have this singular configurations singularities. More so, we also need to know what is the sequence of rotation. So, you have to tell me whether it is X, Y, Z or Z, Y, X or X, Z, Y, because every time we will get different rotation matrices.

Finally, we had Euler parameters and quaternions these are also 4 parameters plus 1 constraint they are similar, but not exactly same as k and ϕ. There are no singularities and its often used in motion planning. Finally, we can convert one representation of orientation to any other representation of orientation for regular cases. So, if there are no problems like singularities and so, on.

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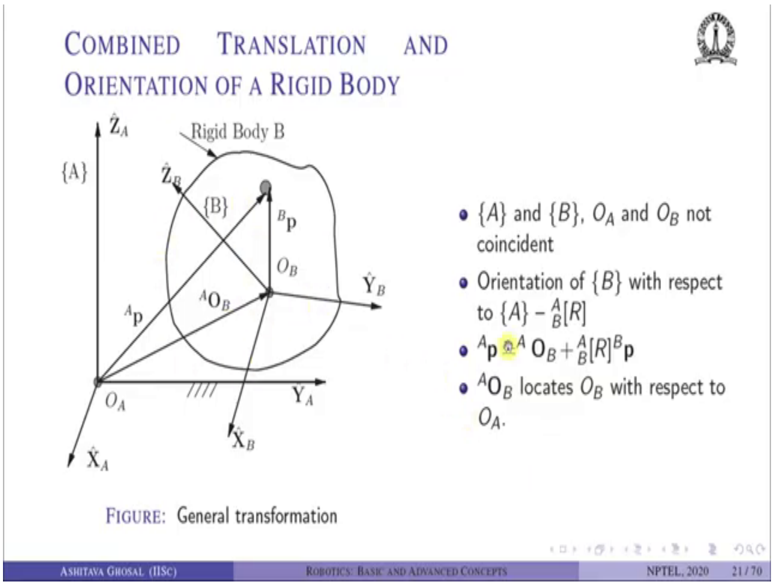

Once you have a representation of orientation of a rigid body now let us look at combined translation and orientation of a rigid body. So, what do we have? We have this rigid body B with a coordinate system X_B, Y_B, Z_B and an origin O_B we have a reference coordinate system X_A, Y_A, Z_A with a origin O_A and labelled {A} and this is labelled {B}.

So, the origins of {A} and {B}, O_A and O_B are not coincident anymore because we are also looking at translation. When we were looking at orientation O_A and O_B were at the same place. So, what is given? The orientation of {B} with respect to {A} this rotation matrix is known or can be obtained. If you take a look at a point on this rigid body given by *Bp*, we can describe this point in the {A} coordinate system by this simple vector addition.

So, ${}^{A}p$ which is this vector is nothing but ${}^{A}O_P$ the vector from origin O_A to O_B plus this vector, but pre multiply by rotation matrix because this vector is described in the {B} coordinate system we cannot add two vectors in two different coordinate system, but fortunately we can just pre multiply by a rotation matrix and get this vector in the {A} coordinate system and then we can add ok.

As meant as mentioned ${}^{A}O_B$ locates the origin O_B with respect to O_A. So, this is the general transformation which consists of a rotation ${}^{A}_{B}[R]$ and a translation ${}^{A}O_B$ ok. So, from this, we can look at this very important concept called homogeneous transformation.

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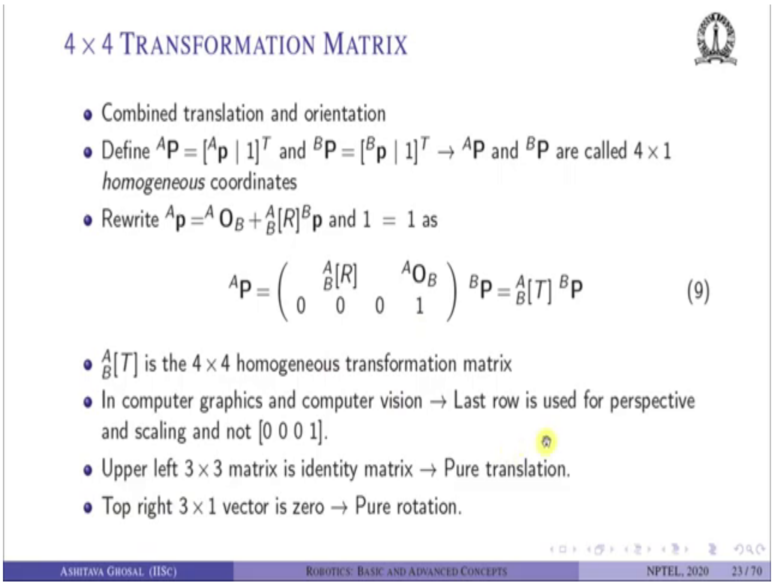

So, we have combined translation and orientation – it is a vector equation, but let us see which we can write it in a more compact form and what do we do? We define a new vector ${}^{A}P$ which is ${}^{A}p$ and we concatenate it with 1. So, ${}^{A}p$ was a 3 by 1 column vector we add one more row of 1.

So, it becomes a 4 by 1 column vector. Likewise B capital P is *Bp*, small *p*, and we have another row which is 1. This ^{A}P and ^{B}P are called 4 by 1 homogeneous coordinates ok.

Strictly speaking in this instead of 1 if it was some other variable called *w*. So, we have X, Y, Z for the top 3 and w then it is homogeneous, but *w* could be equal to 1 which is what we do in robotics.

So, we can rewrite this vector equation *Ap* is equal to $^{A}O_{B}$ + $^{A}{}_{B}[R]$ $\underline{Bp}$, adding one more equation which is obviously true, which is 1 equals 1. And then we have ^{A}P is some matrix into ^{B}P and what is this matrix? The top 3 by 3 is the rotation matrix, the last column is $^{A}O_{B}$ and the last row is 0 0 0 1.

So, if you expand this, you can show that this equation and this equation are exactly the same. So, this 4 by 4 matrix is labelled as [T] and this is called as the 4 by 4 homogeneous transformation matrix. This is a very important matrix in robotics, it is also a very important matrix in many other fields.

So, for example, in computer graphics and computer vision the last row is not 0 0 0 1. If you put some numbers here, the last row can be used to show perspective and if you do not use 1, but you use some numbers along the diagonal and this point and this 4,4 element also we can represent scaling ok.

So, what is perspective? Everybody knows. So, when you look at 2 parallel lines it looks like as if it is meeting at infinity that is the perspective effect and you can show perspective or you can implement perspective and scaling by changing the last row. So, in robotics, the top 3 x 3 $^{A}_{B}[R]$ is the rotation matrix $^{A}O_{B}$ is the translation. If the top 3 by 3 matrix sub-matrix is identity then we have pure translation ok.

If $^{A}O_{B}$ is 0 and we have something here then it is pure rotation ok. So, this transformation matrix contains pure translation, pure rotation and by suitable modification of the last row can be also shown to contain the effect of perspective and scaling.

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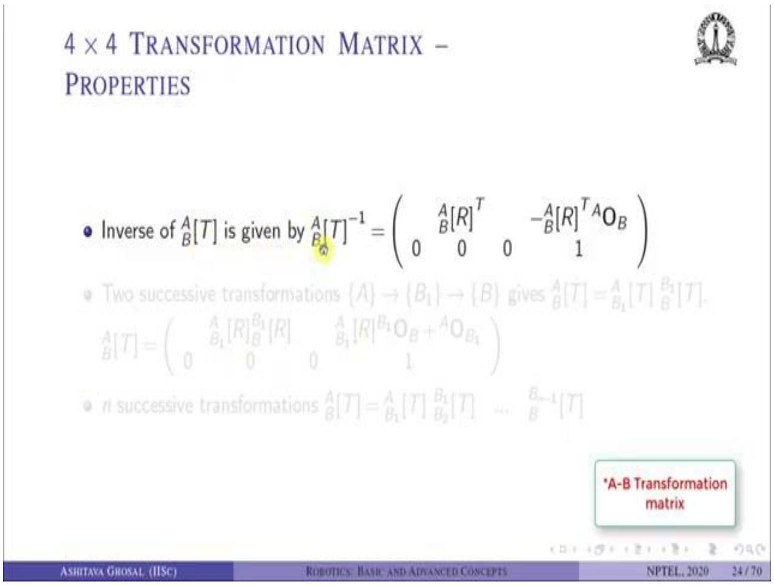

Let us look at some of the properties of this 4 by 4 homogeneous transformation matrix. So, first thing is the inverse of ${}^A_B[T]$ which is denoted by ${}^A_B[T]^{-1}$can be obtained in closed form. I do not need to use brute force elimination for this 4 by 4 transformation matrix.

The top rotation matrix in this ${}^A_B[T]^{-1}$ inverse is the transpose of the rotation matrix and the last column is - ${}^A_B[R]^T \; {}^AO_B$. So, this will be a homework problem you can show that the inverse of the 4 by 4 homogeneous matrix can be easily obtained.

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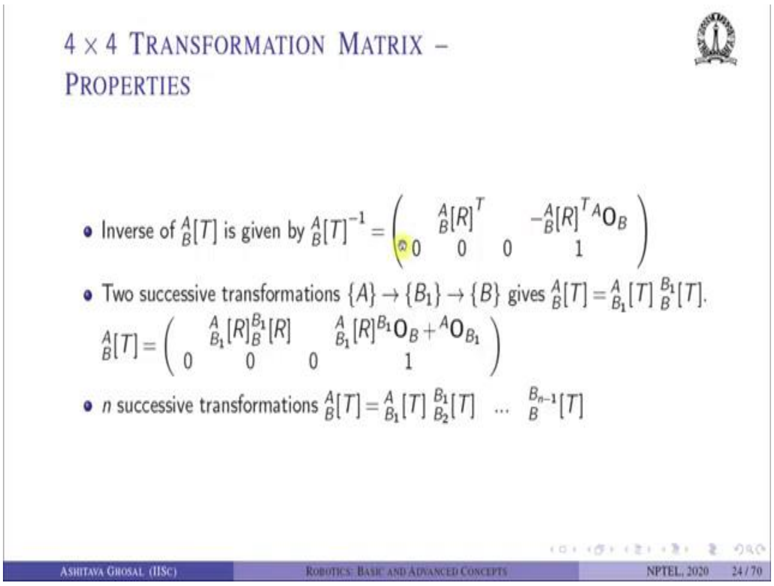

You can have two successive transformations we can go from A to B_1, B_1 to B. So, in this case what is the transformations? It is both translation and rotation. So, then the resultant transformation ${}^{A}_{B}[T]$ is nothing but the product of these two homogeneous transformation matrices. So, ${}^{A}_{B1}[T]{}^{B1}_{B}[T]$.

Moreover, we can have closed form expressions for this resultant successive transformations. The rotation matrix on the top is nothing but the product of the two successive rotation matrices and the translation is again basically summation of two vectors in proper coordinate system. So, ${}^{B1}O_B$ + ${}^{A}O_{B1}$, but ${}^{B1}O_B$ cannot be directly added to ${}^{A}O_{B1}$, we pre multiply by ${}^{A}_{B}[R]$and then we can add.

So, likewise if you have *n* successive transformations A to B_1, B_1 to B_2 finally, B_{n-1} to B, the resultant transformation is nothing but the product of the 4 by 4 homogeneous transformation matrices in the order of the transformations - in order of the translation and orientations that you have done.

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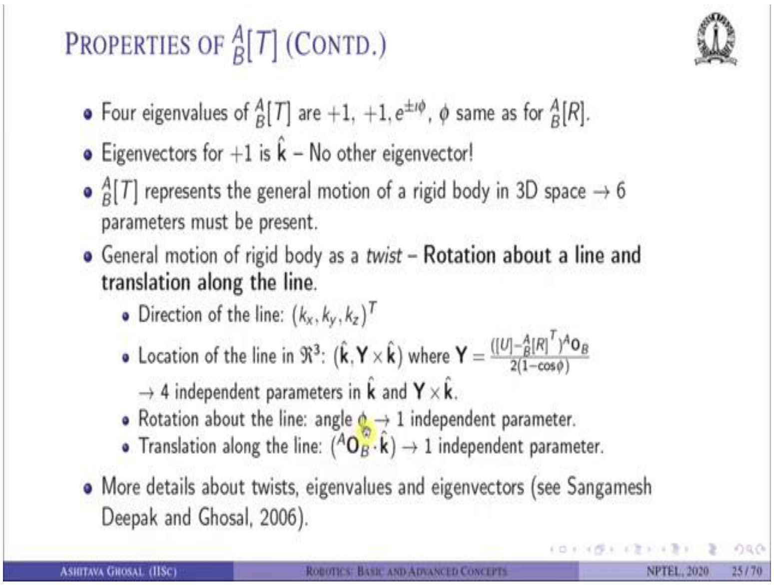

Let us continue the transformation matrix is a 4 by 4 transformation matrix it is of dimension 4 by 4. So, clearly there must be 4 eigenvalues. So, it turns out that the eigenvalues are +1, +1 repeated eigenvalues and *e* to the power $\pm\, i\, \phi$. So, *e* to the power $\pm\, i\, \phi$ is $\cos\phi \pm i \sin\phi$ and what is ϕ? It is the same as what we obtained for the rotation matrix.

So, it is some $\cos^{-1}$ of the trace of matrix and so on minus 1 divided by 2. The eigenvectors corresponding to +1 and + 1 is only one -- this is very interesting property. So, we have two repeated eigenvalues, but there is only one k - one axis of rotation there is no other eigen vector ok.

${}^{A}_{B}[T]$ this homogeneous transformation matrix represents the general motion of a rigid body in 3D space. Why? Because it contains both the translation remember ${}^{A}O_{B}$ and the rotation. So, it contains all the 6 parameters which are 6 degrees of freedom of the rigid body in 3D space.

So, we should be able to figure out what are the 6 parameters in this ${}^{A}_{B}[T]$transformation matrix and the answer is reasonably interesting. That we can show that the general motion of a rigid body can be represent as a twist what is a twist? It is a rotation about a line and translation along the line.

So, the direction of the line is this k_x, k_y, k_z, the axis, the location of the line in 3D space can be shown to be derived from k and some Y x k vector ok. So, where Y is obtained from the identity matrix, transpose of the rotation matrix the translation and 2 into (1 -cos ϕ) division. So, there are four independent parameters in k and Y x k why? There are 2 in this k see unit vector and Y x k is perpendicular to k ok. So, k dot Y x k is equal to 0 ok.

There are four independent parameters in the location of a line, the line in 3D space is located by four independent parameters if you think about it. So, we can have $y = m_1 x + b_1$ and $z = m_2 x + b_2$. So, m_1, m_2, b_1, b_2 four independent parameters describe a line in 3D space and a nice way is to show in terms of k and Y x k ok.

The last two independent parameters are rotation of the angle ϕ which is also an independent parameters and the translation along the line which is ${}^{A}O_{B}$ dot k. So, there are 6 independent parameters here 4 in the line, 1 rotation and 1 translation. So, you can look at more details about twists eigenvalues and eigenvectors in several papers, we can look at a paper which we published in 2006.

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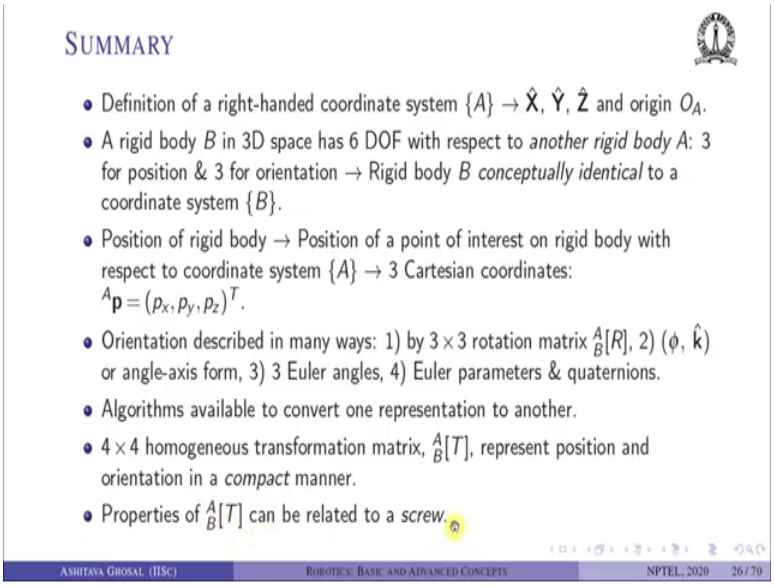

So, in summary, we first need to define a right-handed coordinate system X, Y and Z which is the reference coordinate system to represent a rigid body in 3D space. Coordinate system means X, Y, Z and origin O_A we are going to label all these coordinate systems with {A}, {B}, {C} etcetera because there will be many such coordinate systems.

Rigid body in 3D space has 6 degrees of freedom with respect to another rigid body {A}. So, 3 for position and 3 for orientation. So, the rigid body B is basically conceptually identical to a coordinate system {B}. We are not really interested in the shape and size and weight and other properties of the rigid body at this stage.

The position of a rigid body is nothing but the position of a point of interest on the rigid body with respect to the coordinate system {A}. So, these are most of the time represented by three Cartesian coordinates p_x, p_y, p_z which are nothing but the projections of this vector ${}^Ap$ from the origin of the reference coordinate system to the point and projecting this vector along X, Y and Z axis.

Orientation of a rigid body can be described in many ways, first a 3 by 3 rotation matrix, second angle ϕ and k or the angle axis form. We can also describe using 3 Euler parameters and finally, using sorry 3 Euler angles and we can finally, describe using Euler parameters and quaternions. There are algorithms available to convert one representation to another.

If you combined both the translation and orientation or position and orientation of a rigid body, they can be combined in this 4 by 4 homogeneous transformation matrix which represents the position and orientation in a compact manner and as shown last slide that the properties of ${}^{A}_{B}[T]$ can be related to a screw. Basically a line about which this rigid body rotates and about which it translates.

So, with this we will come to an end of this first lecture, in the next lecture we will look at how to represents the elements of a robot basically the joints and links which make up the robot.