Robotics: Basics and Selected Advanced Concepts
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Lecture - 42
Kinematics and Static Analysis

Welcome to this NPTEL lectures on Robotics: Basic and Advanced Concepts. In this week we are looking at over constrained mechanisms and deployable structures. In the last lecture I had shown you several examples of over constrained mechanisms and deployable structures.

I had also shown you a method based on using natural coordinates or Cartesian coordinates and the Jacobian of the constraint equations to obtain the degrees of freedom of the over constrained mechanisms. In this lecture we look at kinematic and static analysis of over constrained mechanisms.

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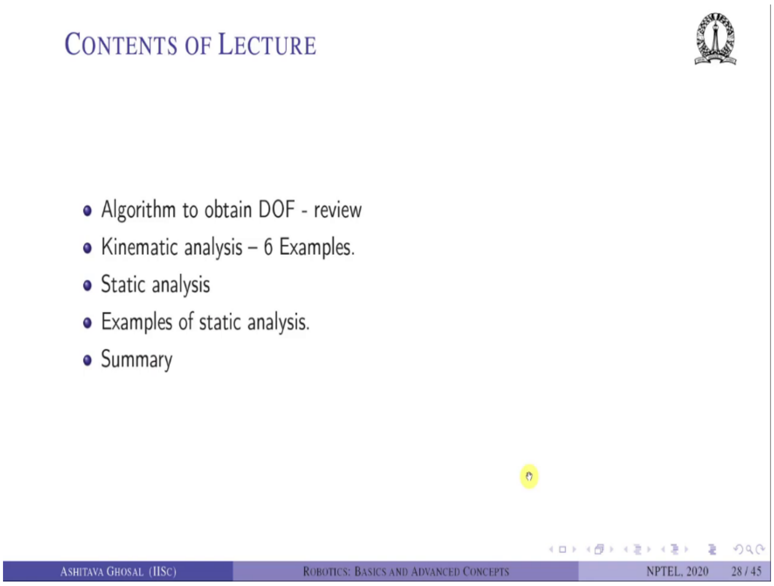

So, the contents of the lecture basically, will quickly review the algorithm to find the actual degree of freedom of an over constrained mechanism using natural coordinates and Jacobian of the constraint equations.

I will show you several examples of kinematic analysis, actually six examples. Then I will show you how we can use this mathematics that we have evolved ok, to do static analysis of over constrained mechanisms. I will show you some examples of static analysis and then we will conclude.

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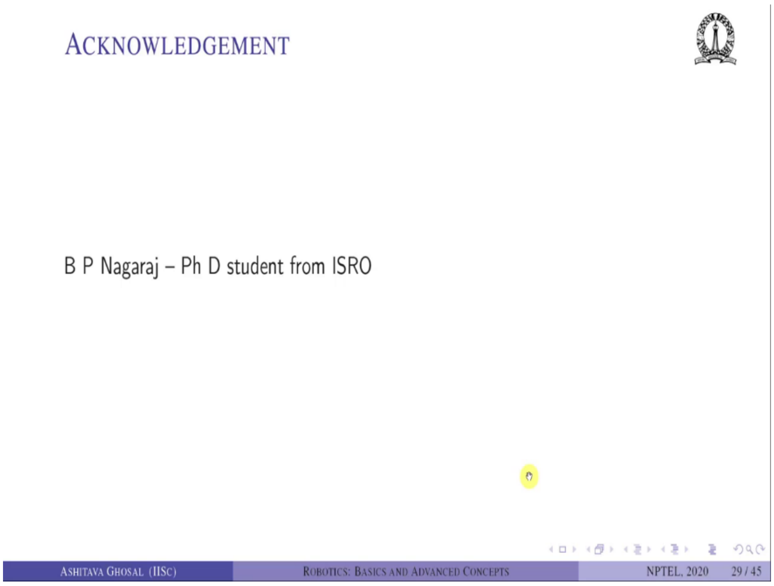

This work is also the result of the research done by Doctor B P Nagaraj who was a Ph D student in our department of long time back. He is now in ISRO.

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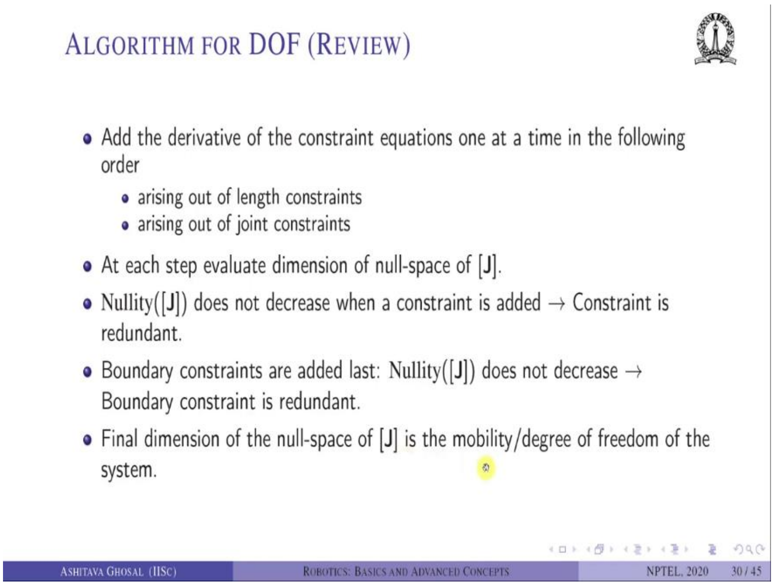

So, let us quickly review the degree of freedom and how to obtain the actual degree of freedom, in an over constrained mechanism. So, as I said we obtained the constraint equations in terms of natural coordinates, we add the derivative of the constraint equations one at a time in the following order.

We first use all the constraint equations arising from the length constraints, find their derivative an added, then we use all the constraint equations arising out of joint constraints and again take the derivatives and added. At each step we evaluate the dimension of the null space of $[J]$.

So, sometimes this evaluation is possible in closed form because if the matrix is very simple otherwise, we have to use numerical techniques. We have some hope of doing it in closed form because the natural constraints are at most quadratic and when you take the derivative, they are linear ok.

So, we can do some linear manipulation of the terms in the Jacobian matrix to find the nullity of the Jacobian matrix. If the nullity of the Jacobian does not decrease when a constraint is added, the constraint is redundant ok.

Finally, the boundary conditions are added last. If the nullity of the Jacobian does not decrease, boundary constraint is redundant ok. So, the final dimension of the null space of $[J]$ is the degree of freedom or mobility of the over constrained mechanism ok.

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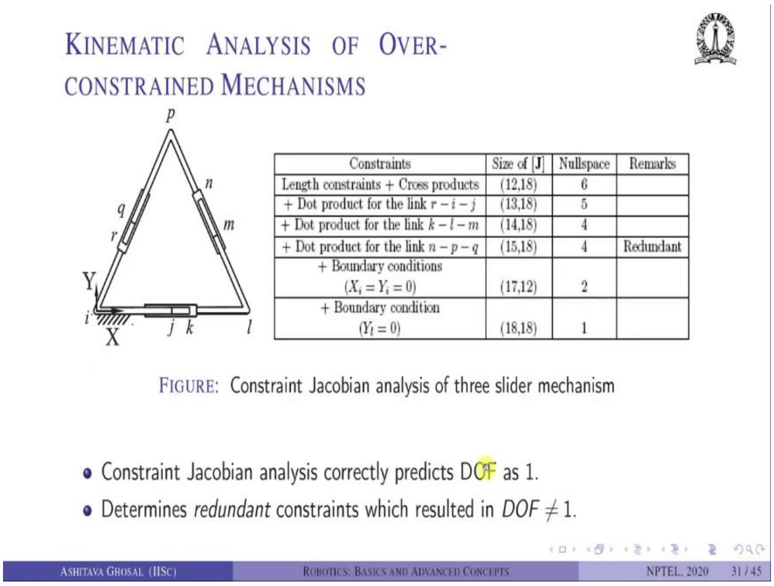

Constraints	Size of [J]	Nullspace	Remarks
Length constraints + Cross products	(12,18)	6	
+ Dot product for the link $r-i-j$	(13,18)	5	
+ Dot product for the link $k-l-m$	(14,18)	4	
+ Dot product for the link $n-p-q$	(15,18)	4	Redundant
+ Boundary conditions ($X_i = Y_i = 0$)	(17,12)	2	
+ Boundary condition ($Y_l = 0$)	(18,18)	1	

So, let us apply this to some simple examples, simple and complex examples of over constrained mechanisms. So, as I had shown you this is an example of an over constrained mechanism with three sliders. So, there are three prismatic joints arranged in an equilateral triangle and if you apply the Grubler Kutzbach criteria you will see that the degree of freedom is less than 1 ok.

So hence, why because there are three prismatic joints and you can check yourself. However, this mechanism can move why because as you pull this outwards this link will this prismatic joint will slide here this will also slide, this will also slide such that always stays as an equilateral triangle ok.

So, let us see if we can predict that this arrangement will have 1 degree of freedom using the algorithm that I discussed last slide. So, if you add the length constraints and the cross product constraint this Jacobian matrix is 12×18 ok, the null space is 6. When you add the dot products for the link $r - i - j$ which is r, i and j, so there is an angle here fixed angle ok, which tells you that some dot products some cosine of something will be equal ok.

So, when you add this the size of the Jacobian is 13×18 and the null space becomes 5. When you add the dot product for the link $k - l - m$ what is $k - l - m$? k, l, m this link, so again there is a fixed angle here. So, the size of the Jacobian is now $(14{,}18)$ ok, 14 rows 18 columns and the null space is 4 dimensional. If you add the dot product for the link which is $q - p - n$ or $n - p - q$ it will become $(15{,}18)$ and the null space remains 4.

So, what does this mean that this dot product which you added which basically says that the angle here remains constant, that is redundant ok. Then if you add boundary conditions which is that the this point is fixed which is $X_i = Y_i = 0$, then the null space becomes 2. If you now add the boundary constraint that this $Y_l = 0$, so this is this is lying along the X axis then you have degrees of freedom 1.

So, what does it tell you, that this mechanism has actually degrees of freedom 1 and then this link or this angle this dot product constraint is redundant ok. So, this is the constraint which is added here which makes the degree of freedom not equal to 1 ok. So, as we said, we can analyze the constraint Jacobian matrix by adding one at a time constraint and wherever the null space does not change we know that is redundant.

And final dimension of the Jacobian matrix, null space of the Jacobian matrix tells you the degree of freedom ok. So, in this example it works whatever the algorithm I showed you, it correctly predicts that the degree of freedom of this three-slider mechanism is 1 and it also tells you which constraint is redundant.

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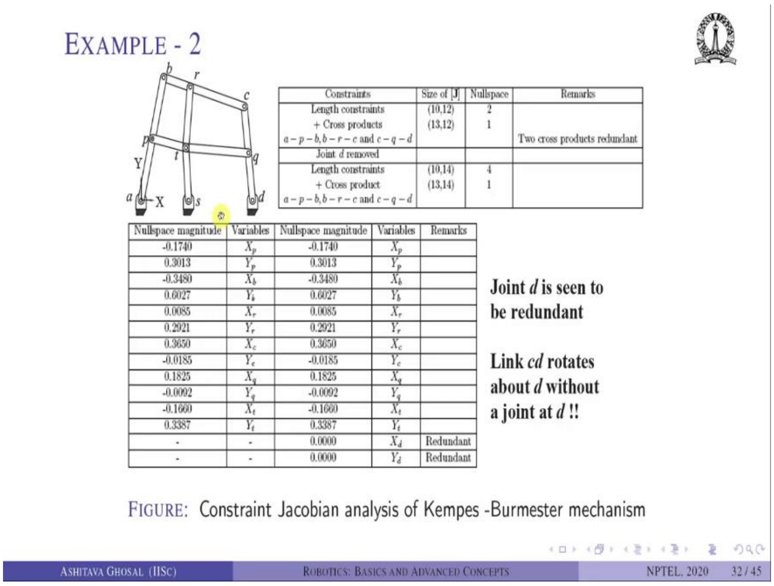

Let us look at this Kempes-Burmester mechanism ok. So, as I said there are many links. So, let us keep on adding.

So, if I put all the length constraints between p and t, p and b, r and b all the various link constraints, you can see that the Jacobian matrix is $(10,12)$, hence the null space is 2. If you add the cross product constraints which is $a - p - b$ that these are lying on the straight line, $b - r - c$ also lying on the straight line and $c - q - d$ also lying on the straight line, then the size of the Jacobian is $(13,12)$ and null space is 1 ok.

If you remove joint d nothing happens to the null space ok. If you now change the length, if you add the length constraints plus cross product constraint which is $a - p - b$, a, p, b and $b - r - c$ and $c - q - d$ what you can see is the size of the Jacobian with joint d removed is $(10,14)$, the null space is 4 and when you add the cross product the size of the Jacobian is $(13,14)$ and the null space is 1.

So, basically even though I have removed this joint d, the degree of freedom is 1 ok. So, here are some results with numbers because the Jacobian matrix otherwise become very big. So, you can see that the null space magnitude is -0.174 when variable X_p is added and so on, and finally, you can see that X_d and Y_d are redundant variables.

The null space magnitude is 0 ok. So, what it means is this joint d is seen to be redundant and link $c-d$ rotates about d without a joint at d ok. So, even though there is no joint this link $c-d$ seems to be rotating about a joint which is not there. So, this is what is called as the focal point mechanism.

So, you can think of suppose I want to polish a glass this is one of the applications of this mechanism. So, I can polish a glass or make a lens out of this glass, but then I want to go on the surface of a sphere or on a curved surface ok. So, I can have a tool here which will polish this glass. So, it will rotate about the point which is inside, below the glass surface ok.

So, hence if you want to just polish the inside of a surface you can do it, but how can you polish the outside of a glass here ok, with the rotating point inside the sphere ok. So, this mechanisms allows you to do that.

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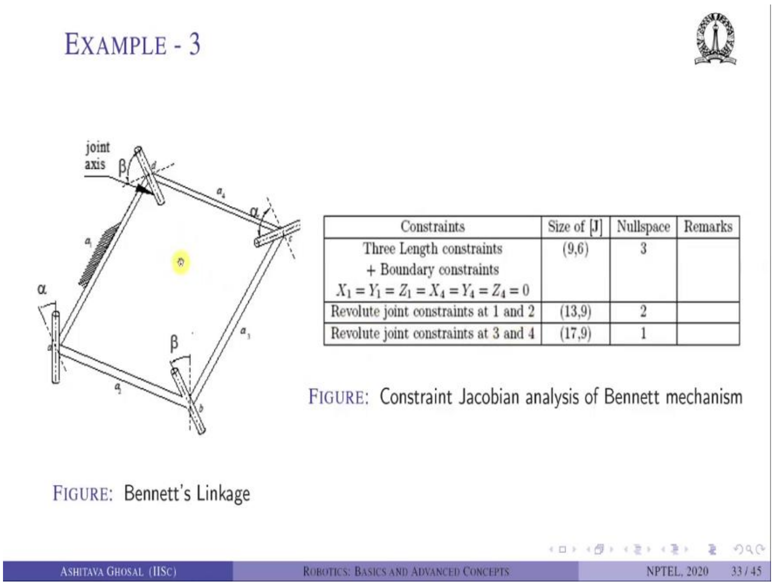

Third example is that of a Bennett mechanism, as I said this is the spatial four jointed mechanism 4R mechanism, but with special geometry. So, there is some $a_1 = a_3$, $a_4 = a_2$ and some relationship between the angles.

So, in this case again if you choose the three length constraints this length, this length or this length plus the boundary constrains that $X_1 = Y_1 = Z_1 = X_4 = Y_4 = Z_4 = 0$. So, these two points are sorry these two points are fixed.

So, then the size of the Jacobian is (9,6), the null space is 3. When you add the derivative of the revolute joint constraints at 1 and 2, the size of the Jacobian is (13,9) and the null space is 2. When you add the revolute joint constraints are 3 and 4 ok.

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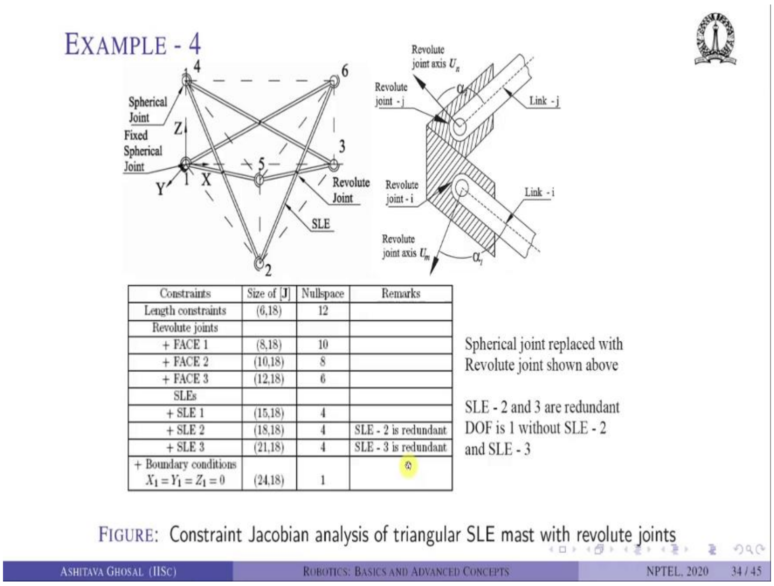

And you take the derivative of the constraint and then add into the Jacobian matrix then the size of the Jacobian is (17,9) and the null space is 1 ok.

So, we can see that the degree of freedom of this Bennett mechanism is actually 1. In this example, when you add these constraints, you have made you know its little bit cooked up, we have added the constraints such that the special geometry is taken into account ok.

So, it is not as if you start from an arbitrary geometry and see that the degree of freedom is 1 that is much harder to do. Let us look at this SLE based mast ok. So, it is a triangular SLE mast, triangular means that there are three faces in the mast and on each face there is an SLE ok.

So, it is a triangle with three faces and each face is an SLE. So, we have one point 2. 1, 2, 3, 4, 5, 6 ok. So, these are the six points in the triangle 1, 2, 3, 4, 5, 6. So, you can think about it there are six of them and then each of these is an SLE in the face and we can think of the joints as either spherical joints or we can think of the joints as revolute joints ok.

If you have revolute joints, then you have to take into account the constraints due to the revolute joint. Remember in a revolute joint we have this revolute joint axis ok. Ok nevertheless, if you were to find all the length constraints between all these points and then take the derivative and put it into the Jacobian matrix, the size of the Jacobian matrix is (6,18) the null space is 12.

If you add the revolute joints in face 1, face 2 and face 3 after adding all of them the null space drops to 6, from 10 to 8 to 6. Then if you add the SLE constraints remember we derived the SLE constraints for SLE 1, 2 and 3 for when you add 1 it becomes 4, when you add 2 it still stays at 4, when you add the 3rd SLE it stays at 4. So, basically the SLE 2 and SLE 3 are redundant.

And finally, when you add the boundary constraints, which is that first point is at (0,0,0). So, then the Jacobian matrix is (24,18) and the null space is 1 ok. So, clearly what it is showing you, this analysis is that the triangular mast with SLE at each face with revolute joints as 1 degree of freedom and there are two SLEs which are redundant meaning we do not need those two SLEs of these two faces, this two SLEs it will still have 1 degree of freedom.

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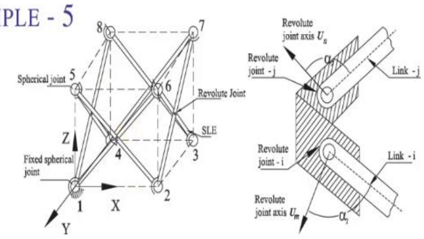
EXAMPLE - 5



Constraints	Size of [J]	Nullspace	Remarks
Length constraints	(8,24)	16	
Revolute joints			
+ FACE 1	(10,24)	14	
+ FACE 2	(12,24)	12	
+ FACE 3	(14,24)	10	
+ FACE 4	(16,24)	8	
SLEs			
+ SLE 1	(19,24)	5	
+ SLE 2	(22,24)	4	two components are redundant
+ SLE 3	(25,24)	4	SLE - 3 is redundant
+ SLE 4	(28,24)	4	SLE - 4 is redundant
+ Boundary conditions $X_1 = Y_1 = Z_1 = 0$	(31,24)	1	

Spherical joint replaced with Revolute joint shown above

SLE - 3 and 4 are redundant DOF is 1 without SLE - 3 and SLE - 4

FIGURE: Constraint Jacobian analysis of box SLE mast with revolute joints

We can also look at a box mast ok. So, basically a box mast is there are four faces which is in the shape of a box and on each face there is an SLE. So, for example, this face this is an SLE, on this face this is an SLE, on the other face this is another SLE and there are four SLEs - scissor like elements.

Again, the spherical joint can be replaced by a revolute joint and we need to take into account the revolute joint axis $\boldsymbol{U}_n$ and $\boldsymbol{U}_m$ and so on. Again, we can add all the length constraints between all the points there are eight points here. So, we will get $(8,24)$ the size of the Jacobian matrix, the null space is 16.

When you add the revolute joints in face 1, 2, 3 and 4 the four faces, then the null space of the Jacobian goes from 14 to 8 ok. When you add the SLEs 1, 2, 3 and 4 you can see the 1st SLE the null space is 5 then it stays at 4. So, what basically it means is the 2nd, 3rd and 4th SLE do not contribute to the degree of freedom of the systems. So, they are redundant ok.

So, it turns out that in SLE 2 there are only two components which are redundant, but 3 and 4 fully redundant and when you add the boundary conditions three points this first point is at $(0,0,0)$, the Jacobian matrix is $(31,24)$, the null space is 1 and we have 1 degree of freedom ok.

So, here also what we have seen is that this box shape mast will have 1 degree of freedom. Although, Grubler's criteria will tell you it is not 1 degree of freedom ok. And it tells you that this SLE 3 and 4 are redundant and two components of SLE 2 are redundant ok.

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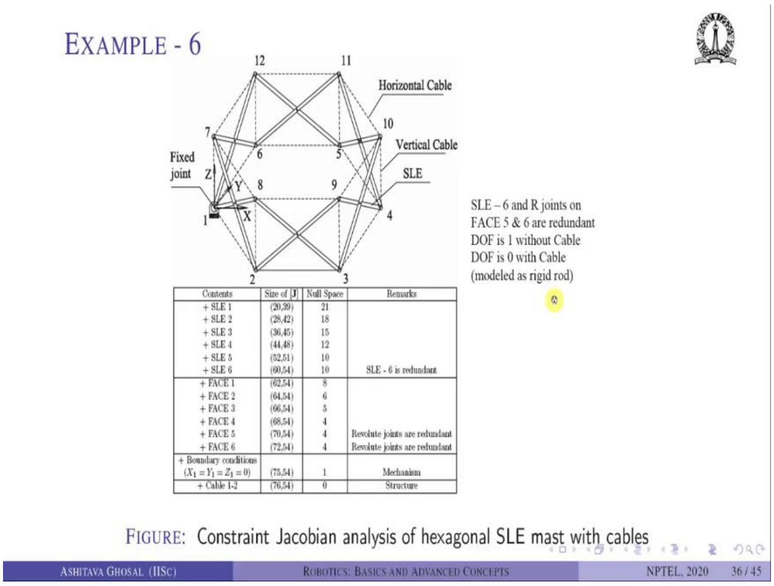

One more example, this is a hexagonal shaped mast each face has an SLE. So, there are six faces of this hexagon and there are six SLEs ok. So, again we can add this SLE 1, 2, 3, 4, 5 and 6 and we can find that the size of the Jacobian is quite large, once the 6^{th} SLE is added it is $(60,54)$.

So, you can see it is a huge matrix ok. The null space of this matrix is 10, when you add these faces the null space goes to 8, 6, 5, 4 and the last two faces 5 and 6 the revolute joints are redundant, in these two faces ok. And the boundary conditions when you add it will become 1.

If you have now a cable which is going around which is used to deployed then and if you assume that the cable can only apply tension it is like a rod in one direction then it becomes a structure. So, remember as we are discussed some of these deployable structures when you fix the cable it becomes a structure, capable of taking load.

So, that is what exactly is happening here ok. So, the degree of freedom with a cable and the cable modeled as the rigid rod is 0.

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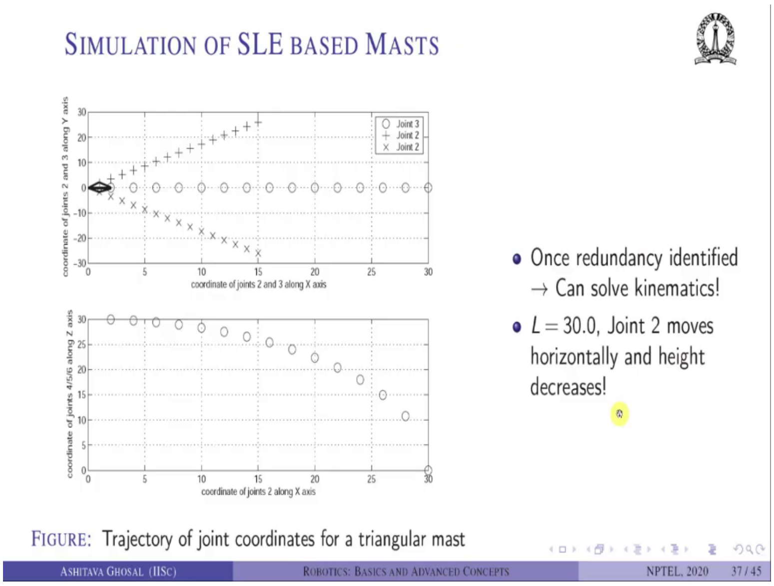

FIGURE: Trajectory of joint coordinates for a triangular mast

So, once we know what are the redundant joints and lengths, we can do kinematic analysis ok. So, for example, in the triangular mast we can plot the joint trajectories ok. So, we can see what the joint 2 is doing, what the joint 3 is doing and what the joint the other joint 2 is doing, what you can see is that it increases and joint three remains at the same place ok.

So, the location of the coordinate of the joint axis 4, 5 and 6 along the Z axis appears in this form and coordinates of joint 2 along the X axis if you plot with respect to coordinates of joint 4, 5 and 6 it looks like this. So, once the redundancy is identified we can actually solve the kinematics because now we know what is the degree of freedom ok.

We can actually use either software tools or we can write the equations, we can take the correct set of equation and solve the kinematics ok. So, for $L = 30$ which is what these two figures, we can see that the joint 2 moves horizontally and the height decreases ok.

So, this is in a triangular mast when all the links are closed to each other it is a long column like thing and when it starts moving height decreases and it becomes wider.

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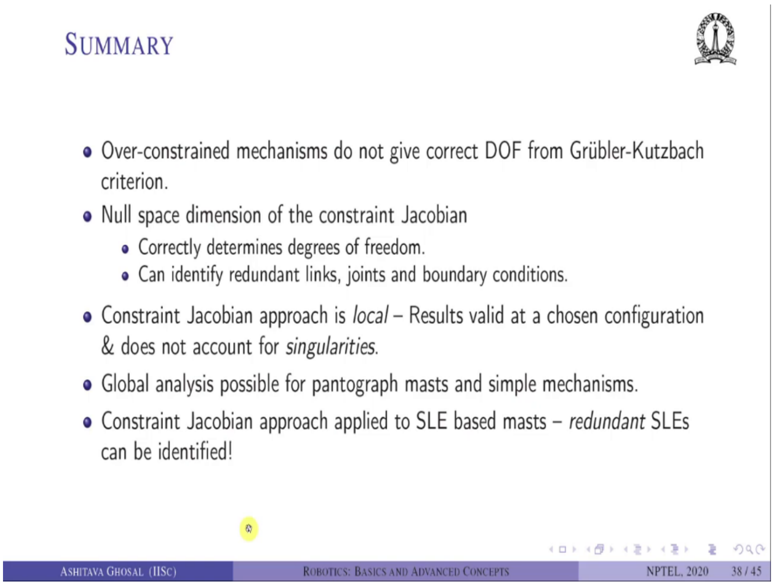

So, in summary we have over constrained mechanisms which do not give the correct DOF according to Grubler Kutzbach criterion. We can use the null space dimension of the constraint Jacobian to correctly determine the degrees of freedom; we can also identify redundant links joints and boundary conditions.

The disadvantage of this approach is the constraint Jacobian is local it is only happening at one place, when you take the derivative of the constraint equation the derivatives are evaluated at one configuration at one set of joint variables or one set of configuration variables.

So, the results are valid at the chosen configuration and it does not account for singularity ok. So, most of these mechanisms will have singular configurations and this constraint Jacobian does not take into account the singularities. Remember in most parallel mechanisms we can have loss and gain type of singularities you have seen it earlier.

So, it does not take into account this loss and gain singularities. In particular, the gain singularities that related to the constraints and that is not taken into account. It is possible to do global analysis for simple pantograph masts and some simple mechanisms.

So, we do not have to actually put in numbers in the Jacobian matrix and find the null space or the dimension of the null space, we can use symbolic algebra tools to find the null space of the Jacobian even when it is written in a symbolic form and it is possible to do because they are the derivatives of at most quadratic equations. So, the terms in the Jacobian matrix are simple because we have used natural coordinates ok.

So, one of the useful things is that if you apply this constraint Jacobian to SLE based masts which are very often used in space craft's in deployable structures, in satellites the redundant SLEs can be identified.

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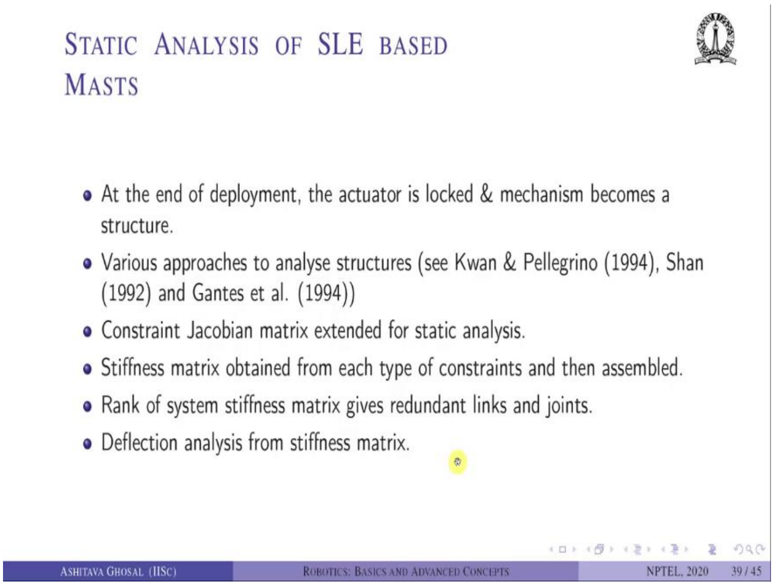

Ok so next continue. So, as I said in this lecture we will also look at static analysis of SLE based masts. So, at the end of the deployment and when the actuator is locked the mechanisms becomes a structure and it can withstand some external loading ok.

So, there are various approaches to analyse these deployable structures. So, this is one such paper very well known authors Kwan and Pellegrino and then you have Shan and then you have Gantes et al. in 94. So, what we are going to do is we are going to extend this constraint Jacobian matrix to static analysis ok.

So, we are going to obtain stiffness matrix obtained from each type of constraints and then we will assemble these stiffness matrix and then we find the rank of the stiffness matrix, it should also give again the same redundant links and joints ok.

And finally, we can obtain the stiffness of the structure and if necessary we can do a deflection analysis, we can apply a load at some point and find how much it deflects.

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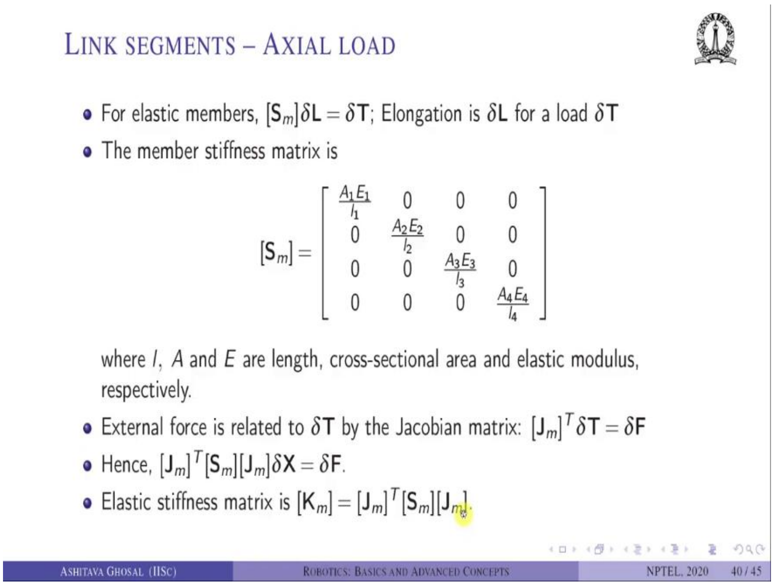

So let us continue. So, if you have an elastic member, we can write a relationship which is like this $[\boldsymbol{S}_m]\delta\boldsymbol{L} = \delta\boldsymbol{T}$. So, the $\delta\boldsymbol{L}$ is the elongation and $\delta\boldsymbol{T}$ is the load. The member stiffness matrix is given in this form for an elastic member. So, it is some AE/l correct. So, it is a long rod and if you pull it then it is under tension or compression and it deflect by some amount and this is a relationship for an elastic member.

So, where I, A and E are the length, cross sectional area and the elastic modulus, respectively. The external force is related to the $\delta\boldsymbol{T}$ by Jacobian matrix. So, $[\boldsymbol{J}_m]^T\delta\boldsymbol{T} = \delta\boldsymbol{F}$ very similar ideas were used in robotics when we said that the external force and the joint force are related using Jacobian transpose.

So, something similar is happening here. So, hence we can show, or we can obtain that $[\boldsymbol{J}_m]^T[\boldsymbol{S}_m][\boldsymbol{J}_m]\delta\boldsymbol{X} = \delta\boldsymbol{F}$. So, here $[\boldsymbol{J}_m]^T\delta\boldsymbol{T} = \delta\boldsymbol{F}$. So, if you use this equation and we use this equation we will get this. So, the elastic stiffness matrix $[\boldsymbol{K}_m] = [\boldsymbol{J}_m]^T[\boldsymbol{S}_m][\boldsymbol{J}_m]$. So, this is one important result.

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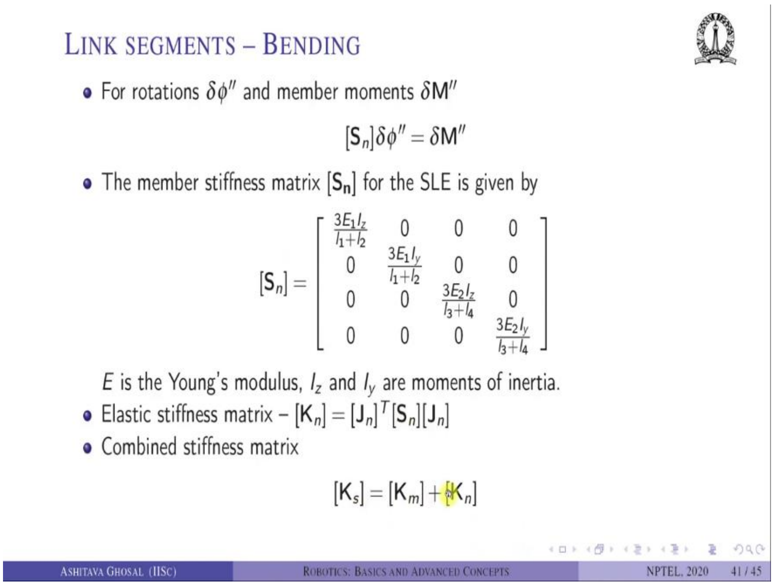

If you have link segments which are also subjected to bending, previously the link segments were axial load as a tension or compression ok. If you have bending, then we can write $[\boldsymbol{S}_n]\delta\phi'' = \delta\boldsymbol{M}''$. So, where this is the moment in the member and this is the rotation of the member and the stiffness matrix $[\boldsymbol{S}_n]$ is given by $\frac{3E_1 I_z}{I_1+I_2}$ and so on ok, this isf from basic mechanics ok.

So, where E is the Young's modulus, I_z and I_y are the moments of inertia ok. So, the elastic stiffness matrix $[\boldsymbol{K}_n] = [\boldsymbol{J}_n]^T[\boldsymbol{S}_n][\boldsymbol{J}_n]$ very similar to the previous case. And then we have a combined stiffness matrix which is due to this bending and due to the axial load $[\boldsymbol{K}_s] = [\boldsymbol{K}_m] + [\boldsymbol{K}_n]$.

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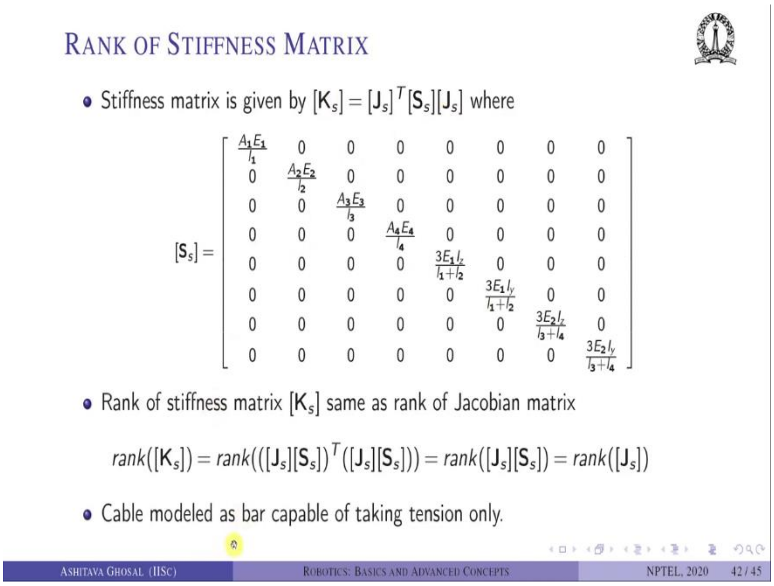

The stiffness matrix is given by $[\boldsymbol{K}_s] = [\boldsymbol{J}_s]^T[\boldsymbol{S}_s][\boldsymbol{S}_s]$, where $[\boldsymbol{S}_s]$ is given by this. So, we have some which are axial loading and some which are bending then we combine all of them and we get this matrix ok. So, the rank of the stiffness matrix $[\boldsymbol{K}_s]$ must be the same as the rank of the Jacobian matrix. So, this is in diagonal ok.

So, this is not singular. So, the rank of $[\boldsymbol{K}_s]$ will be same as the rank of this Jacobian matrix and this is shown here. Just some simple calculation showing the rank of $[\boldsymbol{K}_s]$ is $[\boldsymbol{J}_s]$ times this and so on and it is same as a rank of $[\boldsymbol{J}_s]$. You can just see it from here this matrix is never singular.

So, the rank of this must be related to the rank of the $[\boldsymbol{J}_s]$ matrix. And finally, we will model all cables as bars capable of taking tension only because we have these cables in these deployable structures.

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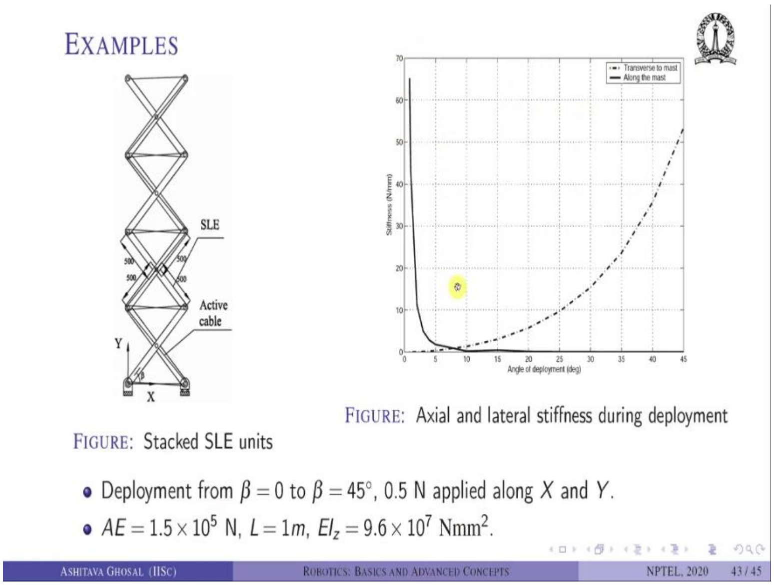

So, let us look at some examples. So, this is the stacked SLE mast. So, we have one active cable which is going like this as shown here, these are this one SLE this is another SLE this is another SLE. So, there are four SLE stacked one on top of each other.

The deployment is from $\beta = 0$ to $\beta = 45$, we apply a 0.5 Newton along X and Y and basically, we want to see what is happening ok, we assume $AE = 1.5 \times 10^5$ Newton's, each length is 1, $EI_z = 9.6 \times 10^7$ Newton millimeter square ok. So, the stiffness of this SLE transverse to the mast and along the mast looks like this.

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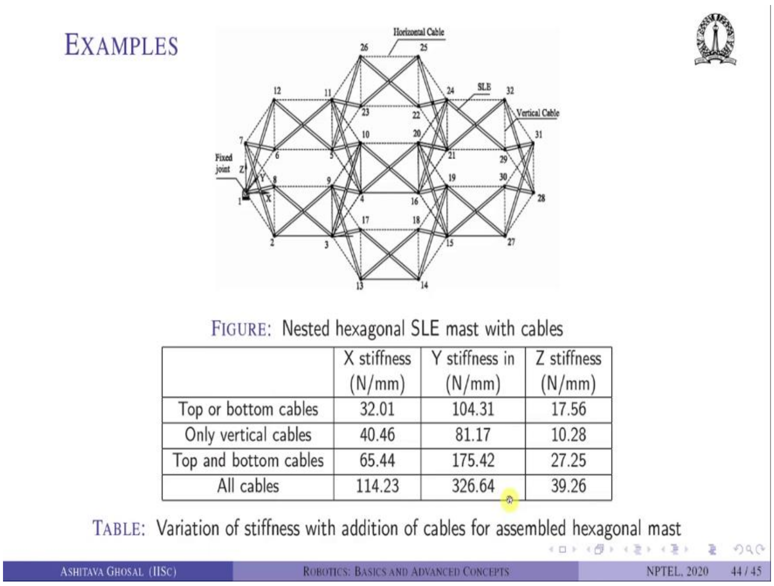

	X stiffness (N/mm)	Y stiffness in (N/mm)	Z stiffness (N/mm)
Top or bottom cables	32.01	104.31	17.56
Only vertical cables	40.46	81.17	10.28
Top and bottom cables	65.44	175.42	27.25
All cables	114.23	326.64	39.26

So, along the mast the stiffness comes down from some 65 or so Newton per millimeter to 0 as the angle increases ok. So, the angle of deployment is 0 to 45 and the stiffness along the mast decreases and the stiffness in the transverse direction keeps on increasing to this quantity ok.

So, does this make sense? yes, when you are deployed fully then you have this stiffness which is very low in the transverse direction whereas, it is reasonably high in the axial direction. This is another example of a hexagonal SLE mast with cables ok. So, we can find after the deployment again we can obtain the stiffness matrix and we can find the X stiffness, the Y stiffness and the Z stiffness.

For top and bottom cables if both the cables are there then you have these values 32.01, 104.31, 17.56 and so on. And if it is only vertical cables, then we get this and if we have top and bottom cables, we have this and we have all cables then you get such stiffness ok. So, either top or bottom, or only vertical, or together top and bottom, or all cable.

So, if you add all these cables the stiffness increases and again if you think about it is correct. The previous example actually matches with the result in a paper by Kwan and Pellegrino. So, this makes sense ok.

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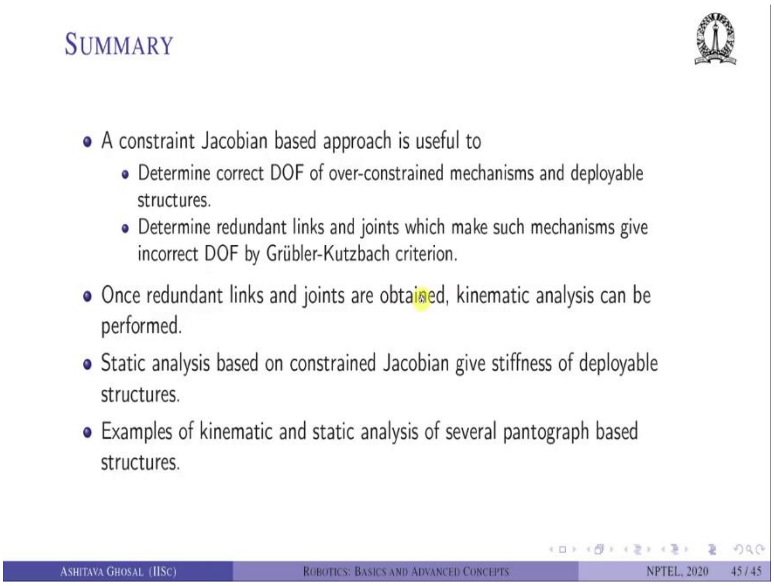

So, in summary we have proposed, or we have used the constraint Jacobian based approach to determine the correct degrees of freedom of an over constrained mechanism and deployable structures.

It tells us which are the redundant links and joints which makes such mechanisms give incorrect DOF by Grubler Kutzbach criterion. Once the redundant links are obtained, once redundant joints are obtained we can perform kinematic analysis we can see how the links and the joints move.

We can also do static analysis based on the constraint Jacobian which gives us stiffness of deployable structures and I have shown you several examples of kinematic and static analysis of several pantograph based structures ok.

So, thank you. So, we will stop here.